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Some Improvements of Hölder's Inequality on Time Scales

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Abstract

The theory and applications of dynamic derivatives on time scales have recently received considerable interest. In this paper, we define a function using the combined diamond-α integral, investigate its monotonicity and give some refinements for Hölder's inequality. We present some applications for $\mathbb{Z}$ and $\mathbb{R}$.

1 Introduction

There have been recent developments of the theory and applications of dynamic derivatives on time scales. This study, which is an unification of the discrete theory with the continuous theory, provides an unification and extension of traditional differential and difference equations. It is also an important tool in many computational and numerical applications. Using the Δ (delta) and ∇ (nabla) dynamic derivatives, a combined dynamic derivative, so called $\Diamond_\alpha$ (diamond-α) dynamic derivative, was introduced as a linear combination of Δ and ∇ dynamic derivatives on time scales. The diamond-α derivative reduces to the Δ derivative for $\alpha = 1$ and to the ∇ derivative for $\alpha = 0$. We refer the reader to [2], [4] and [5] for an account of the calculus associated with the diamond-α dynamic derivatives.

Recently, it has been proven a complete weighted version of diamond-α of Jensen's Inequality (see [3]) :

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Theorem 1. [3, Theorem 2]. *Let $a, b \in \mathbb{T}$ and $m, M \in \mathbb{R}$. If $g \in C([a,b]_{\mathbb{T}}, [m,M])$ and $w \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b w(t)\Diamond_\alpha t > 0$, then the following assertions are equivalent:*

(i) w is an α-SP weight for g on $[a,b]_{\mathbb{T}}$;

(ii) for every $F \in C([m,M], \mathbb{R})$ convex function, we have

$$F\left(\frac{\int_a^b g(t)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right) \leq \frac{\int_a^b F(g(t))w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}. \tag{1}$$

This version of Jensen's inequality is complete since it is true if and only if w is an α-SP weight for g and this special class of weights allows them to take some negative values.

Using Theorem 1, we can obtain stronger versions of many important results, such as Hölder's inequality.

Theorem 2. (Hölder's inequality). *Let $\mathbb{T}$ be a time scale, $a < b \in \mathbb{T}$, $f, g \in C([a,b]_{\mathbb{T}}, [0,+\infty))$ and $w \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ such that wg^q is an α-SP weight for $fg^{-\frac{q}{p}}$, where p and q are Hölder conjugates, (that is, $\frac{1}{p} + \frac{1}{q} = 1$) and $p > 1$. Then, we have*

$$\int_a^b w(t)f(t)g(t)\Diamond_\alpha t \leq \left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{\frac{1}{q}}. \tag{2}$$

If $p < 1$, then the inequality (2) is backward.

Proof. We choose $F(x) = x^p$ in Theorem 2, and F is a convex function on $[0, \infty)$ for $p > 1$. We get

$$\left(\frac{\int_a^b g(t)w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}\right)^p \leq \frac{\int_a^b (g(t))^p w(t)\Diamond_\alpha t}{\int_a^b w(t)\Diamond_\alpha t}. \tag{3}$$

Since wg^q is an α-SP weight for $fg^{-\frac{q}{p}}$, we have

$$\left(\frac{\int_a^b w(t)g^q(t)f(t)g^{-q/p}(t)\Diamond_\alpha t}{\int_a^b w(t)g^q(t)\Diamond_\alpha t}\right)^p \leq \frac{\int_a^b w(t)g^q(t)(f(t)g^{-q/p}(t))^p\Diamond_\alpha t}{\int_a^b w(t)g^q(t\Diamond_\alpha t}. \tag{4}$$

But p and q are Hölder conjugates, and so

$$\int_a^b w(t)f(t)g(t)\Diamond_\alpha t \le \left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{\frac{1}{q}}. \tag{5}$$

□

In section 2, we give our main results, regarding the function defined using Hölder's inequality. Some applications are presented in section 3.

2 Main results

Now, we will define a function based on Hölder's inequality, for time scales.

