DOI: 10.2478/auom-2018-0036
An. Şt. Univ. Ovidius Constanţa **Vol. 26(3),2018, 109–126**

Division hypernear-rings

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Abstract

This paper is a continuation of our work on hypernear-rings with a defect of distributivity D. In particular, here we introduce and study a new subclass of hypernear-rings, called D-division hypernear-rings, establishing a necessary and sufficient condition such that a hypernear-ring with the defect D is a D-division hypernear-ring. Several properties and examples of these two subclasses of hypernear-rings are presented and discussed.

1 Introduction

Canonical hypergroups have been introduced as the additive structures of Krasner hyperrings [17], that have also a multiplicative part-a semigroup-that distributes (from both sides) over the hyperaddition. Mittas [22, 23] was the first one having the idea to study them separately, as independent hypergroups, opening a new theory, developed later on by Corsini [3], Roth [28], Massouros [21], etc. Non commutative canonical hypergroups, called quasicanonical hypergroups by Bonansinga [1], and later on by Massouros [20], and polygroups by Comer [2], have been studied in parallel by the aforementioned researchers (and not only by them) as independent hyperstructures,

Key Words: hypernear-ring; defect of distributivity, d.g. hypernear-ring, division hypernear-ring.
2010 Mathematics Subject Classification: 20N20
Received: 14.11.2017
Accepted: 31.01.2018.

and used later on as the additive structures of a new type of ring-like hyperstructures. An overview of the theory of polygroups is covered by Davvaz's book [8]. Concretely, Dašić [7] defined *hypernear-rings*, as a natural generalisation of near-rings, endowing a quasicanonical hypergroup $(R, +)$ with a multiplication, being distributive with respect to the hyperaddition on the left side, and such that $(R, \cdot)$ is a semigroup with a bilaterally absorbing element. Based on the similar terminology of near-rings, Gontineac [11] called this hypernear-ring a zero-symmetric one. In the same period, the theory of hyperrings enriched itself with different types of hyperrings, having only one or both between addition and multiplication as hyperoperation (for example: additive hyperring [29], multiplicative hyperring [27], superring [22], hyperring in the general case [4, 29]), or substituting the distributivity with the weak distributivity [29] (when the equality is replaced by non void intersection between the left and right side), or with the inclusive distributivity [12, 16] (when the equality symbol is substituted by the inclusion one). A detailed discussion of this terminology is included in [13, 14, 24]. Even if the term "inclusive distributivity" is the most appropriate one for defining this property, it was used only in few papers, while " the weak distributivity" was preferred for the same type of distributivity. Here we keep the initial notation, saying that the multiplication is inclusive distributive with respect to the hyperaddition from the left side, if, for any three elements x, y, z, it holds: $x \cdot (y+z) \subseteq x \cdot y + x \cdot z$. In this paper, as in our previous one [14], by a *hypernear-ring* $(R, +, \cdot)$ we mean a quasicanonical hypergroup $(R, +)$ endowed also with a multiplication, such that $(R, \cdot)$ is a semigroup with a bilaterally absorbing element, and the multiplication is inclusive distributive with respect to the hyperaddition on the left side. If the distributivity holds, i.e. for any theree elelments x, y, z, we have $x \cdot (y + z) = x \cdot y + x \cdot z$, then we call the hypernear-ring a *strongly distributive hypernear-ring.* If the additive structure is a hypergroup and all the other properties related to the multiplication are conserved, we obtain a *general hypernear-ring* [15].

The authors have recently started [14] the study of hypernear-rings R with a defect of distributivity D, where D is the normal subhypergroup of the additive structure $(R, +)$ generated by the elements $d \in -(x\cdot s+y\cdot s)+(x+y)\cdot s$, with $x, y \in R$, while $s \in S$, where $(S, \cdot)$ is a multiplicative subsemigroup of the semigroup $(R, \cdot)$ whose elements generate $(R, +)$. In the present paper we continue in the same direction, introducing the class of *D-division hypernear-*

rings as a subclass of hypernear-rings with a defect D, and that one of *division hypernear-rings*. Another aim of the paper is to state a necessary and sufficient condition under which a hypernear-ring with a defect of distributivity D is a D-division hypernear-ring. A similar condition holds for near-rings, known as the Ligh's theorem for near-rings [18], and it can be also obtained as a consequence of our more general result on hypernear-rings, as we show in the last result of this paper.

2 Preliminaries

The first part of this section gathers together the main properties of division near-rings, most of them presented in Ligh's papers [18, 19]. The theory of hypernear-rings with a defect of distributivity is briefly recalled in the second part of this section, using the authors' paper [14].

