DOI: 10.2478/auom-2018-0020
An. Şt. Univ. Ovidius Constanţa
sciendo
Vol. 26(2),2018, 87–104

A revised moment error expression for the AIRGA algorithm

Heike Faßbender and Julius Mayer

Abstract

The fully adaptive rational global Arnoldi method (AIRGA) for the model-order reduction of second-order multi-input multi-output systems with proportional damping is revisited. The method automatically generates a reduced system approximating the transfer function. It is based on a moment-matching approach. The expansion points are determined iteratively. The reduced order and the number of moments matched per expansion point are determined adaptively using a heuristic based on an error estimation. A revised moment error expression is presented as well as some related findings.

1 Introduction

A continuous time-invariant second-order multi-input multi-output linear dynamical system is of the form

$$\begin{aligned} M\ddot{x}(t) &= -D\dot{x}(t) - Kx(t) + Fu(t), \\ y(t) &= C_p x(t) + C_v \dot{x}(t), \end{aligned} \tag{1}$$

where $M,\ D,\ K \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times m}$, $C_p, C_v \in \mathbb{R}^{q\times n}$ are constant matrices. In (1), $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^q$ is the output. The mass matrix M and the stiffness matrix K need not have any specific property (e.g., symmetry, positive definiteness etc.), but only the special case of proportional damping is

Key Words: Model Order Reduction, Krylov Subspace, Global Arnoldi Algorithm, Moment Matching, Second Order, Proportional Damping.
2010 Mathematics Subject Classification: Primary 65F30; Secondary 70J50.
Received: March, 2017.
Revised: June, 2017.
Accepted: August, 2017.

considered. That is, the damping matrix is chosen as $D = \alpha M + \beta K$ for some choice of real α and β.

In many cases, the original system dimension n is too large to allow for an efficient simulation of (1). Therefore, the goal of model reduction is to generate a low dimensional system that has, as best as possible, the same characteristics as the original system, but whose simulation requires significantly less computational effort. The reduced system of (1) is described by

$$\begin{aligned} \hat{M}\ddot{\hat{x}}(t) &= -\hat{D}\dot{\hat{x}}(t) - \hat{K}\hat{x}(t) + \hat{F}u(t), \\ \hat{y}(t) &= \hat{C}_p\hat{x}(t) + \hat{C}_v\dot{\hat{x}}(t), \end{aligned} \tag{2}$$

where $\hat{M},\ \hat{K},\ \hat{D} \in \mathbb{R}^{r\times r}$, $\hat{F} \in \mathbb{R}^{r\times m}$, $\hat{C}_p,\ \hat{C}_v \in \mathbb{R}^{q\times r}$and $r \ll n$. In order to capture the relevant features of the original model, the damping matrix $\hat{D}$ of the reduced order model is required to be $\hat{D} = \alpha\hat{M} + \beta\hat{K}$.

We will revisit the fully adaptive rational global Arnoldi method (AIRGA) for the model-order reduction of second-order multi-input multi-output systems with proportional damping [3]. This method uses a projection based on a moment-matching approach in order to compute the reduced order system. The AIRGA algorithm is recalled in Section 2. It makes use of a heuristic based on an error estimation of the moment error in order to adaptively determine the number of moments to be matched per expansion point. It turned out that the moment error given in [3] is not correct. In order to present our key findings, a revised moment error, the AIRGA algorithm has to be discussed in some detail. In particular, some technicalities from well-known facts are needed. For that matter, we also present some known facts whose proofs in the existing literature seem to be gappy. Our main result, a revised moment error, is given in Section 3. Some concluding remarks are given in Section 4. As all proofs are very technical, they have been moved to Section A.

2 AIRGA revisited

In this section we briefly review the AIRGA method [3]. This section is longer than usual because we have to present some well-known facts as we need some technical details of their proofs for further discussion. Moreover, we state some known facts whose proofs in the existing literature seem to be gappy.

Moment-Matching and Projection based Model Order Reduction

The objective is to generate a reduced order system (2) for which the first moments of the transfer function match those of the original system. The transfer function $H(s)$ of (1) is the linear mapping of the Laplace transform $U(s)$ of the input $u(t)$ to the

Laplace transform $Y(s)$ of the output $y(t), Y(s) = H(s)U(s)$. It is given by

$$H(s) = (C_p + sC_v)(s^2M + sD + K)^{-1}F =: (C_p + sC_v)T(s). \tag{3}$$

Here and throughout the paper, $s \in \mathbb{C}$ has to be chosen such that $s^2M + sD + K$ is nonsingular. The power series expansion of $T(s)$ around an expansion point $s_i \in \mathbb{C}$ is given by (see, e.g., [10])

$$T(s) = \sum_{k=0}^{\infty} T^{(k)}(s_i)(s - s_i)^k, \tag{4}$$

where the k-th system moments $T^{(k)}(s_i) \in \mathbb{C}^{n\times m}$ at s_i are given by

$$\begin{aligned} T^{(0)}(s_i) &= L_i^{-1}F, \\ T^{(1)}(s_i) &= L_i^{-1}B_iT^{(0)}(s_i), \qquad \text{and for } k = 2,3,\ldots \\ T^{(k)}(s_i) &= L_i^{-1}[B_iT^{(k-1)}(s_i) - MT^{(k-2)}(s_i)] \end{aligned} \tag{5}$$

with

$$L_i = s_i^2M + s_iD + K \quad \text{and} \quad B_i = -(2s_iM + D). \tag{6}$$

From (4) we obtain

$$\begin{aligned} H(s) &= \sum_{k=0}^{\infty} (C_p + sC_v)T^{(k)}(s_i)(s - s_i)^k \\ &= \sum_{k=0}^{\infty} (C_p + s_iC_v)T^{(k)}(s_i)(s - s_i)^k + C_vT^{(k)}(s_i)(s - s_i)^{k+1} \\ &=: \sum_{k=0}^{\infty} h_k(s_i)(s - s_i)^k \end{aligned}$$

with the moments $h_0(s_i) = (C_p + s_iC_v)T^{(0)}(s_i)$, and for $k = 1,2,\ldots$

$$h_k(s_i) = C_vT^{(k-1)}(s_i) + (C_p + s_iC_v)T^{(k)}(s_i) \in \mathbb{C}^{q\times m}.$$

Similarly, the transfer function of the reduced system (2) is given by

$$\hat{H}(s) = (\hat{C}_p + s\hat{C}_v)\hat{T}(s), \tag{7}$$

with $\hat{T}(s) = (s^2\hat{M} + s\hat{D} + \hat{K})^{-1}\hat{F}$. Clearly, here $s \in \mathbb{C}$ has to be chosen such that not only $L = s^2M + sD + K$ is nonsingular, but also such that $\hat{L} = s^2\hat{M} + s\hat{D} + \hat{K}$ is nonsingular as well. In a projection based framework as considered below this will be satisfied automatically, as $\hat{L} = V^HLV$ is nonsingular if L is nonsingular and V is a $n \times r$ matrix with linearly independent columns.