Thus, let f, g and w as in Theorem 2. We consider the function $h : \mathbb{T} \times \mathbb{T}$, given by

$$\begin{aligned} h(x,y) = &\left(\int_x^y w(t)f^p(t)\Diamond_\alpha t\right)^{1/p} \left(\int_x^y w(t)g^q(t)\Diamond_\alpha t\right)^{1/q} \\ &- \int_x^y w(t)f(t)g(t)\Diamond_\alpha t. \end{aligned} \tag{6}$$

This difference is generated by the inequality (2), that gives the nonnegativity of the function h. We will study the monotonicity properties of h, and we will find some refinements of the inequality (2) based on his properties. These results generalize the ones obtained in [12].

The monotonicity properties of h are given by the next theorem.

Theorem 3. *Let $\mathbb{T}$ be a time scale, $a < b \in \mathbb{T}$, $f, g, w \in C([a,b]_\mathbb{T}, [0, +\infty))$, and $w \in C([a,b]_\mathbb{T}, \mathbb{R})$ such that wg^q is an α-SP weight for $fg^{-\frac{q}{p}}$, where p and q are Hölder conjugates. Then,*

(i) if $p > 1$, then the function $h(x,b)$ is monotonously decreasing, while if $p < 1$, then the function $h(x,b)$ is monotonously increasing on $\mathbb{T}$, in relation with x;

(ii) if $p > 1$, then the function $h(a,y)$ is monotonously increasing, while if $p < 1$, then the function $h(a,y)$ is monotonously decreasing on $\mathbb{T}$ in relation with y;

(iii) for all $x \in \mathbb{T}$, $a < x < b$, if $p > 1$, then:

$$\begin{aligned}\int_a^b w(t)f(t)g(t)\Diamond_\alpha t &\le \left(\int_a^x w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^x w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\ &\quad+\int_x^b w(t)f(t)g(t)\Diamond_\alpha t\\ &\le \left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q},\end{aligned} \tag{7}$$

$$\begin{aligned}\int_a^b w(t)f(t)g(t)\Diamond_\alpha t &\le \left(\int_x^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_x^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\ &\quad+\int_a^x w(t)f(t)g(t)\Diamond_\alpha t\\ &\le \left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q},\end{aligned} \tag{8}$$

and

$$\begin{aligned}\int_a^b w(t)f(t)g(t)\Diamond_\alpha t &\le \frac{1}{2}\Bigg(\left(\int_a^x w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^x w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\ &\quad+\int_a^b w(t)f(t)g(t)\Diamond_\alpha t\\ &\quad+\left(\int_a^x w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^x w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\Bigg)\\ &\le \left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^b w(t)g^q(t)\Diamond_\alpha s\right)^{1/q}.\end{aligned} \tag{9}$$

If $p < 1$, then the inequalities (7), (8) and (9) are backwards.

Proof. (i)

(1) Suppose that $p > 1$. Let $x_1, x_2 \in \mathbb{T}$, $x_1 < x_2 < b$. Using the relation $\frac{1}{p} + \frac{1}{q} = 1$, for $i = 1, 2$ we have

$$\left(\int_{x_i}^{b} w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_{x_i}^{b} w(t)g^q(t)\Diamond_\alpha t\right)^{1/q} = \left(\int_{x_i}^{b} w(t)g^q(t)\Diamond_\alpha t\right) \cdot \left(\frac{\int_{x_i}^{b} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_i}^{b} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p}. \tag{10}$$

The function $x \mapsto x^{1/p}$ is concave and, using Jensen's inequality for concave functions, we have

$$\begin{aligned}
&\left(\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t\right)\left(\frac{\int_{x_1}^{b} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p} \\
&\quad= \left(\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t\right)\left(\frac{\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t}{\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t}\cdot\frac{\int_{x_2}^{b} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t}\right. \\
&\qquad \left. +\frac{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}{\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t}\cdot\frac{\int_{x_1}^{x_2} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p} \\
&\quad\geq \left(\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t\right)\left(\frac{\int_{x_2}^{b} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p} \\
&\qquad + \left(\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t\right)\left(\frac{\int_{x_1}^{x_2} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p}.
\end{aligned} \tag{11}$$