2.1 Division near-rings

Division near-rings were first considered by Dickson [9]. Let $(R, +, \cdot)$ be a left near-ring, i.e. $(R, +)$ is a group (not necessarily commutative) with the unit element 0, $(R, \cdot)$ is a semigroup and the left distributivity holds: $x \cdot (y + z) = x \cdot y + x \cdot z$, for any $x, y, z \in R$. It is clear that $x \cdot 0 = 0$, for any $x \in R$, while it might exist $y \in R$ such that $0 \cdot y \neq 0$. If 0 is a bilaterally absorbing element, that is $0 \cdot x = x \cdot 0 = 0$, for any $x \in R$, then R is called a *zero-symmetric near-ring*. In particular, if $(R, +, \cdot)$ is a left near-ring that contains a multiplicative semigroup S, whose elements generate $(R, +)$ and satisfy $(x+y)\cdot s = x\cdot s+y\cdot s$, for all $x, y \in R$ and $s \in S$, then we say that R is a *distributively generated near-ring (d.g. near-ring)*. Regarding the classical example of a near-ring, that one represented by the set of the functions from an additive group G into itself with the pointwise addition and the natural composition of functions, if S is the multiplicative semigroup of the endomorphisms of G and R' is the subnear-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [10]. A near-ring containing more than one element is called a *division near-ring*, if the set $R \setminus \{0\}$ is a multiplicative group [18]. Several examples of division near-rings are given in [10]. It is well known that every division ring is a division near-ring, while there are division near-rings which are not division rings. The main core of Ligh's paper [18] is to show that three fundamental theorems in ring theory can be generalised to d.g. near-rings, while they do not hold for arbitrary near-rings, and they are recalled here below.

1. If e is a unique left identity of a ring, then it is also a right identity.
2. If R is a ring with more than one element such that $a \cdot R = R$, for all nonzero element $a \in R$, then R is a division ring.
3. A ring R with identity $e \neq 0$ is a division ring if and only if it has no proper right ideals.

Lemma 2.1. [18] *If R is a d.g. near-ring, then $0 \cdot x = 0$, for all $x \in R$.*

Theorem 2.2. [18] *A necessary and sufficient condition for a d.g. near-ring with more than one element to be division ring is that for all non-zero elements $a \in R$, it holds $a \cdot R = R$.*

Based on these two results, we can slightly ease the above condition.

Lemma 2.3. *If R is a d.g. near-ring with more than one element, then $a \cdot R = R$, for all $a \in R \setminus \{0\}$, if and only if $a \cdot (R \setminus \{0\}) = R \setminus \{0\}$, for all $a \in R \setminus \{0\}$.*

Proof. Obviously, $a \cdot (R \setminus \{0\}) = R \setminus \{0\}$, for all $a \in R \setminus \{0\}$ implies that $a \cdot R = R$, for all $a \in R \setminus \{0\}$. Suppose now that we have $a \cdot R = R$, for all $a \in R \setminus \{0\}$. First we prove that $a \in a \cdot (R \setminus \{0\}) \subseteq R \setminus \{0\}$, for $a \neq 0$. If there exist $a \neq 0$, $b \neq 0$ such that $a \cdot b = 0$, then since $a \cdot R = R$ and $b \cdot R = R$ it follows that there exist $x, y \in R$ such that $a = a \cdot x$ and $x = b \cdot y$. Therefore, by Lemma 2.1, we have $0 = 0 \cdot y = a \cdot b \cdot y = a \cdot x = a$, which is a contradiction. Thus $a \in a \cdot (R \setminus \{0\}) \subseteq R \setminus \{0\}$. Obviously, $R \setminus \{0\} \subseteq a \cdot R = R$ and since $a \cdot 0 = 0$ it follows that $R \setminus \{0\} \subseteq a \cdot (R \setminus \{0\})$. Therefore, $a \cdot (R \setminus \{0\}) = R \setminus \{0\}$, for all $a \in R \setminus \{0\}$. □

Another example of division ring is given by the following result.

Lemma 2.4. *Every d.g. division near-ring R is a division ring.*

Proof. By Theorem 2.2 [18], the additive group $(R, +)$ of a division near-ring is abelian. From the proof of Theorem 3.4 [18] we know that every element of R is right distributive, i.e. $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$. Thereby, if R is d.g. near-ring, then R is a division near-ring if and only if R is a division ring. □

2.2 Hypernear-rings with a defect of distributivity

Keeping the terminology from our previous paper [14], by a *hypernear-ring* we mean an algebraic system $(R, +, \cdot)$, where R is a non-empty set endowed with a hyperoperation $"+" : R \times R \longrightarrow \mathcal{P}^*(R)$, and an operation $"\cdot" : R \times R \longrightarrow R$, satisfying the following axioms:

I) $(R, +)$ is a quasicanonical hypergroup, i.e. it satisfies the following axioms:

   i) $x + (y + z) = (x + y) + z$, for any $x, y, z \in R$;

   ii) there exists $0 \in R$ such that, for any $x \in R$, $x + 0 = 0 + x = \{x\}$;

   iii) for any $x \in R$, there exists a unique element $-x \in R$, such that $0 \in x + (-x) \cap (-x) + x$;

   iv) for any $x, y, z \in R$, $z \in x+y$ implies that $x \in z+(-y)$, $y \in (-x)+z$.

II) $(R, \cdot)$ is a semigroup endowed with a two-sided absorbing element 0, i.e. for any $x \in R$, $x \cdot 0 = 0 \cdot x = 0$.

