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Best possible estimates of weak solutions of boundary value problems for quasi-linear elliptic equations in unbounded domains

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Abstract

We investigate the behaviour of weak solutions of boundary value problems for quasi-linear elliptic divergence second order equations in unbounded domains. We show the boundedness of weak solutions to our problem. Using barrier function and applying the comparison principle, we find the exact exponent of weak solutions decreasing rate near the infinity.

1 Introduction

Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n,\ n \geq 3$, with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded domain with the smooth boundary ∂G. We consider the following problem :

$$\begin{cases} -\frac{d}{dx_i}\left(r^\tau |u|^q |\nabla u|^{m-2} u_{x_i}\right) + a_0 r^{\tau-m} u|u|^{q+m-2} \\ \qquad\qquad -\mu_0 r^\tau |u|^{q-1} |\nabla u|^m \mathrm{sign} u = f,\ x \in G; \\ \\ \alpha(x) r^\tau |u|^q |\nabla u|^{m-2} \frac{\partial u}{\partial \vec{n}} + \gamma(\omega) r^{\tau-m+1} u|u|^{q+m-2} = g,\ x \in \partial G; \\ \\ \lim\limits_{|x|\to\infty} u(x) = 0; \end{cases} \tag{BVP}$$

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here $a_0 \geq 0,\ 0 \leq \mu_0 < \frac{q+m-1}{m-1},\ q \geq 0,\ 1 < m < n,\ \tau \leq \begin{cases} 0, & \text{if } m \geq 2, \\ m-2, & \text{if } m \in (1,2) \end{cases}$

$\gamma(\omega)$ is a positive bounded function and $\alpha(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \in \mathcal{D}; \\ 1, & \text{if } x \notin \mathcal{D}, \end{array} \right.$ $\mathcal{D} \subseteq \partial G$ is the part of the boundary ∂G, where the Dirichlet boundary condition is posed. Thus, if $\mathcal{D} = \partial G$ then we have the Dirichlet problem, if $\mathcal{D} = \emptyset$ then $\alpha(x) = 1$ and we have the Robin problem, if $\mathcal{D} \subset \partial G$ then we have the mixed problem.

Many problems of mathematical physics lead one to consider the solution of boundary value problems for elliptic second order equations in unbounded domains and to study the behaviour of the solution at infinity, for instance stationary states, travelling waves, homogenization, boundary layer problems, Saint-Venant's principle and so on. Some problems in unbounded domains found applications in to mechanics of inhomogeneous media [9], models in astrophysics (Eddington's model), are used in the analytic theory of polytropic ball model of stellar structures (Lane-Ritter-Emden theory) and have numerous applications in natural sciences, e.g. in scalar field theory, in phase transition theory, in combustion theory, population dynamics [10], pseudo-plastic fluids [6, 8, 12], and ecology models [14].

A number of mathematicians [2, 5, 7, 13, 15, 16] took the study of quasilinear elliptic problems in unbounded domains. The problem of the existence and nonexistence of positive solutions to a weak linear second-order divergence type elliptic equation in an unbounded cone-like domains was studied in [7].

Our aim is to establish the exact exponent of (BVP) weak solutions decreasing rate. For this purpose in Section 2 we formulate and prove the comparison principle. Next, in Section 3, we show the boundedness of weak solutions to problem (BVP). In section 4 we construct of the barrier function and study properties of solutions to the Sturm - Liouville boundary problem. Finally, in Section 5, we prove estimation of weak solutions for the problem (BVP) near the infinity.

For $x = (x_1, \ldots, x_n)$ we introduce the cylindrical coordinates (r, ω, x'), where $x' = (x_3, \ldots, x_n)$, $r = \sqrt{x_1^2 + x_2^2}$, $\omega = \arctan \frac{x_2}{x_1}$. We assume that there exists $d \gg 1$ such that $G = G_0 \cup G_d$, where G_0 is a bounded domain and:

$$G_d = \left\{ (r, \omega, x') \mid d < r < \infty,\ \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right),\ x' \in \mathbb{R}^{n-2} \right\},\ 0 < \omega_0 < 2\pi$$

$\partial G_d = \Gamma_d^+ \cup \Gamma_d^- \cup \Omega_d$, where

$$\Omega_d = \left\{ (r, \omega, x') \mid r = d,\ \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right),\ x' \in \mathbb{R}^{n-2} \right\};$$

$$\Gamma_d^{\pm} = \left\{ \left(r, \pm\frac{\omega_0}{2}, x' \right) | r > d,\ x' \in \mathbb{R}^{n-2} \right\}.$$