Using, once again Jensen's inequality for concave functions on time scales, we get

$$\begin{aligned}
\left(\frac{\int_{x_1}^{x_2} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p} &= \left(\frac{1}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}\right.\\
&\quad \left.\cdot \int_{x_1}^{x_2} w(t)g^q(t)\left(f(t)(g(t))^{-1/(p-1)}\right)^p \Diamond_\alpha t\right)^{1/p}\\
&\geq \frac{1}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}\\
&\quad \cdot \int_{x_1}^{x_2} w(t)g^q(t)\left(\left(f(t)(g(t))^{-1/(p-1)}\right)^p\right)^{1/p} \Diamond_\alpha t\\
&= \frac{\int_{x_1}^{x_2} w(t)f(t)g(t)\Diamond_\alpha t}{\int_{x_1}^{x_2} w(t)g^q(t)\Diamond_\alpha t}.
\end{aligned} \tag{12}$$

Using (10), (11) and (12), it follows

$$\begin{aligned}
h(x_1,b) &= \left(\int_{x_1}^{b} w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_{x_1}^{b} w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\
&\quad - \int_{x_1}^{b} w(t)f(t)g(t)\Diamond_\alpha t\\
&\geq \left(\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t\right)\left(\frac{\int_{x_2}^{b} w(t)f^p(t)\Diamond_\alpha t}{\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t}\right)^{1/p}\\
&\quad + \int_{x_1}^{x_2} w(t)f(t)g(t)\Diamond_\alpha t - \int_{x_1}^{b} w(t)f(t)g(t)\Diamond_\alpha t\\
&= \left(\int_{x_2}^{b} w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_{x_2}^{b} w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\
&\quad - \int_{x_2}^{b} w(t)f(t)g(t)\Diamond_\alpha t\\
&= h(x_2,b).
\end{aligned} \tag{13}$$

If $x_2 = b$, the inequality (7) implies

$$h(x_2,b) = h(b,b) = 0 \leq h(x_1,b). \tag{14}$$

The result of the last inequality (14) is the decreasing monotony of the function $h(x,b)$ on $\mathbb{T}$, in relation with x.

(2) Suppose that $p < 1$. If $0 < p < 1$, then the function $x \mapsto x^{1/p}$ is convex so that the inequalities from (7), (11) and (12) are reversed. This implies that (13) and (14) are also reversed, hence $h(x,b)$ is monotonously increasing over $\mathbb{T}$ in relation with x. If $p < 0$, then $0 < q < 1$ and using the same arguments as for the case $0 < p < 1$, we obtain the increasing monotony of the function $h(x,b)$ over $\mathbb{T}$ in relation with x.

(ii) Using the same arguments as before, we can prove that the function $h(a,y)$ is monotonous in relation with y.

(iii) For every $x \in \mathbb{T}$, $a < x < b$ and $p > 1$, the function $h(a,y)$ is monotonously increasing over $\mathbb{T}$ in relation with y so that, it follows

$$h(a,b) \geq h(a,x) \geq h(a,a) = 0, \tag{15}$$

meaning that

$$\begin{aligned}
&\left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p} \left(\int_a^b w(t)g^q(s)\Diamond_\alpha t\right)^{1/q} \\
&\quad - \int_a^b w(t)f(t)g(t)\Diamond_\alpha t \\
&\quad \geq \left(\int_a^x w(t)f^p(t)\Diamond_\alpha t\right)^{1/p} \left(\int_a^x w(t)g^q(t)\Diamond_\alpha t\right)^{1/q} \\
&\quad - \int_a^x w(t)f(t)g(t)\Diamond_\alpha t \geq 0,
\end{aligned}$$

and, adding $\int_a^b w(t)f(t)g(t)\Diamond_\alpha t$ to each side of the previous inequality, we get (7).