III) The operation $" \cdot "$ is inclusive distributive with respect to the hyperoperation $"+"$ from the left side: for any $x, y, z \in R$, $x \cdot (y+z) \subseteq x \cdot y + x \cdot z$.

If in axiom III) the equality holds, then we get a *strongly distributive hypernear-ring*, called simply by Dašić' [5] a hypernear-ring and by Gontineac [11] a *zero-symmetric hypernear-ring.*

Lemma 2.5. *Let $(R, +, \cdot)$ be a hypernear-ring. For any $x, y \in R$, the following identities are fullfilled:*

i) $-(x + y) = (-y) + (-x)$.

ii) $y \cdot (-x) = -(y \cdot x)$.

Definition 2.6. [11, 20] Let $(R, +, \cdot)$ be a hypernear-ring.

i) A subhypergroup A of the hypergroup $(R, +)$ is called a *normal subhypergroup* if, for all $x \in R$, it holds: $x + A - x \subseteq A$.

ii) A normal subhypergroup A of the hypergroup $(R, +)$ is called a *left hyperideal* of R, if $x \cdot a \in A$, for all $x \in R, a \in A$.

iii) A normal subhypergroup A of the hypergroup $(R,+)$ is called a *right hyperideal* of R if $(x+A)\cdot y - x\cdot y \subseteq A$, for all $x,y \in R$.

iv) If A is a left and a right hyperideal of R, i.e. if $[(x+A)\cdot y - x\cdot y] \cup z\cdot A \subseteq A$, for all $x,y,z \in R$, then we say that A is a *hyperideal* of R.

Remark 2.7. i) If A is a normal subhypergroup of R, then $A = x+A-x$, or equivalently $x+A = A+x$, for any $x \in R$.

ii) It can be easily verified that the condition $(x+A)\cdot y - x\cdot y \subseteq A$ in the previous definition is equivalent to the condition $-(x\cdot y)+(x+A)\cdot y \subseteq A$, for any $x,y \in R$.

Definition 2.8. [14] Let $(R,+,\cdot)$ be a hypernear-ring. If $(S,\cdot)$ is a multiplicative subsemigroup of the semigroup $(R,\cdot)$ such that the elements of S generate $(R,+)$, i.e. for every $r \in R$ there exists a finite sum $\sum_{i=1}^{n} \pm s_i$, where $s_i \in S$, for any $i \in \{1,2,\ldots,n\}$, such that $r \in \sum_{i=1}^{n} \pm s_i$, then we say that S is a *set of generators* of the hypernear-ring R.

The hypernear-ring R with the set of generators S will be denoted by (R,S).

Example 2.9. [14] *Defining on the set* $R = \{0,1,2,3,4,5,6\}$ *the hyperaddition and the multiplication by the following tables*

$+$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	$\{0,6\}$	1
2	2	3	4	5	$\{0,6\}$	1	2
3	3	4	5	$\{0,6\}$	1	2	3
4	4	5	$\{0,6\}$	1	2	3	4
5	5	$\{0,6\}$	1	2	3	4	5
6	6	1	2	3	4	5	0

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	5	4	3	2	1	0
2	0	1	2	3	4	5	0
3	0	0	0	0	0	0	0
4	0	5	4	3	2	1	0
5	0	1	2	3	4	5	0
6	0	0	0	0	0	0	0

one obtains a hypernear-ring. It is simple to check that $S = \{0, 2, 3\}$ *is a system of generators of the hypergroup* $(R, +)$*.*

Now we present another example of hypernear-ring, that later on in this paper we will refer to.

Proposition 2.10. *Let* $(H, +)$ *be a quasicanonical hypergroup and* $T_0(H) = \{f : H \longrightarrow H \mid f(0) = 0\}$*. Define on* $T_0(H)$ *the hyperaddition* $\oplus_T$ *and the multiplication* $\odot_T$ *as follows:*

$$f \oplus_T g = \{h \in T_o(H)\ h(x) \in f(x) + g(x), \forall x \in H\}$$
$$f \odot_T g = g(f(x)), \forall x \in H.$$