Assumptions. The following conditions will be needed regarding to our problem :

(1) $f(x), g(x)$ are measurable functions such that:

$$f(x) \in L_p(G),\ g(x) \in L_\alpha(\partial G);$$
$$\frac{1}{p} < \frac{m}{n} - \frac{1}{t}, \quad \frac{1}{t} < \frac{m}{n} < 1 + \frac{1}{t} < m, \quad \alpha > \frac{n-1}{m-1-\frac{n}{t}}; \tag{1}$$

(2) $|g(x)| \le g_1 r^{\tau-m+1+(q+m-1)\lambda_-},\ |f(x)| \le f_1 r^{\tau-m+(q+m-1)\lambda_-},\ \lambda_- < 0.$

We denote by $\mathfrak{N}^1_{m,q,\tau}(G)$ the set of functions $u(x) \in C^0(\overline{G})$ having first weak derivatives with the finite integral $\int\limits_G (r^\tau |u|^q |\nabla u|^m + r^{\tau-m}|u|^{q+m})\,dx$ for $q \ge 0,\ m > 1,\ \tau \le m-2$.

Definition 1. *A function $u(x)$ is said to be a weak solution of the problem (BVP) provided that $u(x) \in C^0(\overline{G}) \cap \mathfrak{N}^1_{m,q,\tau}(G)$ and satisfies the integral identity*

$$\int\limits_G \left\{ r^\tau |u|^q |\nabla u|^{m-2} u_{x_i} \eta_{x_i} + \left(a_0 r^{\tau-m} u|u|^{q+m-2} - \mu_0 r^\tau |u|^{q-1} |\nabla u|^m signu\right)\eta \right\} dx$$
$$= \int\limits_G f\eta dx + \int\limits_{\partial G} \alpha(x)\left(g - \gamma(\omega) r^{\tau-m+1} u|u|^{q+m-2}\right)\eta ds \tag{II}$$

for all functions $\eta(x) \in C^0(\overline{G}) \cap \mathfrak{N}^1_{m,q,\tau}(G)$ such that $\lim\limits_{|x|\to\infty} \eta(x) = 0$.

Lemma 1.1. *Let $u(x)$ be a weak solution of (BVP). For any function $\eta(x) \in C^0(\overline{G}) \cap \mathfrak{N}^1_{m,q,\tau}(G)$ with $\lim\limits_{|x|\to\infty} \eta(x) = 0$ the equality*

$$\int\limits_{G_R} \left\{ r^\tau |u|^q |\nabla u|^{m-2} u_{x_i} \eta_{x_i} + \left(a_0 r^{\tau-m} u|u|^{q+m-2} - \mu_0 r^\tau |u|^{q-1} |\nabla u|^m signu\right)\eta \right\} dx$$
$$= -\int\limits_{\Omega_R} r^\tau |u|^q |\nabla u|^{m-2} u_{x_i} \eta \cos(r, x_i) d\Omega_R + \int\limits_{G_R} f\eta dx$$
$$+ \int\limits_{\Gamma_R} \alpha(x)\left(g - \gamma(\omega) r^{\tau-m+1} u|u|^{q+m-2}\right)\eta ds. \tag{$II)_{loc}$$

holds for a.e. $R > d \gg 1$.

Proof. The proof is analogous to the proof of Lemma 1.3 [18]. □

We shall consider the substitution

$$u = v|v|^{\zeta-1} \text{ with } \zeta = \frac{m-1}{q+m-1}. \tag{2}$$

Then the identity $(II)_{loc}$ takes the following form

$$\int\limits_{G_R} \left\{\zeta^{m-1} r^{\tau} |\nabla v|^{m-2} v_{x_i} \eta_{x_i} + \left(a_0 r^{\tau-m} v|v|^{m-2} - \mu_0 \zeta^m r^{\tau} |v|^{-1} |\nabla v|^m \mathrm{sign} v\right)\eta\right\} dx$$

$$= -\int\limits_{\Omega_R} \zeta^{m-1} r^{\tau} |\nabla v|^{m-2} v_{x_i} \eta \cos(r, x_i) d\Omega_R + \int\limits_{G_R} f\eta dx$$

$$+ \int\limits_{\Gamma_R} \alpha(x)\left(g - \gamma(\omega) r^{\tau-m+1} v|v|^{m-2}\right)\eta ds \qquad (\widehat{II})_{loc}$$

for a.e. $R > d \gg 1$, $v(x) \in C^0(\overline{G}) \cap \mathfrak{N}^1_{m,0,\tau}(G)$ and any $\eta(x) \in C^0(\overline{G}) \cap \mathfrak{N}^1_{m,0,\tau}(G)$ with $\lim\limits_{|x|\to\infty} \eta(x) = 0$.

