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Bounds on the Third Order Hankel Determinant for Certain Subclasses of Analytic Functions

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Abstract

Let A be the class of analytic functions $f(z)$ in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series expansion about the origin given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \Delta$. The focus of this paper is on deriving upper bounds for the third order Hankel determinant $H_3(1)$ for two new subclasses of A.

1 Introduction

Let A be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is respectively said to be with bounded turning, starlike or convex if and only if for $z \in \Delta$, $Ref'(z) > 0,\ Re\frac{zf'(z)}{f(z)} > 0$ or $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The classes of these functions are respectively denoted by R, S^* and C. For $n \geq 0$ and $q \geq 1$,

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the q^{th} Hankel determinant is defined as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & \vdots & \\ a_{n+q-1} & \cdots & & a_{n+2(q-1)} \end{vmatrix} \tag{2}$$

This determinant has been considered by several authors (see, for example, [1, 2, 3, 10, 17, 18, 22]). In fact Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, upper bounds for the second Hankel determinant were obtained by several authors [6, 9, 12, 20, 21] for different classes of analytic functions. Upper bound on the third Hankel determinant for different classes of functions has been studied recently [1, 2, 3, 10, 16, 22]. In the present investigation, the focus is on the third order Hankel determinant $H_3(1)$ for the classes R_α^β and S_α^β in Δ defined as follows:

Definition 1.1. *Let f be given by (1). Then $f \in R_\alpha^\beta$ if and only if for any $z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1$,*

$$Re\{f'(z) + \alpha z f''(z)\} > \beta. \tag{3}$$

The choice $\alpha = 0$, $\beta = 0$ yields $Re\ f'(z) > 0$, $z \in \Delta$, defining the class R of bounded turning [15] while the choice $\alpha = 0$, yields $Re\ f'(z) > \beta$ [5].

Definition 1.2. *Let f be given by (1). Then $f \in S_\alpha^\beta$ if and only if for any $z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1$,*

$$Re\left\{\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)}\right\} > \beta.$$

The choice $\alpha = 0$, $\beta = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > 0$, $z \in \Delta$, defining the class S^* of starlike functions [19] and the choice of $\alpha = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > \beta$, $z \in \Delta$, defining the class $S^*(\beta)$ starlike functions of order β [19]. Setting $n = 1$ in (2), $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

and for $f \in A$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

Using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{4}$$

In obtaining an upper bound for $|H_3(1)|$, the approach used is to first determine upper bounds for the functionals $|a_2a_3 - a_4|$,$|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$. Furthermore techniques employed in [13, 14] are useful in establishing the results (see, for example [6, 9, 12, 21]).

2 Preliminary Results

Some preliminary results required in the following sections are now listed.
Let P denote the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{5}$$

which are regular in Δ and satisfy $Re\ p(z) > 0,\ z \in \Delta$. Throughout this paper, we assume that $p(z)$ is given by (5) and $f(z)$ is given by (1). To prove the main results, the following known Lemmas are required.

Lemma 2.1. *[4] Let $p \in P$. Then $|c_k| \leq 2,\ k = 1, 2, \ldots$ and the inequality is sharp.*

Lemma 2.2. *[13, 14] Let $p \in P$. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{6}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \tag{7}$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

3 Main Results

For functions $f \in R_\alpha^\beta$, Lemma 3.1- Lemma 3.3 give the upper bounds for the three functionals mentioned earlier while Theorem 3.1 presents an estimate for $|H_3(1)|$.