The structure $(T_0(H), \oplus_T, \odot_T)$ *is a hypernear-ring.*

Proof. Let $f, g \in T_0(H)$. We prove that there exists $h \in T_0(H)$ such that $h(x) \in f(x) + g(x)$, for all $x \in H$. Let $x \in H$. Since $f(x) + g(x) \neq \emptyset$, we can choose $h_x \in f(x) + g(x)$ and we define $h(x) = h_x$. Obviously, $h_0 \in f(0) + g(0) = \{0\}$, i.e. $h(0) = 0$. Now we prove that the hyperoperation $\oplus_T$ is associative. Let $f, g, h \in T_0(H)$. Set $L = (f \oplus_T g) \oplus_T h = \{h" \mid \forall x \in H, h"(x) \in h'(x) + h(x) \wedge h'(x) \in f(x) + g(x)\}$ and $D = f \oplus_T (g \oplus_T h) = \{f" \mid \forall x \in H, f"(x) \in f(x) + f'(x) \wedge f'(x) \in g(x) + h(x)\}$. Thus, if $h" \in L$, then $h"(x) \in (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$. It means that, for any $x \in H$, there exists $a_x \in g(x) + h(x)$ such that $h"(x) \in f(x) + a_x$. Let us define $f'(x) = a_x$. Then $f' \in g \oplus_T h$ and for all $x \in H$, it holds $h"(x) \in f(x) + f'(x)$. Therefore, $h" \in D$. So $L \subseteq D$. Similarly, one gets $D \subseteq L$. Let $\delta : H \longrightarrow H$ be defined by $\delta(x) = 0$, for all $x \in H$. Then, for any $f \in T_0(H)$ it holds $f + \delta = \delta + f = \{f\}$. Moreover, if $f \in T_0(H)$, then the map $-f : H \longrightarrow H$ defined by $(-f)(x) = -(f(x))$ is the unique element in $T_0(H)$

such that $\delta \in f+(-f)\cap(-f)+f$. Indeed, if $\delta \in (f+g)\cap(g+f)$, then, for all $x \in H$, $0 \in (f(x)+g(x))\cap(g(x)+f(x))$, i.e. $g(x)=(-f)(x)$. Besides, for all $f,g,h \in T_0(H)$, $f \in g+h$ implies that, for all $x \in H$, it holds $f(x) \in g(x)+h(x)$ and thereby $g(x) \in f(x)+(-h(x))$, i.e. $g \in f+(-h)$. Similarly, one gets $h \in (-f)+g$. From all above mentioned properties, it follows that $(T_0(H), \oplus_T)$ is a quasicanonical hypergroup. It is clear that $(T_0(H), \odot_T)$ is a semigroup, having δ as a bilaterally absorbing element, since $f \odot_T \delta = f \circ \delta = \delta = \delta \circ f = \delta \odot_T f$.

It remains to prove that the multiplication $\odot_T$ is inclusive distributive with respect to the hyperaddition $\oplus_T$. Let $f,g,h \in T_0(H)$. Set $L = f \odot_T (g \oplus_T h) = \{f \odot_T k \mid k \in g \oplus_T h\}$ and $D = (f \odot_T g) \oplus_T (f \odot_T h) = \{h' \mid \forall x \in H, h'(x) \in g(f(x)) + h(f(x))\}$. Let $k \in g \oplus_T h$. Then, for all $x \in H$, it holds $(f \odot_T k)(x) = k(f(x)) \subseteq g(f(x)) + h(f(x))$. Thus, $f \odot_T k \in D$, meaning that $L \subseteq D$ and now the proof is complete.

□

Proposition 2.11. *With the notation in Proposition 2.10, set* $End_0(H) = \{f : H \longrightarrow H \mid \forall x,y \in H, f(x+y) \subseteq f(x)+f(y), f(0)=0\}$. *Since* $End_0(H)$ *is a multiplicative semigroup of* $T_0(H)$, *define* $E(H) = \{f \mid f \in \sum_{i=1}^n \pm f_i, n \in \mathbb{N}, f_i \in End_0(H)\}$. *Then* $E(H)$ *is a subhypernear-ring of* $T_0(H)$ *generated by* $End_0(H)$.

Proof. If $f,g \in E(H)$, then, for $h \in f \oplus_T g$, it holds $h(x) \in f(x)+g(x)$, for all $x \in H$. So if $f \in \sum_{i=1}^n \pm f_i$ and $g \in \sum_{j=1}^n \pm f'_j$, then, for all $x \in H$, it holds $h(x) \in \sum_{i=1}^n \pm f_i(x) + \sum_{j=1}^n \pm f'_j(x)$, i.e. $h \in E(H)$. Obviously, $\delta \in E(H)$. If $f \in E(H)$, then for all $x \in H$, it holds $f(x) \in \sum_{i=1}^n \pm f_i(x)$ and then $(-f)(x) = -(f(x)) \in -(\sum_{i=1}^n \pm f_i(x)) = \sum_{i=n}^1 \mp f_i(x)$. Thus, $-f \in \sum_{i=n}^1 \mp f_i$, meaning that $-f \in E(H)$.

Let $f,g \in E(H)$ such that $f \in \sum_{i=1}^n \pm f_i$ and $g \in \sum_{j=1}^n \pm f'_j$. Note that, for all $x \in H$, if $f_j \in End_0(H)$, then $f_j(-x) = -f_j(x)$. Indeed, $0 = f_j(0) \in f_j(x+(-x)) \subseteq f_j(x)+f_j(-x)$ and $0 \in f_j((-x)+x) \subseteq f_j(-x)+f_j(x)$. Thus, $f_j(-x) = -f_j(x)$, since the inverse element is unique in H. Now, for all $x \in H$, we have

$$\begin{aligned}
&(f \odot_T g)(x) = g(f(x)) \subseteq \textstyle\sum_{j=1}^n \pm f'_j(f(x)) \subseteq \sum_{j=1}^n \pm f'_j(\sum_{i=1}^n \pm f_i(x)) \subseteq \\
&\subseteq \textstyle\sum_{j=1}^n \pm(\sum_{i=1}^n f'_j(\pm f_i(x))) = \sum_{j=1}^n \pm \sum_{i=1}^n \pm f'_j(f_i(x)) = \\
&= \textstyle\sum_{j=1}^n \pm \sum_{i=1}^n \pm(f'_j \odot_T f_i)(x).
\end{aligned}$$