The power series expansion of $\hat{T}(s)$ around an expansion point $s_i \in \mathbb{C}$ is given by

$$\hat{T}(s) = \sum_{k=0}^{\infty} \hat{T}^{(k)}(s_i)(s-s_i)^k, \tag{8}$$

where $\hat{T}^{(k)}(s_i) \in \mathbb{C}^{r\times m}$ is defined analogously to $T^{(k)}(s_i)$. The moments $\hat{h}_k(s_i)$ of the reduced system are thus defined analogously to $h_k(s_i)$ for $k \in \mathbb{N}_0$.

The goal of the moment-matching approach is to find a reduced order model such that the first few moments of (7) match those of (3), that is,

$$h_k(s_i) = \hat{h}_k(s_i) \quad \text{for} \quad k = 0,1,\ldots,k_i-1$$

for some $k_i \in \mathbb{N}$.

A projection based method to generate a reduced oder model of order r constructs a projection $\Pi = VV^\dagger$ with a full rank matrix $V \in \mathbb{C}^{n\times r}$ and the pseudoinverse $V^\dagger = (V^HV)^{-1}V^H$. Since $\Pi = \Pi^H$ holds, Π is an orthogonal projection. The reduced order model is given by

$$\begin{aligned} &V^\dagger(MV\ddot{\hat{x}}(t) + DV\dot{\hat{x}}(t) + KV\hat{x}(t) - Fu(t)) = 0,\\ &\hat{y}(t) = C_pV\hat{x}(t) + C_vV\dot{\hat{x}}(t). \end{aligned} \tag{9}$$

Thus, we have

$$\hat{M} = V^\dagger MV,\ \hat{D} = V^\dagger DV,\ \hat{K} = V^\dagger KV,\ \hat{F} = V^\dagger F,\ \hat{C}_p = C_pV \text{ and } \hat{C}_v = C_vV. \tag{10}$$

The following well-known theorem [5, 1, 9] states how to choose V in order to achieve the desired moment-matching property. We restate the theorem as we will need the relation (12) later on.

Theorem 2.1. *Assume s_i is chosen such that L_i is nonsingular. Let $V \in \mathbb{C}^{n\times r}$ have linearly independent columns such that*

$$colspan(V) \supset colspan([T^{(0)}(s_i), T^{(1)}(s_i), \ldots, T^{(k_i-1)}(s_i)]). \tag{11}$$

Then for the reduced order system (9) it holds that

$$T^{(k)}(s_i) = V\hat{T}^{(k)}(s_i) \tag{12}$$

and thus the moment-matching property $h_k(s_i) = \hat{h}_k(s_i)$ holds for $k = 0,1,\ldots,k_i-1$.

First and second-order Krylov Subspace

Theorem 2.1 tells us how to choose V. A numerically efficient and stable way to obtain such V makes use of Krylov subspace methods.

A first-order Krylov subspace $\mathcal{K}_k(P,Q)$ of order $k \in \mathbb{N}$ generated by an $n \times n$ matrix P and an $n \times m$ matrix Q is the linear subspace spanned by the columns of the images of Q under powers of P

$$\mathcal{K}_k(P,Q) = \text{colspan}([Q, PQ, P^2Q, \ldots, P^{k-1}Q]).$$

A second-order Krylov subspace $\mathcal{G}_k(P_1,P_2,Q)$ of order k for $n \times n$ matrices P_1, P_2 and an $n \times m$ matrix Q is defined as follows:

$$\mathcal{G}_k(P_1,P_2,Q) = \text{colspan}([G_0, G_1, \ldots, G_{k-1}])$$

with $G_0 = Q$, $G_1 = P_1G_0$ and $G_j = P_1G_{j-1} + P_2G_{j-2}$, $j = 2,3,\ldots,k-1$.

As already observed in [1], the system moments $T^{(k)}(s_i)$ are just the blocks of the second-order Krylov subspace $\mathcal{G}_{k_i}(L_i^{-1}B_i, L_i^{-1}M, L_i^{-1}F)$; that is

$$\text{colspan}([T^{(0)}(s_i), T^{(1)}(s_i), \ldots, T^{(k_i-1)}(s_i)]) = \mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F).$$

This also follows directly from (5).

For the special case of proportionally damped second-order systems, the second-order Krylov subspace is essentially identical to a first order Krylov subspace. This has already been observed, e.g., in [2, 5], but no discussion given so far seems to include all special cases. Let us first consider the following lemma (a similar relation has already been noted in [5, Section 3]); for a proof see the Appendix.

Lemma 2.2. *Assume s_i is chosen such that L_i is nonsingular and $s_i\beta \neq -1$. Then*

$$L_i^{-1}B_i = -(\gamma_{i,1}I_n + \gamma_{i,2}L_i^{-1}M)$$

with

$$\gamma_{i,1} = \frac{\beta}{s_i\beta + 1} \quad \textit{and} \quad \gamma_{i,2} = s_i + \frac{s_i + \alpha}{s_i\beta + 1}.$$

With the help of Lemma 2.2 the following theorem can be proven (see the Appendix).

Theorem 2.3. *Assume s_i is chosen such that L_i is nonsingular and $s_i\beta \neq -1$. Let $\gamma_{i,1}, \gamma_{i,2}$ be defined as in Lemma 2.2. Then, for any $k_i \in \mathbb{N}$, it holds*

$$\mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F), \quad \textit{if } \gamma_{i,2} \neq 0,$$
$$\mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_{\lceil k_i/2 \rceil}(-L_i^{-1}M, L_i^{-1}F), \textit{ if } \gamma_{i,2} = 0.$$

Remark 2.4. *Assume $s_i\beta \neq -1$. Note that $\gamma_{i,2} = 0$ holds iff either $\beta = 0$ and $s_i = -\frac{\alpha}{2}$ or $\beta \neq 0$ and $s_i = -\beta^{-1} \pm \beta^{-1}\sqrt{1-\alpha\beta}$.*

In general, the choice $s_i\beta = -1$ is not feasible. Assume for a moment that $s_i\beta = -1$. This implies $s_i \neq 0$ and $\beta \neq 0$. As $L_i = s_i^2 M + s_i D + K = s_i(s_i + \alpha)M$ must be nonsingular, it further implies that M has to be nonsingular and $s_i \neq -\alpha$. Moreover,

$$\begin{aligned}\mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F) &= \mathcal{K}_{k_i}((s_i^2 + \alpha s_i)^{-1}M^{-1}M, L_i^{-1}F) = \mathcal{K}_{k_i}(I_n, L_i^{-1}F)\\ &= \operatorname{colspan}(M^{-1}F)\end{aligned}$$

and by an easy manipulation

$$\mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_{k_i}(M^{-1}K, M^{-1}F).$$

Therefore, unless $M = \mu K, \mu \in \mathbb{R}$ or $K = 0_{n\times n}$, it follows for $s_i\beta = -1$

$$\mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F) \not\supset \mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F).$$

So, when an expansion point s_i is chosen, it always has to be checked that $s_i\beta \neq -1$ as we will make use of Theorem 2.3 when constructing the matrix V.