Lemma 3.1. *Let $f \in R_\alpha^\beta$. Then*

$$|a_2a_3 - a_4| \leq \frac{(1-\beta)}{2(1+3\alpha)} \tag{8}$$

Proof. Let $f \in R_\alpha^\beta$. Then there exists a p such that

$$f'(z) + \alpha z f''(z) = (1-\beta)p(z) + \beta,\ p(0) = 1,\ Re\ p(z) > 0.$$

Equating the coefficients, we find that

$$a_2 = \frac{c_1(1-\beta)}{2(1+\alpha)}, a_3 = \frac{c_2(1-\beta)}{3(1+2\alpha)}, a_4 = \frac{c_3(1-\beta)}{4(1+3\alpha)}, a_5 = \frac{c_4(1-\beta)}{5(1+4\alpha)}.$$

The functional $|a_2a_3 - a_4|$ is given by

$$|a_2a_3 - a_4| = \left|\frac{c_1c_2(1-\beta)^2}{6(1+\alpha)(1+2\alpha)} - \frac{c_3(1-\beta)}{4(1+3\alpha)}\right|. \tag{9}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$\begin{aligned}
&|a_2a_3 - a_4| = \\
&\frac{(1-\beta)}{48(1+\alpha)(1+2\alpha)(1+3\alpha)}\big|4(1+3\alpha)(1-\beta)c_1(c_1^2 + x(4-c_1^2)) \\
&-3(1+\alpha)(1+2\alpha)(c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2))\big|
\end{aligned}$$

$$\begin{aligned}
=&A(\alpha,\beta)\left|c_1^3(2a-3b) - 2c_1x(4-c_1^2)(3b-a) - 3bx^2c_1(4-c_1^2)\right. \\
&\left.-2y \times 3b(1-|x|^2)(4-c_1^2)\right|
\end{aligned}$$

where $A(\alpha,\beta) = \frac{(1-\beta)}{48(1+\alpha)(1+2\alpha)(1+3\alpha)}, a = 2(1+3\alpha)(1-\beta), b = (1+\alpha)(1+2\alpha)$. Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$\begin{aligned}
|a_2a_3 - a_4| \leq &A(\alpha,\beta)\left\{c^3|2a-3b| + 2c\rho(4-c^2)(3b-a) + 3b\rho^2c(4-c^2)\right. \\
&\left.+2 \times 3b(1-\rho^2)(4-c^2)\right\}
\end{aligned}$$

$$\begin{aligned}
=&A(\alpha,\beta)\left\{c^3|2a-3b| + 2c\rho(4-c^2)(3b-a) + 3b\rho^2(4-c^2)(c-2)\right. \\
&\left.+6b(4-c^2)\right\} = F(\rho).
\end{aligned}$$

Next we maximize the function $F(\rho)$.

$$F'(\rho) = A(\alpha,\beta)\left\{2c(4-c^2)(3b-a) + 6b\rho(4-c^2)(c-2)\right\} \tag{10}$$

$F'(\rho) = 0$ implies $\rho = \frac{c(3b-a)}{3b(2-c)}$. Set $\rho^* = \frac{c(3b-a)}{3b(2-c)}$. Now $0 \leq \rho^* \leq 1$. Also we have $F''(\rho) = A(\alpha,\beta)\{6b(4-c^2)(c-2)\} < 0$, for $c < 2$. Thus ρ^* is the only value in $[0,1]$ at which $F(\rho)$ attains a maximum. Hence $F(\rho) \leq F(\rho^*)$.Thus

$$\begin{aligned} F(\rho) \leq\ & A(\alpha,\beta)\left\{c^3|2a-3b| + \frac{c^2}{3b}(3b-a)^2(2+c) + 6b(4-c^2)\right\} \\ =\ & A(\alpha,\beta)\left\{c^3\{|2a-3b| + \frac{(3b-a)^2}{3b}\} - c^2\{[6b - \frac{2(3b-a)^2}{3b}]\} + 24b\right\} \end{aligned}$$

$$F(\rho) \leq A(\alpha,\beta)\left\{c^3\gamma - c^2\delta + 24b\right\} = G(c),$$

where $\gamma = |2a-3b| + \frac{(3b-a)^2}{3b}, \delta = [6b - \frac{2(3b-a)^2}{3b}]$. $G'(c) = 0$ implies $c = 0$ and at $c = 0$, $G''(c) < 0$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = \rho^*$ and $c = 0$. Hence $|a_2a_3 - a_4| \leq \frac{(1-\beta)}{2(1+3\alpha)}$. □