So $f \odot_T g \in \sum_{i=1}^n \pm(\sum_{j=1}^n \pm(f'_j \odot_T f_i)) \in E(H)$, since $f'_j \odot_T f_i \in End_0(H)$. □

Definition 2.12. [14] Let $(R, +, \cdot)$ be a hypernear-ring with the set of generators S and set

$$\begin{aligned} D_S &= \{d \mid d \in -(x \cdot s + y \cdot s) + (x + y) \cdot s, x, y \in R, s \in S\} = \\ &= \bigcup_{\substack{x,y \in R \\ s \in S}} [-(x \cdot s + y \cdot s) + (x + y) \cdot s]. \end{aligned}$$

The normal subhypergroup D of the hypergroup $(R, +)$ generated by D_S is called the *defect of distributivity* of the hypernear-ring (R, S). Moreover, we say that (R, S) is a *hypernear-ring with the defect* D.

Example 2.13. *Let us continue with the hypernear-ring in Example 2.9. In [14] we have found that its defect of distributivity is $D = \{0, 3, 6\}$.*

Lemma 2.14. [14] *Let (R, S) be a hypernear-ring with the defect D. For all $x, y \in R$ and $s \in S$, it holds:*

$$(x + y) \cdot s \subseteq x \cdot s + y \cdot s + D.$$

Lemma 2.15. [14] *If (R, S) is a hypernear-ring with the defect D, then:*

1. $(-x) \cdot s \subseteq -(x \cdot s) + D$,
2. $(x - y) \cdot s \subseteq x \cdot s - y \cdot s + D$,

for all $x, y \in R$ and each $s \in S$.

We conclude this section with some properties of the defect of the distributivity of a hypernear-ring.

Corollary 2.16. [14] *If (R, S) is a hypernear-ring with the defect D, then:*

1. $R \cdot D \subseteq D$
2. $D \cdot S \subseteq D$
3. *D is a hyperideal of R.*

Definition 2.17. [14] If (R, S) is a hypernear-ring with the defect $D = \{0\}$, then we say that (R, S) is a *distributively generated hypernear-ring* (by short, *d.g. hypernear-ring*).

Example 2.18. *If $(H,+)$ is a quasicanonical hypergroup such that $x+(-x) = (-x)+x = \{0\}$, for all $x \in H$, then the hypernear-ring $E(H)$ in Proposition 2.11 is distributively generated by $End(H)$.*

First, we note that $End_0(H) = End(H) = \{f : H \longrightarrow H \mid \forall x, y \in H, f(x+y) \subseteq f(x) + f(y)\}$. Indeed, for $f \in End(H)$, we have $f(0) \subseteq f(0) + f(0)$ and thus $0 = f(0) + (-f(0)) \subseteq (f(0) + f(0) + (-f(0))$, i.e. $0 \in f(0) + (f(0) + (-f(0)) = f(0) + \{0\} = f(0)$. Thus, $f(0) = 0$.

The hypernear-ring $E(H)$ is generated by $S = End(H)$ and the defect of distributivity of $E(H)$ is $D = \{0\}$. Indeed,

$$D_S = \bigcup_{\substack{f,g \in E(H)\\ h \in End(H)}} \left[-(f \odot_T h + g \odot_T h) + (f+g) \odot_T h\right].$$

If $l \in D_S$, then $l \in (-p) + r \circ h$, for some $p \in f \odot_T h \oplus_T g \odot_T h$ and $r \in f+g$. So, for all $x \in H$, it holds $p(x) \in h(f(x)) + h(g(x))$ and $r(x) \in f(x) + g(x)$ and $l(x) \in (-p)(x) + h(r(x)) = -(p(x)) + h(r(x)) \subseteq -\left[h(f(x)) + h(g(x))\right] + h(f(x)+g(x)) = -h(g(x)) - h(f(x)) + h(f(x)+g(x)) \subseteq -h(g(x)) - h(f(x)) + h(f(x)) + h(g(x)) = \{0\}$. Thus $D_S = \{0\}$ and thereby $D = \{0\}$.

Theorem 2.19. [14] *Let (R,S) be a hypernear-ring with the defect D and let A be a hyperideal of R. Then the factor hypernear-ring $(\bar{R} = R/A, \oplus, \odot)$ has a set of generators $\bar{S} = \{C(s) = s + A \mid s \in S\}$ and the defect $\bar{D} = \{C(d) = d + A \mid d \in D\}$.*

Corollary 2.20. [14] *Let (R,S) be a hypernear-ring with the defect D and let A be a hyperideal of R. Then the factor hypernear-ring $(\bar{R}, \bar{S})$ is a d.g. hypernear-ring if and only if $D \subseteq A$.*

Example 2.21. *Let (R,S) be a hypernear-ring with the defect D. Since D is a hyperideal of R, it follows immediately that the factor hypernear-ring $\bar{R} = R/D$, having the set of generators $\bar{S} = \{s + D \mid s \in S\}$, is distributively generated.*

3 D-division hypernear-rings

In this section, after defining the notion of D-division hypernear-ring, we investigate when a hypernear-ring with the defect D is a D-division hypernear-ring.