The Global Arnoldi Method

Theorem 2.3 suggests to generate the desired matrix V from $\mathcal{K}_{k_i}\ (-L_i^{-1}M,\ \ L_i^{-1}F)$; in case $\gamma_{i,2} = 0$, only $\mathcal{K}_{\lceil k_i/2\rceil}(-L_i^{-1}M, L_i^{-1}F)$ has to be considered. Standard efficient and numerically sound methods to compute a basis (and thus V) of a Krylov subspace are, e.g., the block or the global Arnoldi algorithm [7, 6, 8].

The AIRGA method uses the global Arnoldi method. It constructs a basis $W_{i,1}$, $W_{i,2}$, …, $W_{i,k_i} \in \mathbb{C}^{n\times m}$ of the Krylov subspace $\mathcal{K}_{k_i}(P_i, Q_i)$ with $P_i = -L_i^{-1}M \in \mathbb{C}^{n\times n}$ and $Q_i = L_i^{-1}F \in \mathbb{C}^{n\times m}$ which is block-orthonormal in the following sense

$$\begin{aligned}\langle W_{i,j}, W_{i,p}\rangle &= 0 \qquad j \neq p,\\ \langle W_{i,j}, W_{i,p}\rangle &= 1 \qquad j = p\end{aligned} \qquad \text{for } \ j, p = 1, \ldots, k_i. \tag{13}$$

Here, $< Y, Z >= \operatorname{trace}(Y^H Z)$ where $Y, Z \in \mathbb{C}^{n\times s}$. The associated norm is the Frobenius norm $\|\cdot\|_F$.

In order to simplify the discussion, we assume that k_i is chosen such that the global Arnoldi algorithm does not break down; that is, for each s_i it produces a matrix $W_i = [W_{i,1} \ \ \cdots \ \ W_{i,k_i}] \in \mathbb{C}^{n\times k_i\cdot m}$, representing a block-orthonormal basis of the block Krylov subspace $\mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F)$. Then the following relation holds for the block-orthonormal matrix W_i

$$P_i W_i = W_i(H_{k_i}^{(i)} \otimes I_m) + h_{k_i+1,k_i}^{(i)}[0, \ldots, 0, W_{i,k_i+1}], \tag{14}$$

with

$$W_{i,1} = Q_i / h_{1,0}^{(i)}, \qquad h_{1,0}^{(i)} = \|Q_i\|_F. \tag{15}$$

Here $H_{k_i}^{(i)}$ is an unreduced $k_i \times k_i$ upper Hessenberg matrix and $\otimes$ denotes the usual Kronecker product. If $m = 1$, the global Arnoldi algorithm reduces to the standard Arnoldi algorithm.

Multiple expansion points

In order to ensure a good reduced model in the entire problem dependent frequency domain of interest, one usually employs not just one expansion point, but a set of ℓ expansion points. That is, one considers a set $S = \{s_1, \ldots, s_\ell\}$ of ℓ expansion points and the corresponding block Krylov subspaces

$$\mathcal{K}_{k_i}(P_i, Q_i) = \mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F) \quad \text{for} \quad i = 1, \ldots, \ell$$

together with the associated block-orthonormal basis $W_i \in \mathbb{C}^{n \times k_i \cdot m}$ (computed by the global Arnoldi algorithm such that each W_i satisfies (14).). Recall that the expansion points s_i, $i = 1, \ldots, \ell$, have to be chosen such that $L_i = s_i^2 M + s_i D + K$ is nonsingular and $s_i \beta \neq -1$.

Generating the projection Π

Let

$$W = [W_1 \;\; W_2 \;\; \ldots \;\; W_\ell] \in \mathbb{C}^{n \times r_{max}}, \qquad r_{max} = m \sum_{i=1}^{\ell} k_i. \tag{16}$$

Clearly,

$$\text{colspan}(W) \supset \text{colspan}(W_i) \supset \text{colspan}(T^{(k)}(s_i))$$

for $i = 1, 2, \ldots, \ell$ and $k = 0, 1, \ldots, k_i - 1$. Now $\Pi = VV^\dagger$ can be set up using any full rank matrix $V \in \mathbb{C}^{n \times r}$ which has the same column space as W. Then, due to Theorem 2.1, the first k_i moments at the expansion point $s_i, i = 1, \ldots, \ell$ of the reduced order system (9) generated with V are matching those of the original system (1), that is,

$$h_j(s_i) = \hat{h}_j(s_i) \quad \text{holds for} \quad j = 0, \ldots, k_i - 1 \;\text{ and }\; i = 1, 2, \ldots, \ell.$$

Choosing the expansion points iteratively

Given the number ℓ of expansion points, the set $S = \{s_1, \ldots s_\ell\}$ of expansion points and the number k_i of moments to be matched at each s_i, the algorithm sketched so far will compute the desired reduced order model. As it is a priori not obvious how to choose k_i, the AIRGA algorithm [3] chooses the k_i adaptively given a fixed set S and the total number of number of moments to be matched, r_{max}/m. Thus, unlike as described so far, the algorithm does not generate W_i corresponding to $\mathcal{K}_{k_i}(L_i^{-1}M, L_i^{-1}F)$ at once. Instead, the following approach is used: The expansion points are picked iteratively. The first time s_i is picked, just $\mathcal{K}_1(L_i^{-1}M, L_i^{-1}F)$ is used to generate $W_i \in \mathbb{C}^{n \times m}$ and just one moment is matched at s_i. The next time s_i is picked, this is expanded to $\mathcal{K}_2(L_i^{-1}M, L_i^{-1}F)$ and $W_i \in \mathbb{C}^{n \times 2m}$ matching two moments at s_i, and so forth. Assume that the algorithm has picked the expansions points such that the first k_i moments are matched at expansion point s_i, that is, $h_k(s_i) = \hat{h}_k(s_i)$ holds for

$k = 0, 1, \ldots, k_i - 1$. The choice of the next expansion point to be considered is based on the k_i-th moment error at expansion point s_i

$$||h_{k_i}(s_i) - \hat{h}_{k_i}(s_i)||_F = \varepsilon_{k_i}(s_i). \tag{17}$$

The expansion point s_p chosen next is the one corresponding to the maximum moment error by $s_p = \text{argmax}_{s_i} \varepsilon_{k_i}(s_i)$.