Because $h(x,b)$ is monotonously decreasing over $\mathbb{T}$ in relation with x, we have

$$h(a,b) \geq h(x,b) \geq h(b,b) = 0, \tag{16}$$

meaning

$$\begin{aligned}
&\left(\int_a^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\
&\quad-\int_a^b w(t)f(t)g(t)\Diamond_\alpha t\\
&\geq\left(\int_x^b w(t)f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_x^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q}\\
&\quad-\int_x^b w(t)f(t)g(t)\Diamond_\alpha t\geq 0,
\end{aligned}$$

and, adding $\int_a^b w(t)f(t)g(t)\Diamond_\alpha t$ to each side of the previous inequality, we get (8). Adding the inequalities from (7) and (8), we obtain (9).

If $p<1$, the inequalities from (7), (8) and (9) are reversed, which implies that the inequalities from (15) and 16) are also reversed.

□

3 Applications

If $\mathbb{T}=\mathbb{R}$ and $w\equiv 1$ then Theorem 3 becomes Theorem 1.2 from [12]. If $\mathbb{T}=\mathbb{Z}$, $a=0$, $b=n$, $\alpha=0$ and $f(i)=a_i$, $g(i)=b_i$, $w(i)=b_i$ for $i=1,...,n$, we have the following corollary that improves Theorem 1.1 from [12], if we note $H(n)=h(0,n)$:

Corollary 4. *Let $a_i>0$, $b_i>0$, $w_i>0$ $(i=1,2,...,n;\ n>1)$ and consider p, q as Hölder conjugates. If $p>1$, then it follows*

$$H(n)\geq H(n-i) \tag{17}$$

If we define C and D by

$$C(k)=\left(\sum_{i=1}^{k} w_i a_i^p\right)^{1/p}\left(\sum_{i=1}^{k} w_i b_i^q\right)^{1/q}+\sum_{i=k+1}^{n} w_i a_i b_i,$$

and

$$D(k)=\left(\sum_{i=k+1}^{n} w_i a_i^p\right)^{1/p}\left(\sum_{i=k+1}^{n} w_i b_i^q\right)^{1/q}+\sum_{i=1}^{k} w_i a_i b_i,$$

for $k = 1, 2, ..., n$, *then*

$$\sum_{i=1}^{n} w_i a_i b_i = C(1) \leq C(2) \leq ... \leq C(n-1) = \left(\sum_{i=1}^{k} w_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{k} w_i b_i^q\right)^{1/q} \tag{18}$$

and

$$\sum_{i=1}^{n} w_i a_i b_i = D(n) \leq D(n-1) \leq ... \leq D(1) = \left(\sum_{i=1}^{k} w_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{k} w_i b_i^q\right)^{1/q}. \tag{19}$$

Also, for all $m = 1, ..., n-1$, *we have*

$$\sum_{i=1}^{n} w_i a_i b_i \leq \frac{1}{2}(C(m) + C(n-m)) \leq \left(\sum_{i=1}^{k} w_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{k} w_i b_i^q\right)^{1/q} \tag{20}$$

and

$$\sum_{i=1}^{n} w_i a_i b_i \leq \frac{1}{2}(D(m) + D(n-m)) \left(\sum_{i=1}^{k} w_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{k} w_i b_i^q\right)^{1/q}. \tag{21}$$

If $p < 1$, *then the above inequalities are reversed.*

If $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = n$ and $f(i) = a_i$, $g(i) = b_i$, $w(i) = w_i$, for $i = 0, ..., n$, then, for all $\alpha \in [0, 1]$, we get:

$$\begin{aligned} \int_a^b w(t) f(t) g(t) \Diamond_\alpha t &= \alpha \sum_{i=0}^{n-1} w_i a_i b_i + (1-\alpha) \sum_{1}^{n} w_i a_i b_i \\ &= \alpha w_0 a_0 b_0 + (1-\alpha) w_n a_n b_n + \sum_{1}^{n-1} w_i a_i b_i, \end{aligned}$$