Corollary 3.1. *Choosing $\alpha = 0$, $\beta = 0$ in (8), we get $|a_2a_3 - a_4| \leq \frac{1}{2}$. This result coincides with the corresponding result in [3].*

Lemma 3.2. *Let $f \in R_\alpha^\beta$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}\frac{(1-\beta)^2}{(1+2\alpha)^2} \tag{11}$$

Proof. Let $f \in R_\alpha^\beta$. In a manner similar to the proof of Lemma 3.1, we can derive

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3(1-\beta)^2}{8(1+\alpha)(1+3\alpha)} - \frac{c_2^2(1-\beta)^2}{9(1+2\alpha)^2}\right| \tag{12}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$\begin{aligned} = & \left|\frac{c_1(1-\beta)^2}{32(1+\alpha)(1+3\alpha)}[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2)\right. \\ & \left. +2y(1-|x|^2)(4-c_1^2)] - \frac{(1-\beta)^2}{36(1+2\alpha)^2}[c_1^2 + x(4-c_1^2)]^2\right|. \end{aligned}$$

$$\begin{aligned} = & \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left|9c_1(1+2\alpha)^2[c_1^3 + 2xc_1(4-c_1^2)\right. \\ & -x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)] \\ & \left.-8(1+\alpha)(1+3\alpha)[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)]\right|. \end{aligned}$$

Let $N = \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$,$a = 9(1+2\alpha)^2$, $b = 8(1+\alpha)(1+3\alpha)$ and $a-b = 9(1+2\alpha)^2 - 8(1+\alpha)(1+3\alpha) = 1+12\alpha^2+4\alpha \geq 0$, since $\alpha \geq 0$.

$$\begin{aligned}|a_2a_4 - a_3^2| &= N\left|ac_1[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right.\\ &\quad \left.-b[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)]\right|\\ &= N\left|c_1^4(a-b) + 2xc_1^2(4-c_1^2)(a-b) - x^2(4-c_1^2)[ac_1^2 + b(4-c_1^2)]\right.\\ &\quad \left.+2yac_1(1-|x|^2)(4-c_1^2)\right|.\end{aligned}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq N\left\{c^4(a-b) + 2\rho c^2(4-c^2)(a-b)\right.\\ &\quad \left.+\rho^2(4-c^2)[c^2(a-b) - 2ac + 4b] + 2ac(4-c^2)\right\}\\ &= N\left\{c^4(a-b) + 2\rho c^2(4-c^2)(a-b)\right.\\ &\quad \left.+\rho^2(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)}) + 2ac(4-c^2)\right\} = F(\rho).\end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = N[2c^2(4-c^2)(a-b) + 2\rho(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)})] \geq 0,$$

since $a - b > 0$, $2b/(a-b) > 2$ so that $c - 2b/(a-b) < c - 2 < 0$ and $(c-2)(c - \frac{2b}{(a-b)}) > 0$ for all $c \in [0,2]$. This implies that $F(\rho)$ is an increasing function of ρ on the closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is,

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq N\left\{c^4(a-b) + 2c^2(4-c^2)(a-b)\right.\\ &\quad \left.+(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)}) + 2ac(4-c^2)\right\}\\ &= N\left\{-2c^4(a-b) - 4c^2(4b-3a) + 16b\right\} = G(c).\end{aligned}$$

$G'(c) = 0$ implies $c = 0$ so that at $c = 0$,$G''(c) < 0$. Therefore $c = 0$ is a point of maximum for $G(c)$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence, $|a_2a_4 - a_3^2| \leq \frac{4}{9}\frac{(1-\beta)^2}{(1+2\alpha)^2}$. □

Corollary 3.2. *Choosing $\alpha = 0$, $\beta = 0$ in (11), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. This result coincides with [7].*

Corollary 3.3. *Choosing $\alpha = 0$ in (11), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}(1-\beta)^2$. This result coincides with [11].*