3 AIRGA revised

The idea for the adaptive choice of the expansion points is based on an expression which describes the k_i-th moment error $\varepsilon_{k_i}(s_i)$. The expression given in [3] is not correct. Here is the revised version of the result.

Theorem 3.1. *Assume that s_i is chosen for all $i = 1, \ldots, \ell$, such that $L_i = s_i^2 M + s_i D + K$ is nonsingular and $s_i \beta \neq -1$. Let $W_i, i = 1, \ldots, \ell$, be computed by the global Arnoldi method such that (14) and (15) hold. Let $W = [W_1 \;\; W_2 \;\; \cdots \;\; W_\ell] \in \mathbb{C}^{n \times r_{max}}$ be as in (16). Let $V \in \mathbb{C}^{n \times r}$ be a full rank matrix which has the same column space as W. Let the reduced order system (9) be generated via (10). Then the error of the $k_i + 1$-th moment at s_i can be expressed as*

$$\begin{aligned}\varepsilon_{k_i}(s_i) &:= ||h_{k_i}(s_i) - \hat{h}_{k_i}(s_i)||_F \\ &= |\gamma_{i,2}^{k_i}| \cdot \left(\prod_{k=0}^{k_i} h_{k+1,k}^{(i)} \right) \cdot ||(C_p + s_i C_v) \left(I_n - V \left(V^\dagger L_i V \right)^{-1} V^\dagger L_i \right) W_{i,k_i+1}||_F.\end{aligned}$$

In order to be able to prove Theorem 3.1 the following observation which is inspired by [4, Theorem 2] will be useful.

Lemma 3.2. *Let $P_i \in \mathbb{C}^{n \times n}$, $Q_i \in \mathbb{C}^{n \times m}$, $\widetilde{H}_i = H_{k_i}^{(i)} \otimes I_m$ and $E_i = e_i \otimes I_m$, where $e_i \in \mathbb{R}^{k_i}$ denotes the k_i-th unit vector. Let W_i be computed by the global Arnoldi method such that (14) and (15) hold. Then it holds*

$$P_i^{k_i} Q_i = h_{1,0}^{(i)} W_i \widetilde{H}_i^{k_i} E_1 + \left(\prod_{k=0}^{k_i} h_{k+1,k}^{(i)} \right) W_{i,k_i+1}.$$

Theorem 3.1 gives rise to Algorithm 1. It starts with an initial set of expansion points and automatically and adaptively chooses the number of moments to be matched at each expansion point s_i based on Theorem 3.1. This is controlled by the inner while loop starting at line 10 where V is computed.

One can use different methods to obtain a full rank matrix $V \in \mathbb{C}^{n \times r}$ which has the same column space as $W \in \mathbb{C}^{n \times r_{max}}$. A numerically safe way to generate the matrix V from W is the use of the rank-revealing QR-decomposition of W. The relevant part

Algorithm 1 Revised Adaptive Iterative Rational Global Arnoldi Algorithm

Input: M, D, K, F, C_p, C_v, α, β such that $D = \alpha M + \beta K$,
$r_{\max}$, initial set of exp. points $S = \{s_1, \ldots, s_\ell\}$
Output: r, $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{F}$, $\hat{C}_p$, $\hat{C}_v$, such that $\hat{D} = \alpha\hat{M} + \beta\hat{K}$
while no convergence **do**
  $W = [\,], \quad seq = [\,], \quad r_W = 0, \quad [n,m] = \text{size}(F)$
  for $i = 1 : \ell$ **do**
    $L_i = s_i^2 M + s_i D + K$
    $R_i = L_i^{-1} F$
    $h_i = ||R_i||_F, \quad R_i = R_i/h_i, \quad h_{i,\Pi} = h_i, \quad \gamma_{i,\Pi} = 1, \quad \gamma_{i,2} = s_i + \frac{s_i+\alpha}{s_i\beta+1}$
  end for
  while $r_W \leq r_{max} - m$ and no convergence **do**
    if $r_W = 0$ **then**
      $p = \text{argmax}_i \, ||\gamma_{i,\Pi} h_{i,\Pi} (C_p + s_i C_v) R_i||_F$
    else
      $p = \text{argmax}_i \, ||\gamma_{i,\Pi} h_{i,\Pi} (C_p + s_i C_v) \left(I_n - V \left(V^H L_i V\right)^{-1} V^H L_i\right) R_i \, ||_F$
    end if
    $seq = [seq,\, p], \quad W = [W \; R_p], \quad r_W = r_W + m$
    $R_p = -L_p^{-1} M R_p$
    for $k = 1 : \text{length}(seq)$ **do**
      if $seq(k) = p$ **then**
        $h = \text{trace}(W(:, (k-1)m + 1 : km)^H R_p)$
        $R_p = R_p - hW(:, (k-1)m + 1 : km)$
      end if
    end for
    $h_p = ||R_p||_F, \quad h_{p,\Pi} = h_{p,\Pi} \cdot h_p, \quad \gamma_{p,\Pi} = \gamma_{p,\Pi} \cdot \gamma_{p,2}$
    if $h_p \neq 0$ **then**
      $R_p = R_p/h_p$
    end if
    Determine V from [Q,R,E] = qr(W,0) (to deflate all linear
      dependent columns)
  end while
  Choose new set of expansion points $S = \{s_1, \ldots, s_\ell\}$
end while
Determine V from [Q,R,E] = qr([Re(V), Im(V)],0)
r = size(V,2)
Compute the reduced order system as in (18)

of its unitary factor is then used as V such that V has orthonormal columns. Thus it holds $V^\dagger = V^H$ and the projection $\Pi = VV^\dagger = VV^H$ becomes a Galerkin projection. The system matrices of the reduced order system are given by

$$\hat{M} = V^H M V,\ \hat{D} = V^H D V,\ \hat{K} = V^H K V,\ \hat{F} = V^H F,\ \hat{C}_p = C_p V \text{ and } \hat{C}_v = C_v V. \quad (18)$$

The size of the resulting reduced system can not be predetermined. At the end, V will have r_{max} or less columns.

Please note that h_p in line 24 corresponds to a lower subdiagonal element of the associated Hessenberg matrix. In case $h_p = 0$, we have $R_p = 0_{n\times m}$. The algorithm does not break down, as this implies that the corresponding moment error is equal to zero. Thus, the corresponding expansion point will not be chosen again.