Definition 3.1. Let (R, S) be a hypernear-ring with the defect of distributivity $D \neq R$. The structure $(R \setminus D, \cdot)$ is a *D-multiplicative group* if the following assertions hold.

1. The set $R \setminus D$ is closed under the multiplication.
2. There exists $e \in R \setminus D$ such that, for each $x \in R \setminus D$ it holds $x \cdot e \in x + D$ and $e \cdot x \in x + D$. A such element e is called the *identity element*.
3. For each $x \in R \setminus D$ there exists $x' \in R \setminus D$, such that $x \cdot x' \in e + D$ and $x' \cdot x \in e + D$.

Definition 3.2. Let (R, S) be a hypernear-ring with the defect of distributivity D. We say that R is a *D-division hypernear-ring (a hypernear-ring of D-fractions)* if $(R \setminus D, \cdot)$ is a D-multiplicative group.

Definition 3.3. Let (R, S) be a hypernear-ring with the defect of distributivity D. We say that (R, S) is a *hypernear-ring without D-divisors* if, for all $x, y \in R$, $x \cdot y \in D$ implies that $x \in D$ or $y \in D$. Otherwise, we say that R *has D-divisors* if there exist $x, y \in R \setminus D$ such that $x \cdot y \in D$.

Example 3.4. *The hypernear-ring in Example 2.9 is without D-divisors.*

Proposition 3.5. *Let (R, S) be a hypernear-ring with the defect of distributivity $D \neq R$. If $a \cdot (R \setminus D) + D = R \setminus D + D$, for all $a \in R \setminus D$, then R is a hypernear-ring without D-divisors.*

Proof. Suppose there exist $x, y \in R \setminus D$ such that $x \cdot y \in D$. Since $x \in R \setminus D \subseteq R \setminus D + D = x \cdot (R \setminus D) + D$, it follows that there exists $x' \in R \setminus D$ and $d_1 \in D$ such that $x \in x \cdot x' + d_1$. Moreover, from $x' \in R \setminus D \subseteq R \setminus D + D = y \cdot (R \setminus D) + D$, it follows that there exist $y' \in R \setminus D$ and $d_2 \in D$ such that $x' \in y \cdot y' + d_2$. Therefore, $x \in x \cdot (y \cdot y' + d_2) + d_1 \subseteq x \cdot y \cdot y' + x \cdot d_2 + d_1$. Since D is a hyperideal of R, for all $r, r' \in R$, it holds: $(r + D) \cdot r' - r \cdot r' \subseteq D$. In particular, for $r = 0$ we obtain $(0 + D) \cdot r' - 0 \cdot r' \subseteq D$, i.e. $D \cdot r' - 0 \subseteq D$, meaning that $D \cdot R \subseteq D$. Therefore, $(x \cdot y) \cdot y' \in D$ as $x \cdot y \in D$. Besides, $x \cdot d_2 \in D$, as $d_2 \in D$. It follows that $(x \cdot y) \cdot y' + x \cdot d_2 + d_1 \subseteq D$, i.e. $x \in D$, which contradicts the initial assumption. We conclude that R is a hypernear-ring without D-divisors. $\square$

Corollary 3.6. *If (R, S) is a hypernear-ring with the defect of distributivity $D \neq R$ such that $a \cdot (R \setminus D) + D = R \setminus D + D$, for all $a \in R \setminus D$, then the set $R \setminus D$ is closed under the multiplication.*

Proof. Since R is a hypernear-ring without D-divisors, it means that $a, b \in R \setminus D$ implies that $a \cdot b \in R \setminus D$. $\square$

Theorem 3.7. *Let (R, S) be a hypernear-ring with the defect $D \neq R$. A necessary and sufficient condition for the hypernear-ring R to be a D-division hypernear-ring is that $a \cdot (R \setminus D) + D = R \setminus D + D$, for all $a \in R \setminus D$.*

Proof. Necessity. Let $R \setminus D$ be a D-multiplicative group with the identity element e. Let $a \in R \setminus D$. Obviously, $a \cdot (R \setminus D) + D \subseteq R \setminus D + D$. We prove now the other inclusion $R \setminus D + D \subseteq a \cdot (R \setminus D) + D$. Suppose $x \in R \setminus D$. Since $R \setminus D$ is a D-multiplicative group, it follows that there exist $a' \in R \setminus D$ and $d_1 \in D$ such that $a \cdot a' \in e + d_1$. Besides there exists $d_2 \in D$ such that $x \in e \cdot x + d_2 \subseteq (a \cdot a' - d_1) \cdot x + d_2$. Since D is a hyperideal of R, we have $(a \cdot a' - d_1) \cdot x - (a \cdot a') \cdot x \subseteq D$, and therefore $(a \cdot a' - d_1) \cdot x \subseteq ((a \cdot a' - d_1) \cdot x - (a \cdot a') \cdot x + (a \cdot a') \cdot x \subseteq D + (a \cdot a') \cdot x$. It follows that $x \in D + a \cdot (a' \cdot x) + d_2 = a \cdot (a' \cdot x) + D \subseteq a \cdot (R \setminus D) + D$. We conclude that $R \setminus D \subseteq a \cdot (R \setminus D) + D$, i.e. $R \setminus D + D \subseteq a \cdot (R \setminus D) + D$.