Lemma 3.3. *Let $f \in R_\alpha^\beta$. Then for $1/3 \leq \beta < 1$,*

$$|a_3 - a_2^2| \leq \frac{2(1-\beta)}{3(1+2\alpha)} \tag{13}$$

Proof. Let $f \in R_\alpha^\beta$. Then by proceeding as in Lemma 3.1, we have

$$|(a_3 - a_2^2)| = \left| \frac{c_2(1-\beta)}{3(1+2\alpha)} - \frac{c_1^2(1-\beta)^2}{4(1+\alpha)^2} \right| \tag{14}$$

$$|(a_3 - a_2^2)| = \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2} \left|4(1+\alpha)^2 c_2 - 3(1+2\alpha)(1-\beta)c_1^2\right|.$$

Substituting for c_2 from (6) of Lemma 2.2, we obtain

$$\begin{aligned}
&|(a_3 - a_2^2)| \\
&= \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2} \left|2(1+\alpha)^2[c_1^2 + x(4-c_1^2)] - 3(1+2\alpha)(1-\beta)c_1^2\right| \\
&= M|k_1c_1^2 + k_1x(4-c_1^2) - k_2c_1^2|,
\end{aligned}$$

where $M = \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2}, k_1 = 2(1+\alpha)^2, k_2 = 3(1+2\alpha)(1-\beta)$. Set $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0, 2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$|a_3 - a_2^2| \leq M[c^2|k_1 - k_2| + k_1\rho(4 - c^2)] = F(\rho).$$

Differentiating $F(\rho)$, we get $F'(\rho) = M[k_1(4-c^2)] \geq 0$, implying that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is,

$$|(a_3 - a_2^2)| \leq M[c^2|k_1 - k_2| + k_1(4 - c^2)] = G(c).$$

By hypothesis, $\beta \geq 1/3$ and hence $k_1 - k_2 = 2\alpha^2 - 2\alpha - 1 + 3\beta(1+2\alpha) \geq 2\alpha^2$. Hence $G(c) = M[4k_1 - c^2k_2]$, $G'(c) = -2Mk_2c$ and $G''(c) = -2Mk_2$. Since $c \in [0, 2]$, it follows that G(c) attains the maximum at $c = 0$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|a_3 - a_2^2| \leq M[4k_1] = \frac{2(1-\beta)}{3(1+2\alpha)}$. ∎

Corollary 3.4. *Choosing* $\alpha = 0$ *in* (13), *we get* $|a_3 - a_2^2| \leq \frac{2}{3}(1-\beta)$. *This result coincides with [10], for* $1/3 \leq \beta < 1$.

Remark 3.1. *Let* $f \in R_\alpha^\beta$. *By Lemma 2.1, we have*

$$
\begin{aligned}
|a_3| &= \left|\frac{c_2(1-\beta)}{3(1+2\alpha)}\right| \leq \frac{2(1-\beta)}{3(1+2\alpha)},\\
|a_4| &= \left|\frac{c_3(1-\beta)}{4(1+3\alpha)}\right| \leq \frac{(1-\beta)}{2(1+3\alpha)},\\
|a_5| &= \left|\frac{c_4(1-\beta)}{5(1+4\alpha)}\right| \leq \frac{2(1-\beta)}{5(1+4\alpha)}.
\end{aligned}
$$

Using the above results, the upper bound for $|H_3(1)|$, $f \in R_\alpha^\beta$ is immediately obtained.

Theorem 3.1. *Let* $f \in R_\alpha^\beta$. *Then for* $1/3 \leq \beta < 1$,

$$
|H_3(1)| \leq \frac{8(1-\beta)^3}{27(1+2\alpha)^3} + \frac{(1-\beta)^2}{4(1+3\alpha)^2} + \frac{4(1-\beta)^2}{15(1+2\alpha)(1+4\alpha)}.
$$

In the following results, with similar approach and technique, an upper bound for $|H_3(1)|$ is attained for $f \in S_\alpha^\beta$. As before, we first derive estimates for the functionals $|a_2a_3 - a_4|$, $|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$. Their estimates are given in Lemmas 3.4, 3.5, and 3.6.