The quality of the reduced order system heavily depends on the choice of the expansion points. As a good set of expansion points is usually not available, typically such a set is determined iteratively. This is controlled by the outer while loop starting at line 3 of Algorithm 1. One starts with an initial set of expansion points, computes the corresponding reduced order model and selects a new set of expansion points, e.g., based on the eigenvalues of $\lambda^2\hat{M} + \lambda\hat{D} + \hat{K}$. The actual selection criterion has to be based on the problem considered, see, e.g., the discussion in [3] and the references therein. This process is repeated till convergence, measured, e.g., in terms of the H_2-error between the previously computed reduced order system and the current one

$$\text{err} = \|\hat{H}_{previous} - \hat{H}_{current}\|_{H_2}.$$

Here $\|H - G\|_{H_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(\imath\omega) - G(\imath\omega)\|_F d\omega$, where the transfer functions H and G belong to systems with the same input and output dimension. For a thorough discussion on how to determine convergence as well as a new set of expansion points in the outer loop see [3].

Finally note that allowing complex-valued expansion points s_i leads to $W \in \mathbb{C}^{n\times r}$. Thus V and the reduced order system (9) is complex-valued, even though usually a real-valued one is desired. Using complex-conjugate pairs of expansion points, at least theoretically, the entire computations can be done in real arithmetic. A different option is to split the complex-valued matrix V into its real and imaginary part and to use a rank-revealing QR decomposition of $[\text{Re}(V), \text{Im}(V)]$ to obtain a real matrix with orthonormal columns and the same column space. This real-valued matrix may have twice the number of columns as desired. Hence, the dimension of the reduced order system will be doubled. The number of moments matched does not change.

4 Concluding Remarks

In [3] the AIRGA algorithm for model-order reduction of second-order multi-input multi-output systems with proportional damping has been proposed. The method

relies on the moment error $\varepsilon_{k_i}(s_i)$as in (17). Unfortunately the expression for the moment error given in [3] is not correct. Section 3 presents our main contribution: the revised moment error given in Theorem 3.1 as well as Lemma 3.2 which is needed to proof Theorem 3.1. The idea of the AIRGA algorithm and all further details necessary to proof Theorem 3.1 have been summarized in Section 2. In doing so, we have rounded off some of the results employed (by explicitily stating all assumptions and findings which may have been obscured in previous publications). In [3] numerical examples have been considered. Repeating those with the revised moment error do not reveal any major differences in the results. Thus, these experiments have not been included here.

A Proofs

In this section we provide the details of the proofs for the theorems and lemmas of Sections 2 and 3. For ease of reference, the theorems and lemmas are restated except for Theorem 2.1 which is well-known. Lemma 2.2 and Theorem 2.3 appeared in slightly different form in, e.g., [2, 5], while Lemma 3.2 and Theorem 3.1 are new.

Lemma 2.2 *Assume s_i is chosen such that L_i is nonsingular and $s_i\beta \neq -1$. Then*

$$L_i^{-1}B_i = -(\gamma_{i,1}I_n + \gamma_{i,2}L_i^{-1}M)$$

with

$$\gamma_{i,1} = \frac{\beta}{s_i\beta+1} \quad \textit{and} \quad \gamma_{i,2} = s_i + \frac{s_i+\alpha}{s_i\beta+1}.$$

Proof.

$$\begin{aligned}
-L_i^{-1}B_i = L_i^{-1}(2s_iM+D) &= L_i^{-1}(2s_iM+\alpha M+\beta K)\\
&= (2s_i+\alpha)L_i^{-1}M+\beta L_i^{-1}K\\
&= (2s_i+\alpha)L_i^{-1}M+\frac{\beta}{s_i\beta+1}(s_i\beta+1)L_i^{-1}K\\
&= (2s_i+\alpha)L_i^{-1}M+\frac{\beta}{s_i\beta+1}L_i^{-1}\left(-(s_i^2+s_i\alpha)M+(s_i^2+s_i\alpha)M+(s_i\beta+1)K\right)\\
&= \left(2s_i+\alpha-\frac{\beta(s_i^2+s_i\alpha)}{s_i\beta+1}\right)L_i^{-1}M+\frac{\beta}{s_i\beta+1}L_i^{-1}\left(s_i^2M+s_iD+K\right)\\
&= \left(s_i+\frac{s_i+\alpha}{s_i\beta+1}\right)L_i^{-1}M+\frac{\beta}{s_i\beta+1}L_i^{-1}L_i\\
&= \gamma_{i,2}L_i^{-1}M+\gamma_{i,1}I_n.
\end{aligned}$$

□

Theorem 2.3 *Assume s_i is chosen such that L_i is nonsingular and $s_i\beta \neq -1$. Let $\gamma_{i,1}, \gamma_{i,2}$ be defined as in Lemma 2.2. Then for any $k_i \in \mathbb{N}$ it holds*

$$\begin{aligned}\mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) &= \mathcal{K}_{k_i}(-L_i^{-1}M, L_i^{-1}F), \quad \text{if } \gamma_{i,2} \neq 0,\\ \mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) &= \mathcal{K}_{\lceil k_i/2 \rceil}(-L_i^{-1}M, L_i^{-1}F), \text{ if } \gamma_{i,2} = 0.\end{aligned}$$

Proof. From $\gamma_{i,2} = s_i + \frac{s_i+\alpha}{s_i\beta+1}$ it follows that for $\beta \neq 0$

$$\gamma_{i,2} = \frac{s_i^2\beta + 2s_i + \alpha}{s_i\beta + 1} = \frac{s_i^2\beta^2 + 2s_i\beta + \alpha\beta}{\beta(s_i\beta+1)} = 0$$

iff the numerator $s_i^2\beta^2 + 2s_i\beta + \alpha\beta$ is zero. This yields $s_i\beta = -1 \pm \sqrt{1-\alpha\beta}$.

Assume $\gamma_{i,2} \neq 0$. Set $P := -L_i^{-1}M$, $Q := L_i^{-1}F$ such that the blocks of the Krylov subspace $\mathcal{K}_k(-L_i^{-1}M, L_i^{-1}F)$ are given as $Q,\ PQ,\ P^2Q, \ldots, P^{k-1}Q$. Since

$$T^{(0)}(s_i) = L_i^{-1}F = Q,$$

we have $\mathcal{G}_1(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_1(-L_i^{-1}M, L_i^{-1}F)$. Next with Lemma 2.2, it holds

$$T^{(1)}(s_i) = L_i^{-1}B_iL_i^{-1}F = -(\gamma_{i,1}I_n + \gamma_{i,2}L_i^{-1}M)L_i^{-1}F = -\gamma_{i,1}Q - \gamma_{i,2}PQ.$$

Thus, as $\gamma_{i,2} \neq 0$ we have $\mathcal{G}_2(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_2(-L_i^{-1}M, L_i^{-1}F)$.