We need some auxiliary lemmas:

Lemma 1.2. Stampacchia's Lemma. (See Lemma 3.11 of [11]) *Let $\varphi : [k_0, \infty) \to \mathbb{R}$ be a non-negative and non-increasing function which satisfies*

$$\varphi(h) \le \frac{C}{(h-k)^{\alpha}} [\varphi(k)]^{\beta} \quad \text{for} \quad h > k > k_0,$$

where C, α, β are positive constants with $\beta > 1$. Then $\varphi(k_0 + d) = 0$, where $d^{\alpha} = C\varphi^{\beta-1}(k_0) 2^{\frac{\alpha\beta}{\beta-1}}$.

Lemma 1.3. (see Lemma 2.1 in [3]). *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1 & x \ge 0 \\ -e^{-\varkappa x} + 1 & x \le 0, \end{cases}$$

where $\varkappa > 0$. Let a, b be positive constants, $m > 1$. If $\varkappa > \frac{2b}{a} + m$, then we have

$$a\eta'(x) - b|\eta(x)| \ge \frac{a}{2} e^{\varkappa x}, \quad \forall x \ge 0. \qquad (3)$$

Moreover, there exist some $d \ge 0$ and $M > 0$ such that

$$\eta(x) \le M \left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \eta'(x) \le M \left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \ge d; \qquad (4)$$

$$|\eta(x)| \ge x, \quad \forall x \in \mathbb{R}. \qquad (5)$$

Proof. See the proof of Lemma 1.23 in [1]. □

Lemma 1.4. *(see [4], Examples 1.5, 1.6, p. 29) Let $m^{\#}$ denote the number associated to m by the relation*

$$\frac{1}{m^{\#}} = \frac{1}{m}\left(1+\frac{1}{t}\right) - \frac{1}{n} \tag{6}$$

and assume that (1) holds. Then there exist positive constants c_1, c_2 (depending only on meas$D$, n, m, t, τ) such that

$$\int\limits_D r^{\tau-m}|v|^m dx \le c_1 \int\limits_D r^{\tau}|\nabla v|^m dx,$$

$$\left(\int\limits_D |v|^{m^{\#}} dx\right)^{\frac{m}{m^{\#}}} \le c_2 \int\limits_D \left(r^{\tau-m}|v|^m + r^{\tau}|\nabla v|^m\right) dx$$

for any $v(x) \in \mathfrak{N}^1_{m,0,\tau}(D)$.

Lemma 1.5. *There exists a constant $c_4 > 0$ depending only on $n, m, t, D, \partial D$ such that for any $v(x) \in \mathfrak{N}^1_{m,0,\tau}(D)$*

$$\left(\int\limits_{\partial D} |v|^{\alpha^*} ds\right)^{\frac{1}{\alpha^*}} \le c_4 \left\{\int\limits_D (r^{\tau-m}|v|^m + r^{\tau}|\nabla v|^m) dx\right\}^{\frac{1}{m}}, \quad \alpha^* = \frac{m(n-1)}{n-m+\frac{n}{t}}. \tag{7}$$

This lemma is a simple implication of Theorem 3.9 of [11].

Corollary 1.6. *From Lemmas 1.4 and 1.5*

$$\left(\int\limits_D |v|^{m^{\#}} dx\right)^{\frac{m}{m^{\#}}} + \left(\int\limits_{\partial D} |v|^{\alpha^*} ds\right)^{\frac{m}{\alpha^*}} \le c_3 \left\{\int\limits_D (r^{\tau-m}|v|^m + r^{\tau}|\nabla v|^m) dx\right\} \tag{8}$$

for any $v(x) \in \mathfrak{N}^1_{m,0,\tau}(D)$, where $c_3 > 0$ depends on $n, m, t, D, \partial D$, $\|r^{m-\tau}\|_{L_t(D)}, \|r^{-\tau}\|_{L_t(D)}$.