Lemma 3.4. *Let* $f \in S_\alpha^\beta$. *Then*

$$
|a_2a_3 - a_4| \leq \frac{2(1-\beta)}{3(1+4\alpha)} \tag{15}
$$

Proof. Let $f \in S_\alpha^\beta$. Then there exists a $p \in P$ such that

$$
zf'(z) + \alpha z^2 f''(z) = [(1-\beta)p(z) + \beta]f(z),
$$

for some $z \in \Delta$. Equating the coefficients, we have

$$
a_2 = \frac{c_1(1-\beta)}{1+2\alpha}, \quad a_3 = \frac{c_2(1-\beta)}{2(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{2(1+2\alpha)(1+3\alpha)},
$$

$$
a_4 = \frac{c_3(1-\beta)}{3(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)},
$$

and

$$
\begin{aligned}
a_5 &= \frac{c_1^4(1-\beta)^4}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_2^2(1-\beta)^2}{8(1+3\alpha)(1+5\alpha)} \\
&+ \frac{c_1^2c_2(1-\beta)^3(20\alpha+6)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_1c_3(1-\beta)^2(4+14\alpha)}{12(1+2\alpha)(1+4\alpha)(1+5\alpha)} \\
&+ \frac{c_4(1-\beta)}{4(1+5\alpha)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|a_2a_3 - a_4| = \frac{(1-\beta)}{6(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} &\left|c_1c_2(1-\beta)4\alpha(1+2\alpha)\right. \\
&\left. +2c_1^3(1-\beta)^2(1+5\alpha) - 2c_3(1+2\alpha)^2(1+3\alpha)\right|
\end{aligned} \tag{16}
$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we have

$$
\begin{aligned}
&|a_2a_3 - a_4| \\
&= B(\alpha,\beta)\left|\frac{4\alpha(1+2\alpha)(1-\beta)c_1}{2}(c_1^2 + x(4-c_1^2))\right. \\
&\quad +c_1^3 2(1-\beta)^2(1+5\alpha) - \frac{2(1+2\alpha)^2(1+3\alpha)}{4}[c_1^3 + 2xc_1(4-c_1^2) \\
&\quad \left. -x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right| \\
&= B(\alpha,\beta)\left|r_1c_1^3 + r_2c_1x(4-c_1^2) + \frac{r_3}{4}x^2c_1(4-c_1^2) - \frac{r_3}{2}y(1-|x|^2)(4-c_1^2)\right|,
\end{aligned}
$$

where

$$
\begin{aligned}
B(\alpha,\beta) &= \frac{(1-\beta)}{6(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}, \\
r_1 &= 2\alpha(1+2\alpha)(1-\beta) + 2(1-\beta)^2(1+5\alpha) - \frac{(1+2\alpha)^2(1+3\alpha)}{2}, \\
r_2 &= 2\alpha(1+2\alpha)(1-\beta) - (1+2\alpha)^2(1+3\alpha), r_3 = (1+2\alpha)^2(1+3\alpha).
\end{aligned}
$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get,

$$
\begin{aligned}
&|a_2a_3 - a_4| \\
&\leq \beta(\alpha,\beta)\{|r_1|c^3 + |r_2|\rho c(4-c^2) + \frac{r_3}{2}\rho^2 c(4-c^2) + r_3(4-c^2) - r_3\rho^2(4-c^2)\} \\
&= \beta(\alpha,\beta)\{|r_1|c^3 + |r_2|\rho c(4-c^2) + \frac{r_3}{2}\rho^2(c-2)(4-c^2) + r_3(4-c^2)\} = F(\rho).
\end{aligned}
$$