Now, assume $\mathcal{G}_j(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_j(-L_i^{-1}M, L_i^{-1}F)$ for $j = 1, 2, \ldots, p$. Then we can find $\mu_i^{(j-1,k)} \in \mathbb{C}$ for $k = 0, 1, \ldots, j-1$ and $j = 1, 2, \ldots, p$ such that $T^{(j-1)}(s_i) = \sum_{k=0}^{j-1} \mu_i^{(j-1,k)} P^kQ$. With Lemma 2.2 it follows

$$\begin{aligned}T^{(p)}(s_i) &= L_i^{-1}B_iT^{(p-1)}(s_i) - L_i^{-1}MT^{(p-2)}(s_i)\\ &= -(\gamma_{i,1}I_n + \gamma_{i,2}L_i^{-1}M)T^{(p-1)}(s_i) - L_i^{-1}MT^{(p-2)}(s_i)\\ &= -\gamma_{i,1}T^{(p-1)}(s_i) - \gamma_{i,2}PT^{(p-1)}(s_i) - PT^{(p-2)}(s_i)\\ &= -\gamma_{i,1}\sum_{k=0}^{p-1}\mu_i^{(p-1,k)}P^kQ - \gamma_{i,2}P\sum_{k=0}^{p-1}\mu_i^{(p-1,k)}P^kQ - P\sum_{k=0}^{p-2}\mu_i^{(p-2,k)}P^kQ\\ &= -\gamma_{i,1}\sum_{k=0}^{p-1}\mu_i^{(p-1,k)}P^kQ - \gamma_{i,2}\sum_{k=1}^{p}\mu_i^{(p-1,k-1)}P^kQ - \sum_{k=1}^{p-1}\mu_i^{(p-2,k-1)}P^kQ\\ &= -\gamma_{i,1}\mu_i^{(p-1,0)}Q - \sum_{k=1}^{p-1}(\gamma_{i,1}\mu_i^{(p-1,k)} + \gamma_{i,2}\mu_i^{(p-1,k-1)} + \mu_i^{(p-2,k-1)})P^kQ\end{aligned}$$

$$
\begin{aligned}
&- \gamma_{i,2}\mu_i^{(p-1,p-1)}P^pQ \\
&=: \sum_{k=0}^{p} \mu_i^{(p,k)}P^kQ. \qquad (19)
\end{aligned}
$$

The above directly reveals the recursion formula

$$
\begin{aligned}
\mu_i^{(p,0)} &= -\gamma_{i,1}\mu_i^{(p-1,0)} \\
\mu_i^{(p,k)} &= -\gamma_{i,1}\mu_i^{(p-1,k)} - \gamma_{i,2}\mu_i^{(p-1,k-1)} - \mu_i^{(z-2,k-1)}, \quad \text{for } k = 1,2,\ldots,p-1 \\
\mu_i^{(p,p)} &= -\gamma_{i,2}\mu_i^{(p-1,p-1)}
\end{aligned}
$$

for any $p \geq 2$ with $\mu_i^{(0,0)} = 1$, $\mu_i^{(1,0)} = -\gamma_{i,1}$ and $\mu_i^{(1,1)} = -\gamma_{i,2}$. Particularly, it holds

$$\mu_i^{(p,p)} = (-\gamma_{i,2})^p. \qquad (20)$$

As $\gamma_{i,2} \neq 0$, we immediately have $\mathcal{G}_k(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_k(-L_i^{-1}M, L_i^{-1}F)$, so that the first equation of the theorem is proven by induction.

In order to prove the second statement of the theorem, assume $\gamma_{i,2} = 0$. With Lemma 2.2, it follows

$$\mathcal{G}_{k_i}(L_i^{-1}B_i, -L_i^{-1}M, L_i^{-1}F) = \mathcal{G}_{k_i}(-\gamma_{i,1}I_n, -L_i^{-1}M, L_i^{-1}F) = \mathcal{K}_{\lceil k_i/2 \rceil}(-L_i^{-1}M, L_i^{-1}F).$$

□

Lemma 3.2 *Let $P_i \in \mathbb{C}^{n\times n}$, $Q_i \in \mathbb{C}^{n\times m}$, $\widetilde{H}_i = H_{k_i}^{(i)} \otimes I_m$ and $E_i = e_i \otimes I_m$, where $e_i \in \mathbb{R}^{k_i}$ denotes the k_i-th unit vector. Let W_i be computed by the global Arnoldi method such that (14) and (15) hold. Then it holds*

$$P_i^{k_i}Q_i = h_{1,0}^{(i)}W_i\widetilde{H}_i^{k_i}E_1 + \left(\prod_{k=0}^{k_i} h_{k+1,k}^{(i)}\right) W_{i,k_i+1}.$$

Proof. Observe

$$P_iW_i = W_i(H_{k_i}^{(i)} \otimes I_m) + h_{k_i+1,k_i}^{(i)}[0,\ldots,0,W_{i,k_i+1}] = W_i\tilde{H}_i + h_{k_i+1,k_i}^{(i)}W_{i,k_i+1}E_{k_i}^T. \qquad (21)$$

Multiplication from the left by $P_i^{k_i-1}$ and repeated use of (21) yields

$$
\begin{aligned}
P_i^{k_i}W_i &= P_i^{k_i-2}\left(P_iW_i\right)\widetilde{H}_i + P_i^{k_i-1}h_{k_i+1,k_i}^{(i)}W_{i,k_i+1}E_{k_i}^T \\
&= P_i^{k_i-2}\left(W_i\widetilde{H}_i + h_{k_i+1,k_i}^{(i)}W_{i,k_i+1}E_{k_i}^T\right)\widetilde{H}_i + h_{k_i+1,k_i}^{(i)}P_i^{k_i-1}W_{i,k_i+1}E_{k_i}^T
\end{aligned}
$$

$$
\begin{aligned}
&= P_i^{k_i-3}\left(P_iW_i\right)\widetilde{H}_i^2 + h_{k_i+1,k_i}^{(i)}\sum_{k=0}^{1} P^{k_i-1-k}W_{i,k_i+1}E_{k_i}^T\widetilde{H}_i^k \\
&= P_i^{k_i-3}\left(W_i\widetilde{H}_i + h_{k_i+1,k_i}^{(i)}W_{i,k_i+1}E_{k_i}^T\right)\widetilde{H}_i^2 \\
&\qquad + h_{k_i+1,k_i}^{(i)}\sum_{k=0}^{1} P^{k_i-1-k}W_{i,k_i+1}E_{k_i}^T\widetilde{H}_i^k \\
&= P_i^{k_i-4}\left(P_iW_i\right)\widetilde{H}_i^3 + h_{k_i+1,k_i}^{(i)}\sum_{k=0}^{2} P^{k_i-1-k}W_{i,k_i+1}E_{k_i}^T\widetilde{H}_i^k \\
&= \dots \\
&= W_i\widetilde{H}_i^{k_i} + h_{k_i+1,k_i}^{(i)}\sum_{k=0}^{k_i-1} P^{k_i-1-k}W_{i,k_i+1}E_{k_i}^T\widetilde{H}_i^k.
\end{aligned}
$$