while,

$$\begin{aligned} \left(\int_a^b w(t) f^p(t) \Diamond_\alpha t\right)^{1/p} &= \left(\alpha \sum_{i=0}^{n-1} w_i {a_i}^p + (1-\alpha) \sum_{i=1}^{n} w_i {a_i}^p\right)^{1/p} \\ &= \left(\alpha w_0 {a_0}^p + (1-\alpha) w_n {a_n}^p + \sum_{i=1}^{n-1} {a_i}^p\right)^{1/p} \end{aligned}$$

and

$$\begin{aligned}\left(\int_a^b w(t)g^q(t)\Diamond_\alpha t\right)^{1/q} &= \left(\alpha\sum_{i=0}^{n-1} w_i {b_i}^q + (1-\alpha)\sum_{i=1}^{n} w_i {b_i}^q\right)^{1/q}\\ &= \left(\alpha w_0 {b_0}^q + (1-\alpha)w_n {b_n}^q + \sum_{1}^{n-1} w_i {b_i}^q\right)^{1/q}.\end{aligned}$$

Then, Theorem 3 proves, for every $0<k<n$ that

$$\begin{aligned}&\alpha\sum_{i-0}^{n-1} w_i a_i b_i + (1-\alpha)\sum_{i=1}^{n} w_i a_i b_i\\ &\qquad\le \left(\alpha\sum_{i=0}^{k-1} w_i {a_i}^p + (1-\alpha)\sum_{i=1}^{k} w_i {a_i}^p\right)^{1/p}\\ &\qquad\quad\cdot\left(\alpha\sum_{i=0}^{k-1} w_i {b_i}^q + (1-\alpha)\sum_{i=1}^{k} w_i {b_i}^q\right)^{1/q}\\ &\qquad\quad+\alpha\sum_{i=k}^{n-1} w_i a_i b_i + (1-\alpha)\sum_{i=k+1}^{n} w_i a_i b_i\\ &\qquad\le \left(\alpha\sum_{i=0}^{n-1} w_i {a_i}^p + (1-\alpha)\sum_{i=1}^{n} w_i {a_i}^p\right)^{1/p}\\ &\qquad\quad\cdot\left(\alpha\sum_{i=0}^{n-1} w_i {b_i}^q + (1-\alpha)\sum_{i=1}^{n} w_i {b_i}^q\right)^{1/q},\end{aligned} \tag{22}$$

hence,

$$
\begin{aligned}
\alpha w_0 a_0 b_0 + &(1-\alpha) w_0 a_n b_n + \sum_{i=1}^{n-1} w_i a_i b_i \\
&\leq \left(\alpha w_0 {a_0}^p + (1-\alpha) w_k {a_k}^p + \sum_{i=1}^{k-1} w_i {a_i}^p \right)^{1/p} \\
&\qquad \cdot \left(\alpha w_0 {b_0}^q + (1-\alpha) w_n {b_n}^q + \sum_{i=1}^{k-1} w_i {b_i}^q \right)^{1/q} \\
&\qquad + \alpha \sum_{i=k}^{n-1} w_i a_i b_i + (1-\alpha) \sum_{i=k+1}^{n} w_i a_i b_i \\
&\leq \left(\alpha w_0 {a_0}^p + (1-\alpha) w_n {a_n}^p + \sum_{i=1}^{n-1} w_i {a_i}^p \right)^{1/p} \\
&\qquad \cdot \left(\alpha w_0 {b_0}^q + (1-\alpha) w_n {b_n}^q + \sum_{i=1}^{n-1} {b_i}^q \right)^{1/q}
\end{aligned}
\tag{23}
$$