Sufficiency. Let $a \cdot (R \setminus D) + D = R \setminus D + D$, for all $a \in R \setminus D$. By Corollary 3.6, it follows that the set $R \setminus D$ is closed under the multiplication. Note that there exists $s \in R \setminus D$ such that $s \in S$. To the contrary, if $S \subseteq D$, then $R = \langle S \rangle \subseteq D$, meaning that $R = D$, which contradicts our assumption. Thus, let $s \in R \setminus D$ such that $s \in S$. Since $s \in (R \setminus D) + D = s \cdot (R \setminus D) + D$, it follows that there exist $e \in R \setminus D$ and $d_1 \in D$ such that $s \in s \cdot e + d_1$. Hence, $s \cdot (e \cdot s - s) \subseteq (s \cdot e) \cdot s - s \cdot s \subseteq (s - d_1) \cdot s - s \cdot s \subseteq D$, since D is a hyperideal. By Proposition 3.5, R is a hypernear-ring without D-divisors and since $s \in R \setminus D$, we get $e \cdot s - s \subseteq D$. Thus $e \cdot s \in D + s = s + D$, i.e. $e \cdot s \in s + D$.

If $x \in R \setminus D$, then

$$
\begin{aligned}
&(x \cdot e - x) \cdot s \subseteq x \cdot (e \cdot s) - x \cdot s + D \subseteq x \cdot (s + D) - x \cdot s + D \subseteq \\
&\subseteq x \cdot s + x \cdot D - x \cdot s + D \subseteq x \cdot s + D - x \cdot s + D \subseteq D + D = D.
\end{aligned}
$$

Since $s \notin D$, we have $x \cdot e - x \subseteq D$, meaning that $x \cdot e \in D + x = x + D$. Besides, $s \cdot (e \cdot x - x) \subseteq (s \cdot e) \cdot x - s \cdot x \subseteq (s - d_1) \cdot x - s \cdot x \subseteq D$, since D is a hyperideal. Again, since $s \notin D$, we obtain $e \cdot x - x \subseteq D$, implying that $e \cdot x \in D + x = x + D$. Thereby e is the identity element.

Suppose now that $a \in R \setminus D$. Since $e \in R \setminus D \subseteq R \setminus D + D = a \cdot (R \setminus D) + D$, then there exist $a' \in R \setminus D$ and $d \in D$ such that $e \in a \cdot a' + d$. Besides, $a \cdot (a' \cdot a - e) \subseteq (a \cdot a') \cdot a - a \cdot e \subseteq (e - d) \cdot a - (a + D)$. Since D is a hyperideal of

R, we have $(e-d)\cdot a-e\cdot a\subseteq D$, i.e. $(e-d)\cdot a\subseteq D+e\cdot a=e\cdot a+D$. Therefore, $a\cdot(a'\cdot a-e)\subseteq e\cdot a+D-(a+D)=e\cdot a+D-D-a$. Besides, $e\cdot a\in a+D$ and thus $a\cdot(a'\cdot a-e)\subseteq a+D+D-D-a\subseteq a+D-a\subseteq D$. Since $a\notin D$, it follows that $a'\cdot a-e\subseteq D$, meaning that $a'\cdot a\in D+e=e+D$. Hence, we have shown that $R\setminus D$ is a multiplicative group, implying that (R,S) is a D-division hypernear-ring. □

Example 3.8. *Considering the hypernear-ring R in Example 2.9 and following [14], we know that $(R,+,\cdot)$ is a hypernear-ring with the set of generators $S=\{0,2,3\}$ and the defect of distributivity $D=\{0,3,6\}$. It can be easily verified that $(R\setminus D,\cdot)$ is a D-multiplicative group. Indeed, $R\setminus D=\{1,2,4,5\}$ is closed under the multiplication. Moreover, $e=2$ is the identity element. Finally, one gets, for any $a\in R\setminus D$, that $a\cdot a\in 2+D=\{2,5\}$, meaning that the inverse of each element $a\in R\setminus D$ is a itself. So (R,S) is a D-division hypernear-ring.*

Example 3.9. *Let $G=(\mathbb{Z}_4,+)$ the additive group of the integers modulo 4 and set $R=P_G=G\cup\{4\}$. Define on R the hyperaddition "$+$" as in Example 2.5. [14] (this is the general method to construct a quasicanonical hypergroup [8]), i.e. it has the following table*