As $Q_i = h_{1,0}^{(i)}W_{i,1} = h_{1,0}^{(i)}W_iE_1$, we have

$$
P_i^{k_i}Q_i = h_{1,0}^{(i)}P_i^{k_i}W_iE_1 = h_{1,0}^{(i)}W_i\widetilde{H}_i^{k_i}E_1 + h_{1,0}^{(i)}h_{k_i+1,k_i}^{(i)}\sum_{k=0}^{k_i-1} P^{k_i-1-k}W_{i,k_i+1}E_{k_i}^T\widetilde{H}_i^kE_1. \quad (22)
$$

Since H_i is an upper Hessenberg matrix, $\widetilde{H}_i$ is a block upper Hessenberg matrix. It follows that

$$
\begin{aligned}
E_{k_i}^T\widetilde{H}_i^pE_1 &= 0 \quad \text{for } p = 0,1,\dots,k_i-2, \qquad \text{and} \\
E_{k_i}^T\widetilde{H}_i^{k_i-1}E_1 &= \prod_{k=1}^{k_i-1} h_{k+1,k}^{(i)}I_m.
\end{aligned}
$$

Substituting this into (22) gives

$$
\begin{aligned}
P_i^{k_i}Q_i &= h_{1,0}^{(i)}W_i\widetilde{H}_i^{k_i}E_1 + h_{1,0}^{(i)}h_{k_i+1,k_i}^{(i)}W_{i,k_i+1}\prod_{k=1}^{k_i-1} h_{k+1,k}^{(i)}I_m \\
&= h_{1,0}^{(i)}W_i\widetilde{H}_i^{k_i}E_1 + \prod_{k=0}^{k_i} h_{k+1,k}^{(i)}W_{i,k_i+1}.
\end{aligned}
$$

□

Theorem 3.1 *Assume that s_i is chosen for all $i = 1,\dots,\ell$, such that $L_i = s_i^2M + s_iD + K$ is nonsingular and $s_i\beta \neq -1$. Let $W_i, i = 1,\dots,\ell$, be computed by the global Arnoldi method such that (14) and (15) hold. Let $W = [W_1 \;\; W_2 \;\; \cdots \;\; W_\ell] \in \mathbb{C}^{n\times r_{max}}$ be as in (16). Let $V \in \mathbb{C}^{n\times r}$ be a full rank matrix which has the same column space as W. Let*

the reduced order system (9) be generated via (10). Then the error of the k_i+1-th moment at s_i can be expressed as

$$\begin{aligned}\varepsilon_{k_i}(s_i) &:= ||h_{k_i}(s_i)-\hat{h}_{k_i}(s_i)||_F \\ &= |\gamma_{i,2}^{k_i}|\cdot\left(\prod_{k=0}^{k_i} h_{k+1,k}^{(i)}\right)\cdot||(C_p+s_iC_v)\left(I_n-V\left(V^\dagger L_iV\right)^{-1}V^\dagger L_i\right)W_{i,k_i+1}||_F.\end{aligned}$$

Proof. First consider $k_i=0$. Recall $\hat{T}^{(0)}(s_i)=\hat{L}_i^{-1}\hat{F}$.

$$\begin{aligned}\varepsilon_0(s_i) &= ||(C_p+s_iC_v)T^{(0)}(s_i)-(\hat{C}_p+s_i\hat{C}_v)\hat{T}^{(0)}(s_i)||_F \\ &= ||(C_p+s_iC_v)T^{(0)}(s_i)-(C_p+s_iC_v)V\hat{T}^{(0)}||_F \\ &= ||(C_p+s_iC_v)\left(T^{(0)}(s_i)-V\hat{L}_i{}^{-1}\hat{F}\right)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(0)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}(V^\dagger F)\right)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(0)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}V^\dagger(L_iL_i^{-1})F\right)||_F \\ &= ||(C_p+s_iC_v)\left(I_n-V\left(V^\dagger L_iV\right)^{-1}V^\dagger L_i\right)T^{(0)}(s_i)||_F. \end{aligned}\tag{23}$$

Next consider $k_i=1$. Recall $\hat{T}^{(1)}(s_i)=\hat{L}_i^{-1}\hat{B}_i\hat{T}^{(0)}(s_i)$, and $T^{(0)}(s_i)=V\hat{T}^{(0)}(s_i)$, but in general $T^{(1)}(s_i)\neq V\hat{T}^{(1)}(s_i)$.

$$\begin{aligned}\varepsilon_1(s_i) &= ||(C_p+s_iC_v)T^{(1)}(s_i)-(\hat{C}_p+s_i\hat{C}_v)\hat{T}^{(1)}(s_i)||_F \\ &= ||(C_p+s_iC_v)T^{(1)}(s_i)-(C_p+s_iC_v)V\hat{L}_i^{-1}\hat{B}_i\hat{T}^{(0)}(s_i)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(1)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}\left(V^\dagger B_iV\right)\hat{T}^{(0)}(s_i)\right)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(1)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}\left(V^\dagger B_i\right)\left(V\hat{T}^{(0)}(s_i)\right)\right)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(1)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}\left(V^\dagger B_i\right)T^{(0)}(s_i)\right)||_F \\ &= ||(C_p+s_iC_v)\left(T^{(1)}(s_i)-V\left(V^\dagger L_iV\right)^{-1}V^\dagger(L_iL_i^{-1})B_iT^{(0)}(s_i)\right)||_F \\ &= ||(C_p+s_iC_v)\left(I_n-V\left(V^\dagger L_iV\right)^{-1}V^\dagger L_i\right)T^{(1)}(s_i)||_F. \end{aligned}\tag{24}$$

Consider $k_i>1$. Recall $\hat{T}^{(k_i)}(s_i)=\hat{L}_i^{-1}\left(\hat{B}_i\hat{T}^{(k_i-1)}(s_i)-\hat{M}\hat{T}^{(k_i-2)}(s_i)\right)$ and $T^{(j)}(s_i)=V\hat{T}^{(j)}(s_i)$, for $j=1,\ldots,k_i-1$, but in general $T^{(k_i)}(s_i)\neq V\hat{T}^{(k_i)}(s_i)$.