Next we maximize the function $F(\rho)$. Differentiating $F(\rho)$, we get

$$F'(\rho) = B(\alpha,\beta)[|r_2|c(4-c^2) + r_3\rho(4-c^2)(c-2)].$$

$F'(\rho) = 0$ implies $\rho = \frac{|r_2|c}{r_3(2-c)}$. Set $\rho^* = \frac{|r_2|c}{r_3(2-c)}$. Now, $0 \le \rho^* \le 1$. Also we have $F''(\rho) = B(\alpha,\beta)r_3(4-c^2)(c-2) \le 0$. Thus ρ^* is the only value in $[0,1]$ at which $F(\rho)$ attains maximum. Hence $F(\rho) \le F(\rho^*)$.Thus

$$\begin{aligned} F(\rho) &\le B(\alpha,\beta)[|r_1|c^3 + \frac{r_2^2c^2(2+c)}{2r_3} + 4r_3 - r_3c^2] \\ &= B(\alpha,\beta)[c^3\gamma - c^2\delta + 4r_3] = G(c), \end{aligned}$$

where $\gamma = |r_1| + \frac{r_2^2}{2r_3}$, $\delta = r_3 - \frac{r_2^2}{r_3} \ge 0$, $G'(c) = 0$ implies $c = 0$ and at $c = 0$, $G''(c) < 0$. Therefore $c = 0$ is a point of maximum of $G(c)$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = \rho^*$ and $c = 0$. Hence $|a_2a_3 - a_4| \le \frac{2(1-\beta)}{3(1+4\alpha)}$. □

Corollary 3.5. *Choosing $\alpha = 0$, $\beta = 0$ in (15), we get $|a_2a_3 - a_4| \le \frac{2}{3}$.*

Corollary 3.6. *Choosing $\alpha = 0$, in (15), we get*

$$|a_2a_3 - a_4| \le \frac{2(1-\beta)}{3}.$$

Lemma 3.5. *Let $f \in S_\alpha^\beta$. Then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+3\alpha)^2} \tag{17}$$

Proof. Let $f \in S_\alpha^\beta$. Then by proceeding as in Lemma 3.4, we have

$$\begin{aligned} &|a_2a_4 - a_3^2| = \\ &\left|\frac{c_1c_3(1-\beta)^2}{3(1+2\alpha)(1+4\alpha)} - \frac{c_2^2(1-\beta)^2}{4(1+3\alpha)^2} - \frac{c_1^4(1-\beta)^4(1+6\alpha)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\right. \\ &\left.-\frac{c_1^2c_2(1-\beta)^3(2\alpha)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\right| \end{aligned} \tag{18}$$

$$\begin{aligned} &= \frac{(1-\beta)^2}{48(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\left|16(1+2\alpha)(1+3\alpha)^2c_1c_3\right. \\ &\left.-12c_2^2(1+2\alpha)^2(1+4\alpha) - 4c_1^4(1-\beta)^2(1+6\alpha) - 4(1-\beta)2\alpha c_1^2c_2\right|. \end{aligned}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$|a_2a_4 - a_3^2| = M\left|k_1c_1[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right.$$
$$\left.-k_2[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)] - k_3c_1^4 - k_4c_1^2[c_1^2 + x(4-c_1^2)\right|,$$

where $M = \frac{(1-\beta)^2}{48(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}$,
$k_1 = 4(1+2\alpha)(1+3\alpha)^2$,$k_2 = 3(1+2\alpha)^2(1+4\alpha)$, $k_3 = 4(1-\beta)^2(1+6\alpha)$ and $k_4 = 8\alpha(1-\beta)$.

$$|a_2a_4 - a_3^2| =$$
$$M\left|c_1^4[k_1 - k_2 - k_3 - k_4] + xc_1^2(4-c_1^2)[2k_1 - 2k_2 - k_4] - x^2c_1^2(4-c_1^2)k_1\right.$$
$$\left.-x^2(4-c_1^2)^2k_2 + 2yc_1k_1(1-|x|^2)(4-c_1^2)\right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying triangle inequality with $\rho = |x| \leq 1$, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq M\left\{c^4|k_1 - k_2 - k_3 - k_4| + \rho c^2(4-c^2)|2k_1 - 2k_2 - k_4|\right.\\ &\left.+\rho^2(4-c^2)(c^2(k_1-k_2) - 2ck_1 + 4k_2) + 2ck_1(4-c^2)\right\}\\ &= M\left\{c^4|k_1 - k_2 - k_3 - k_4| + \rho c^2(4-c^2)|2k_1 - 2k_2 - k_4|\right.\\ &\left.+\rho^2(4-c^2)(k_1-k_2)(c-2)(c - \frac{2k_2}{k_1-k_2}) + 2ck_1(4-c^2)\right\} = F(\rho).\end{aligned}$$