and, also

$$
\begin{aligned}
\alpha w_0 a_0 b_0 + &(1-\alpha) w_n a_n b_n + \sum_{i=1}^{n-1} w_i a_i b_i \\
&\leq \left(\alpha w_k {a_k}^p + (1-\alpha) w_n {a_n}^p + \sum_{i=k+1}^{n} w_i {a_i}^p \right)^{1/p} \\
&\qquad \cdot \left(\alpha w_k {b_k}^q + (1-\alpha) w_n {b_n}^q + \sum_{i=k+1}^{n} w_i {b_i}^q \right)^{1/q} \\
&\qquad + \alpha \sum_{i=0}^{k-1} w_i a_i b_i + (1-\alpha) \sum_{i=1}^{k} w_i a_i b_i \\
&\leq \left(\alpha w_0 {a_0}^p + (1-\alpha) w_n {a_n}^p + \sum_{i=1}^{n-1} w_i {a_i}^p \right)^{1/p} \\
&\qquad \cdot \left(\alpha w_0 {b_0}^q + (1-\alpha) w_n {b_n}^q + \sum_{i=1}^{n-1} w_i {b_i}^q \right)^{1/q} .
\end{aligned}
\tag{24}
$$

The inequalities from (22) and (23) are improvements of the inequalities from (18), (19), (20) and (21).

(iv) If $\mathbb{T} = 2^{\mathbb{N}}$ with $a = 0$, $b = n$ and $f(i) = 2^{a_i}$, $g(i) = 2^{b_i}$, $w(i) = 2^{w_i}$, where $a_i, b_i, w_i \in \mathbb{R}$, for $i = 0, ..., n$, then for all $\alpha \in [0, 1]$, we get:

$$\begin{aligned}
\alpha 2^{w_0+a_0+b_0} &+ (1-\alpha)2^{w_n+a_n+b_n} + \sum_{i=1}^{n-1} 2^{w_i+a_i+b_i} \\
&\leq \left(\alpha 2^{w_0+pa_0} + (1-\alpha)2^{w_k+pa_k} + \sum_{i=1}^{k-1} 2^{w_i+pa_i}\right)^{1/p} \\
&\quad \cdot \left(\alpha 2^{w_0+qb_0} + (1-\alpha)2^{w_n+qb_n} + \sum_{i=1}^{k-1} 2^{w_i+qb_i}\right)^{1/q} \\
&\quad + \alpha \sum_{i=k}^{n-1} 2^{w_i+a_i+b_i} + (1-\alpha) \sum_{i=k+1}^{n} 2^{w_i a_i b_i} \\
&\leq \left(\alpha 2^{w_0+pa_0} + (1-\alpha)2^{w_n+pa_n} + \sum_{i=1}^{n-1} 2^{w_i+pa_i}\right)^{1/p} \\
&\quad \cdot \left(\alpha 2^{w_0+qb_0} + (1-\alpha)2^{w_n+qb_n} + \sum_{i=1}^{n-1} 2^{w_i+qb_i}\right)
\end{aligned} \tag{25}$$

and, also

$$
\begin{aligned}
\alpha 2^{w_0 a_0 b_0} + (1-\alpha) 2^{w_n a_n b_n} + \sum_{i=1}^{n-1} 2^{w_i a_i b_i} \\
\leq \left(\alpha 2^{w_k + p a_k} + (1-\alpha) 2^{w_n + p a_n} + \sum_{i=k+1}^{n} 2^{w_i + p a_i} \right)^{1/p} \\
\cdot \left(\alpha 2^{w_k q b_k} + (1-\alpha) 2^{w_n + q b_n} + \sum_{i=k+1}^{n} 2^{w_i + q b_i} \right)^{1/q} \\
+ \alpha \sum_{i=0}^{k-1} 2^{w_i a_i b_i} + (1-\alpha) \sum_{i=1}^{k} 2^{w_i a_i b_i} \\
\leq \left(\alpha 2^{w_0 + p a_0} + (1-\alpha) 2^{w_n + p a_n} + \sum_{i=1}^{n-1} 2^{w_i + p a_i} \right)^{1/p} \\
\cdot \left(\alpha 2^{w_0 + q b_0} + (1-\alpha) 2^{w_n + q b_n} + \sum_{i=1}^{n-1} 2^{w_i + q b_i} \right)^{1/q} .
\end{aligned} \tag{26}
$$

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