2 The comparison principle

Taking into account $(\widehat{II})_{loc}$ we consider the second order quasi-linear degenerate operator Q of the form

$$
\begin{aligned}
Q(v,\eta) \equiv & \int\limits_{G_d} (\zeta^{m-1} r^{\tau} |\nabla v|^{m-2} v_{x_i}\eta_{x_i} + a_0 r^{\tau-m} v|v|^{m-2}\eta(x) \\
& - \mu_0 r^{\tau}\zeta^m |\nabla v|^m |v|^{-1}\eta(x)\cdot \mathrm{sign} v)dx + \int\limits_{\Gamma_d} \alpha(x)\gamma(\omega) r^{\tau-m+1} v|v|^{m-2}\eta(x)ds \\
& + \int\limits_{\Omega_d} r^{\tau}\zeta^{m-1}|\nabla v|^{m-2} v_{x_i}\eta(x)\cos(r,x_i)d\Omega_d \\
& - \int\limits_{\Gamma_d}\alpha(x)g(x)\eta(x)ds - \int\limits_{G_d} f(x)\eta(x)dx
\end{aligned}
\tag{9}
$$

for $v(x) \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d)$ and for all non-negative $\eta \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d)$ under the following assumptions: $f(x) \in L_1(G_d), g(x) \in L_1(\Gamma_d)$.

Proposition 2.1. *Let $d \gg 1$, functions $u, w \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d)$ and satisfy the inequality $Q(v,\eta) \le Q(w,\eta)$ for all non-negative $\eta(x) \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d)$ and also the inequality $v(x) \le w(x)$ holds on Ω_d. Then $v(x) \le w(x)$ in G_d.*

Proof. Let us define $z = v - w$ and $v^t = tv + (1-t)w,\ t \in [0,1]$. Then we have

$$
\begin{aligned}
& Q(v,\eta) - Q(w,\eta) \\
& = \int\limits_{G_d}\Bigg\{ \zeta^{m-1} r^{\tau}\eta_{x_i} z_{x_j} \int\limits_0^1 \frac{\partial\left(|\nabla v^t|^{m-2} v^t_{x_i}\right)}{\partial v^t_{x_j}} dt + a_0 r^{\tau-m}\eta z(x) \int\limits_0^1 \frac{\partial\left(v^t|v^t|^{m-2}\right)}{\partial v^t} dt \\
& - \mu_0 r^{\tau}\zeta^m \eta(x) z_{x_i} \int\limits_0^1 \frac{\partial\left(|\nabla v^t|^m |v^t|^{-1}\mathrm{sign} v^t\right)}{\partial v^t_{x_i}} dt \\
& - \mu_0 r^{\tau}\zeta^m \eta(x) z(x) \int\limits_0^1 \frac{\partial\left(|\nabla v^t|^m |v^t|^{-1}\mathrm{sign} v^t\right)}{\partial v^t} dt \Bigg\} dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int\limits_{\Gamma_d} \alpha(x)\gamma(\omega) r^{\tau-m+1}\left(\int\limits_0^1 \frac{\partial(v^t|v^t|^{m-2})}{\partial v^t}dt\right) z(x)\eta(x)ds \\
&+\zeta^{m-1}\int\limits_{\Omega_d} r^{\tau}\left(\int\limits_0^1 \frac{\partial\left(|\nabla v^t|^{m-2}v^t_{x_i}\right)}{\partial v^t_{x_j}}dt\right)\cos(r,x_i)\cdot z_{x_j}\eta(x)d\Omega_d.
\end{aligned}
\tag{10}
$$

In fact, for example

$$
\begin{aligned}
|\nabla v|^{m-2}v_{x_i}-|\nabla w|^{m-2}w_{x_i} = |\nabla v^t|^{m-2}v^t_{x_i}\Big|_0^1 &= \int\limits_0^1 \frac{\partial(|\nabla v^t|^{m-2}v^t_{x_i})}{\partial v^t_{x_j}}\cdot\frac{\partial v^t_{x_j}}{\partial t}dt \\
&= z_{x_j}\int\limits_0^1 \frac{\partial(|\nabla v^t|^{m-2}v^t_{x_i})}{\partial v^t_{x_j}}dt.
\end{aligned}
$$

Now we define following sets

$$
\begin{aligned}
(G_d)^+ &:= \{x\in G_d \mid v(x)>w(x)\}\subset G_d, \\
(\Gamma_d)^+ &:= \{x\in \Gamma_d \mid v(x)>w(x)\}\subset \Gamma_d
\end{aligned}
$$

and assume that $(G_d)^+\neq\emptyset$. Let $k\geq 1$ be any odd number. As a test function in (10) we choose $\eta=\max\{(v-w)^k,0\}$. In virtue of the fact that

$$
\int\limits_0^1 \frac{\partial(v^t|v^t|^{m-2})}{\partial v^t}dt=(m-1)\int\limits_0^1 |v^t|^{m-2}dt>0
$$