Differentiating $F(\rho)$, we get

$$\begin{aligned}F'(\rho) = M[&c^2(4-c^2)|2k_1 - 2k_2 - k_4|\\ &+ 2\rho(4-c^2)(k_1-k_2)(c-2)(c - \frac{2k_2}{(k_1-k_2)}) \geq 0,\end{aligned}$$

since $2k_2/(k_1-k_2) > 2$ so that $c - 2k_2/(k_1-k_2) < c - 2 < 0$ and $k_1 - k_2 = (1+2\alpha)(36\alpha^2 + 12\alpha + 1) > 0$ as $\alpha > 0$, and so $(c-2)(c - \frac{2k_2}{(k_1-k_2)} > 0$ for all $c \in [0,2]$. This implies that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is,

$$F(\rho)$$
$$\leq M\left\{c^4|k_1 - k_2 - k_3 - k_4| + (4-c^2)[c^2|2k_1 - 2k_2 - k_4| + (c^2(k_1-k_2) + 4k_2)]\right\}$$
$$= M\left\{[c^4[|k_1 - k_2 - k_3 - k_4| - (|2k_1 - 2k_2 - k_4| - (k_1 - k_2))]\right.$$
$$\left.-c^2[4k_2 - 4(|2k_1 - 2k_2 - k_4| - 4(k_1 - k_2))] + 16k_2\right\} = G(c).$$

$G'(c) = 0$ implies $c = 0$ so that at $c = 0, G''(c) < 0$. Therefore $c = 0$ is a point of maximum for $G(c)$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{(1+3\alpha)^2}$. □

Corollary 3.7. *Choosing* $\alpha = 0$, $\beta = 0$ *in* (17), *we get* $|a_2a_4 - a_3^2| \leq 1$. *This result coincides with [8].*

Corollary 3.8. *Choosing* $\alpha = 0$, *in* (17), *we get* $|a_2a_4 - a_3^2| \leq (1-\beta)^2$.

Lemma 3.6. *Let* $f \in S_\alpha^\beta$. *Then for* $1/2 \leq \beta < 1$,

$$|a_3 - a_2^2| \leq \frac{1-\beta}{1+3\alpha} \tag{19}$$

Proof. Let $f \in S_\alpha^\beta$. Then by proceeding as in Lemma 3.4, we have

$$|a_3 - a_2^2| = \left| \frac{c_2(1-\beta)}{2(1+3\alpha)} - \frac{c_1^2(1-\beta)^2(1+4\alpha)}{2(1+2\alpha)^2(1+3\alpha)} \right| \tag{20}$$

Substituting for c_2 from Lemma 2.2 we obtain

$$\begin{aligned} |a_3 - a_2^2| &= \left| \frac{(1-\beta)}{2(1+3\alpha)} \frac{1}{2}[c_1^2 + x(4-c_1^2)] - \frac{c_1^2(1-\beta)^2(1+4\alpha)}{2(1+3\alpha)(1+2\alpha)^2} \right| \\ &= \frac{(1-\beta)}{4(1+2\alpha)^2(1+3\alpha)} \left| [c_1^2 + x(4-c_1^2)](1+2\alpha)^2 - 2c_1^2(1-\beta)(1+4\alpha) \right| \\ &= M \left| b_1[c_1^2 + x(4-c_1^2)] - b_2c_1^2 \right|, \end{aligned}$$

where $M = \frac{(1-\beta)}{4(1+2\alpha)^2(1+3\alpha)}, b_1 = (1+2\alpha)^2, b_2 = 2(1-\beta)(1+4\alpha)$. Therefore

$$|a_3 - a_2^2| = M \left| b_1c_1^2 + b_1x(4-c_1^2) - b_2c_1^2 \right| = M \left| (b_1 - b_2)c_1^2 + b_1x(4-c_1^2) \right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying triangle inequality with $\rho = |x| \leq 1$, we obtain