$$\varepsilon_{k_i}(s_i)=||(C_p+s_iC_v)T^{(k_i)}(s_i)-(\hat{C}_p+s_i\hat{C}_v)\hat{T}^{(k_i)}(s_i)||_F$$

$$
\begin{aligned}
&= ||(C_p + s_i C_v)T^{(k_i)}(s_i) - (C_p + s_i C_v)V\hat{L}_i^{-1}\left(\hat{B}_i\hat{T}^{(k_i-1)}(s_i) - \hat{M}\hat{T}^{(k_i-2)}(s_i)\right)||_F \\
&= ||(C_p + s_i C_v)\left\{T^{(k_i)}(s_i) - V\left(V^\dagger L_i V\right)^{-1}V^\dagger\left(B_i T^{(k_i-1)}(s_i) - MT^{(k_i-2)}(s_i)\right)\right\}||_F \\
&= ||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)T^{(k_i)}(s_i)||_F. \qquad (25)
\end{aligned}
$$

Consider now $k_i \in \mathbb{N}_0$ arbitrarily. With (19) and (23), (24) or (25) we obtain further

$$
\varepsilon_{k_i}(s_i) = ||\sum_{k=0}^{k_i}\mu_i^{(k_i,k)}(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)P_i^k Q_i||_F.
$$

As W is built as in (16) and the column space of V and W are the same, we have

$$
\operatorname{colspan}(V) \supset \operatorname{colspan}(W_i) \supset \operatorname{colspan}([Q_i, P_i Q_i, P_i^2 Q_i, \ldots, P_i^{k_i-1}Q_i]).
$$

Thus, we can write $P_i^j Q_i$ as a linear combination of the blocks $W_{i,1}, \ldots, W_{i,j+1}$ and further as

$$
P_i^j Q_i = V A_i^{(j)}, \qquad j = 0, \ldots, k_i - 1
$$

for some $A_i^{(j)} \in \mathbb{C}^{r\times m}$. Applying this relation and Lemma 3.2 we obtain

$$
\begin{aligned}
\varepsilon_{k_i}(s_i) &= ||\left(\sum_{k=0}^{k_i-1}\mu_i^{(k_i,k)}(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)VA_i^{(k)}\right) \\
&\qquad + \mu_i^{(k_i,k_i)}(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)P_i^{k_i}Q_i||_F \\
&= ||\mu_i^{(k_i,k_i)}(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)P_i^{k_i}Q_i||_F \\
&= |\mu_i^{(k_i,k_i)}|\cdot||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)\left(h_{1,0}^{(i)}W_i\widetilde{H}_i^{k_i}E_1 + \prod_{k=0}^{k_i}h_{k+1,k}^{(i)}W_{i,k_i+1}\right)||_F \\
&= |\mu_i^{(k_i,k_i)}|\cdot||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)W_i h_{1,0}^{(i)}\widetilde{H}_i^{k_i}E_1 \\
&\qquad + (C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)\prod_{k=0}^{k_i}h_{k+1,k}^{(i)}W_{i,k_i+1}||_F
\end{aligned}
$$

Since $\operatorname{colspan}(V) \supset \operatorname{colspan}(W_i)$ we have

$$
W_i = VA_{W_i}
$$

for some $A_{W_i} \in \mathbb{C}^{r\times(k_i\cdot m)}$. Recalling $\mu_i^{(k_i,k_i)} = (-\gamma_{i,2})^{k_i}$ from (20) it holds

$$
\varepsilon_{k_i}(s_i) = |\gamma_{i,2}^{k_i}|\cdot||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1}V^\dagger L_i\right)VA_{W_i}h_{1,0}^{(i)}\widetilde{H}_i^{k_i}E_1
$$

$$+ (C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1} V^\dagger L_i\right) \prod_{k=0}^{k_i} h^{(i)}_{k+1,k} W_{i,k_i+1}||_F$$

$$= |\gamma^{k_i}_{i,2}| \cdot ||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1} V^\dagger L_i\right) \prod_{k=0}^{k_i} h^{(i)}_{k+1,k} W_{i,k_i+1}||_F$$

$$= |\gamma^{k_i}_{i,2}| \prod_{k=0}^{k_i} h^{(i)}_{k+1,k} \cdot ||(C_p + s_i C_v)\left(I_n - V\left(V^\dagger L_i V\right)^{-1} V^\dagger L_i\right) W_{i,k_i+1}||_F$$

□

References

[1] Z. Bai and Y. Su. Dimension reduction of large-scale second-order dynamical systems via a second-order Arnoldi method. *SIAM Journal on Scientific Computing,*, 26(5):1692–1709, 2005.

[2] C. Beattie and S. Gugercin. Krylov-based model reduction of second-order systems with proportional damping. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 2278–2283, 2005.

[3] T. Bonin, H. Faßbender, A. Soppa, and M. Zaeh. A fully adaptive rational global Arnoldi method for the model-order reduction of second-order MIMO systems with proportional damping. *Elsevier Mathematics and Computers in Simulation,*, 122:1–19, 2016.

[4] C.-C. Chu, M.-H. Lai, and W.-S. Feng. Model-order reductions for MIMO systems using global Krylov subspace methods. *Mathematics and Computers in Simulation*, 79:1153–1164, 2008.

[5] R. Eid, B. Salimbahrami, B. Lohmann, E.B. Rudny, and J.G. Korvink. Parametric Order Reduction of Proportionally Damped Second Order Systems. Technical Reports on Automatic Control vol. TRAC-1, Technische Universität München, München, Germany, 2006.

[6] K. Jbilou, A. Messaoudi, and H. Sadok. Global FOM and GMRES algorithms for matrix equations. *Applied Numerical Mathematics*, 31(1):49 – 63, 1999.

[7] Y. Saad. *Iterative Methods for Sparse Linear Systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2003.

[8] M. Sadkane. Block-Arnoldi and Davidson methods for unsymmetric large eigenvalue problems. *Numerische Mathematik*, 64(1):195–211, 1993.

[9] B. Salimbahrami and B. Lohmann. Order reduction of large scale second-order systems using Krylov subspace methods. *Linear Algebra and its Applications*, 415(2):385 – 405, 2005.

[10] J. M. Wang, C. C. Chu, Q. Yu, and E. S. Kuh. On projection-based algorithms for model-order reduction of interconnects. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(11):1563–1585, 2002.

Heike Faßbender
TU Braunschweig
Institut Computational Mathematics
Universitätsplatz 2
38106 Braunschweig, Germany
Email: h.fassbender@tu-braunschweig.de

Julius Mayer (corresponding author)
TU Braunschweig
Institut Computational Mathematics
Universitätsplatz 2
38106 Braunschweig, Germany
juliusmayer@gmx.net