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	$\{0,4\}$	1
2	2	3	$\{0,4\}$	1	2
3	3	$\{0,4\}$	1	2	3
4	4	1	2	3	0

and then define the multiplication by the table

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	0
2	0	0	0	0	0
3	0	3	2	1	0
4	0	0	0	0	0

So $(R,+)$ is a canonical hypergroup, while $(R,\cdot)$ is a semigroup, having 0 as a bilaterally absorbing element. It can be verified that, for any $x,y,z\in R$, it *holds $x \cdot (y+z) \subseteq x \cdot y + x \cdot z$, meaning that $(R, +, \cdot)$ is a hypernear-ring. Take $S = \{1\}$. Obviously, S is a subsemigroup of $(R, \cdot)$ and it generates $(R, +)$. The set D_S in Definition 2.12 is $D_S = \{-(x \cdot 1 + y \cdot 1) + (x+y) \cdot 1 \mid x, y \in R\} = \{0, 2, 4\}$. The hyperoperation on D_S has the table*

$+$	0	2	4
0	0	2	4
2	2	$\{0,4\}$	2
4	4	2	0

following that D_S is a normal subhypergroup of $(R, +)$, so $D_S = D$. Thereby the hypernear-ring (R, S) has the defect $D = \{0, 2, 4\}$. Moreover, we can immediately see that the multiplicative structure $(R \setminus D, \cdot)$ is a group, so $R \setminus D$ is a D-multiplicative group, i.e. (R, S) is a D-division hypernear-ring.

Definition 3.10. A hypernear-ring containing more than one element is called a *division hypernear-ring* if the set of all its non-zero elements is a multiplicative group.

Example 3.11. *Let $(R, +, \cdot)$ be the hypernear-ring in Example 2.9. Taking $S = \{0, 2, 3\}$, we get the defect $D = \{0, 3, 6\}$. Let $\bar{R} = R/D$ be the factor hypernear-ring, i.e. $\bar{R} = \{\bar{0} = D, \bar{1} = \{1, 4\}, \bar{2} = \{2, 5\}\}$, having the following tables of the (hyper)operations:*

$\oplus$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{1}$
$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

Obviously, $\bar{R} \setminus \{\bar{0}\} = \{\bar{1}, \bar{2}\}$ is a group, thereby $\bar{R}$ is a division (hyper)near-ring (the addition is a commutative operation), so it is a division ring. If we take $\bar{S} = \{\bar{2}\}$, then $\bar{S}$ is a subsemigroup of $(\bar{R}, \cdot)$ and $D_{\bar{S}} = \{\bar{0}\}$, since $-(\bar{x} \cdot \bar{2} + \bar{y} \cdot \bar{2}) + (\bar{x} + \bar{y}) \cdot \bar{2} = -(\bar{2} \cdot \bar{x} + \bar{2} \cdot \bar{y}) + \bar{2} \cdot (\bar{x} + \bar{y}) = -(\bar{2} \cdot \bar{y}) - (\bar{2} \cdot \bar{x}) + \bar{2} \cdot \bar{x} + \bar{2} \cdot \bar{y} = \{\bar{0}\}$, meaning that the defect of distributivity of $(\bar{R}, \bar{S})$ is $\bar{D} = \{\bar{0}\}$.

Remark 3.12. If (R, S) is a d.g. hypernear-ring, then R is a division hypernear-ring if and only if R is a D-division hypernear-ring, since $D = \{0\}$.

Theorem 3.13. *Let* (R, S) *be a non-zero d.g. hypernear-ring. A necessary and sufficient condition for the hypernear-ring* R *to be a division hypernear-ring is that* $a \cdot (R \setminus \{0\}) = R \setminus \{0\}$*, for all* $a \in R \setminus \{0\}$.

Proof. It follows from Theorem 3.7, since $D = \{0\}$. □

If R is a d.g. near-ring, then by Lemma 2.1, R is a zero symmetric near-ring, that is a special case of d.g. hypernear-rings. Thus, from Theorem 3.13 it follows the next result.

Corollary 3.14. *A necessary and sufficient condition for a d.g. near-ring* R *with more than one element to be a division near-ring is that* $a \cdot (R \setminus \{0\}) = R \setminus \{0\}$*, for all* $a \in R \setminus \{0\}$.

Based on Corollary 3.14, Lemma 2.3 and Lemma 2.4, one gets the Ligh's result, i.e. Theorem 2.2 in this note.

4 Conclusions

Naturally extended the properties of near-rings related to distributivity to the similar algebraic hyperstructures, i.e. hypernear-rings, we notice that they hold under the same conditions, and more over they can be obtained as generalisations of those for structures. In particular, in this note we have shown that each hypernear-ring with the defect of distributivity D, satisfying a certain property (see Theorem 3.7) is a D-division hypernear-ring, and viceversa. Another important property of division near-rings says that the additive part of a division near-ring is commutative [25]. This is one open problem in hypernear-rings theory, that we intend to investigate in our future research, and also to extend it to the case of general hypernear-rings [15].

Acknowledgements

The second author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1 - 0285).

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