$$|a_3 - a_2^2| \leq M[c^2|b_1 - b_2| + b_1\rho(4-c^2)] = F(\rho).$$

Differentiating $F(\rho)$, we get $F'(\rho) = Mb_1(4-c^2) > 0$, implying that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is

$$|(a_3 - a_2^2)| \leq M[c^2|b_1 - b_2| + b_1(4-c^2)] = G(c).$$

By hypothesis, $\beta \geq 1/2$ and hence $b_1 - b_2 = 4\alpha^2 - 4\alpha - 1 + 2\beta(1+4\alpha) \geq 4\alpha^2$. Hence $G(c) = M[4b_1 - b_2c^2]$, $G'(c) = -2b_2Mc$ and $G''(c) = -2b_2M$. Since $c \in [0,2]$, it follows that $G(c)$ attains a maximum at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|(a_3 - a_2^2)| \leq \frac{(1-\beta)}{(1+3\alpha)}$. □

Corollary 3.9. *Choosing* $\alpha = 0$ *in* (19),*we get* $|a_3 - a_2^2| \leq (1-\beta)$.

Using Lemma 2.1,the following estimates can be deduced.

Remark 3.2. *Let* $f \in S_\alpha^\beta$. *By Lemma 2.1, we have*

$$\begin{aligned}
|a_3| &= \left|\frac{c_2(1-\beta)}{2(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{2(1+2\alpha)(1+3\alpha)}\right|, \\
&\leq \frac{(1-\beta)(3+2\alpha-2\beta)}{(1+2\alpha)(1+3\alpha)}, \\
|a_4| &= \left|\frac{c_3(1-\beta)}{3(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)}\right|, \\
&\leq \frac{(1-\beta)[12+12\alpha^2+4\beta^2-16\alpha\beta-14\beta+26\alpha]}{3(1+2\alpha)(1+3\alpha)(1+4\alpha)} \quad \text{and} \\
|a_5| &= \left|\frac{c_1^4(1-\beta)^4}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}\right. \\
&\quad + \frac{c_2^2(1-\beta)^2}{8(1+3\alpha)(1+5\alpha)} + \frac{c_1^2c_2(1-\beta)^3(20\alpha+6)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} \\
&\quad \left. + \frac{c_1c_3(1-\beta)^2(4+14\alpha)}{12(1+2\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_4(1-\beta)}{4(1+5\alpha)}\right| \\
&\leq \frac{(1-\beta)\left\{\begin{array}{r} 120+288\alpha^2-16\beta^3+744\alpha^2+548\alpha-188\beta \\ +96\beta^2-600\alpha\beta+144\alpha^2\beta+160\alpha\beta^2 \end{array}\right\}}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

Finally, using the above results, an upper bound for $|H_3(1)|$, $f \in S_\alpha^\beta$ is immediately obtained. 3.2.

Theorem 3.2. *Let* $f \in S_\alpha^\beta$. *Then for* $1/2 \leq \beta < 1$,

$$\begin{aligned}
|H_3(1)| \leq\ & \frac{(1-\beta)^3(3+2\alpha-2\beta)}{(1+2\alpha)(1+3\alpha)^3} \\
& + \frac{2(1-\beta)^2[12+12\alpha^2+4\beta^2-16\alpha\beta-14\beta+26\alpha]}{9(1+2\alpha)(1+3\alpha)(1+4\alpha)^2} \\
& + \frac{(1-\beta)^2\left\{\begin{array}{r} 120+288\alpha^2-16\beta^3+744\alpha^2+548\alpha-188\beta \\ +96\beta^2-600\alpha\beta+144\alpha^2\beta+160\alpha\beta^2 \end{array}\right\}}{24(1+2\alpha)(1+3\alpha)^2(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

Remark 3.3. *The determination of the sharp estimates for* $|H_3(1)|$ *for functions belonging to the classes* R_α^β *and* S_α^β *remain to be explored.*

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