and because $\eta\Big|_{\Omega_d}=0$, we obtain from (10) that

$$
\begin{aligned}
\int\limits_{(G_d)^+}\Bigg\{& k\zeta^{m-1}r^{\tau}z^{k-1}z_{x_i}z_{x_j}\int\limits_0^1 \frac{\partial\left(|\nabla v^t|^{m-2}v^t_{x_i}\right)}{\partial v^t_{x_j}}dt \\
&- \mu_0 r^{\tau}\zeta^m z^k z_{x_i}\int\limits_0^1 \frac{\partial\left(|\nabla v^t|^m (v^t)^{-1}\right)}{\partial v^t_{x_i}}dt \\
&- \mu_0 r^{\tau}\zeta^m z^{k+1}v^t\int\limits_0^1 \frac{\partial\left(|\nabla v^t|^m (v^t)^{-1}\right)}{\partial v^t}dt\Bigg\}dx\leq 0.
\end{aligned}
\tag{11}
$$

Now, because :

$$\frac{\partial\left(|\nabla v^t|^{m-2}v^t_{x_i}\right)}{\partial v^t_{x_j}} z_{x_i}z_{x_j} = \left(|\nabla v^t|^{m-2}\delta_i^j + (m-2)|\nabla v^t|^{m-4}v^t_{x_i}v^t_{x_j}\right) z_{x_i}z_{x_j}$$
$$\geq \gamma_m |\nabla v^t|^{m-2}|\nabla z|^2, \quad \text{where } \gamma_m = \begin{cases} 1, & m \geq 2; \\ m-1, & 1 < m \leq 2; \end{cases}$$
$$\frac{\partial\left(|\nabla v^t|^m (v^t)^{-1}\right)}{\partial v^t} = -\frac{|\nabla v^t|^m}{(v^t)^2}; \ \frac{\partial\left(|\nabla v^t|^m (v^t)^{-1}\right)}{\partial v^t_{x_i}} = m|\nabla v^t|^{m-2}v^t_{x_j}(v^t)^{-1},$$

then (11) yields

$$\int\limits_{(G_d)^+} r^\tau \left\{ k\gamma_m \zeta^{m-1} z^{k-1}\left(\int\limits_0^1 |\nabla v^t|^{m-2}dt\right)|\nabla z|^2 \right.$$
$$\left. + \mu_0 \zeta^m z^{k+1}\left(\int\limits_0^1 |v^t|^{-2}|\nabla v^t|^m dt\right)\right\}dx$$
$$\leq \int\limits_{(G_d)^+} m\mu_0 r^\tau \zeta^m z^k \left(\int\limits_0^1 |v^t|^{-1}|\nabla v^t|^{m-1}dt\right)|\nabla z| dx. \quad (12)$$

On account of the Cauchy inequality with $\varepsilon = 2$

$$mz^k|\nabla z||v^t|^{-1}|\nabla v^t|^{m-1} = \left(|v^t|^{-1}z^{\frac{k+1}{2}}|\nabla v^t|^{\frac{m}{2}}\right)\cdot\left(mz^{\frac{k-1}{2}}|\nabla z||\nabla v^t|^{\frac{m}{2}-1}\right)$$
$$\leq |v^t|^{-2}z^{k+1}|\nabla v^t|^m + \frac{1}{4}m^2 z^{k-1}|\nabla z|^2|\nabla v^t|^{m-2}$$

we obtain from (12) that

$$\left(k\gamma_m - \frac{1}{4}m^2\mu_0\zeta\right)\int\limits_{(G_d)^+} r^\tau z^{k-1}|\nabla z|^2\left(\int\limits_0^1 |\nabla v^t|^{m-2}dt\right)dx \leq 0.$$

Thus, choosing the odd number $k \geq \max\left(1, \frac{m^2\mu_0\zeta}{2\gamma_m}\right)$, in view of $z(x) \equiv 0$ on $\partial(G_d)^+$, we get $z(x) \equiv 0$ in $(G_d)^+$. We have arrived to a contradiction to our definition of the set $(G_d)^+$. By this fact, the proposition is proved. □

3 The boundedness of weak solutions

In this section we consider one of the possible cases of deriving $L_\infty(G)$ *a priori* estimate of the weak solution to problem (BVP).

We first observe that there exists $R^* > 1$ such that $|u(x)| < 1$ for all $x \in G_{R^*}$. We denote $G^* \equiv G \setminus G_{R^*}$ and introduce the set $A(k) = \{x \in \overline{G^*},\ \ |u(x)| > k\}$. We shall use the notation: $(|u|-k)_+ := \max(|u|-k, 0)$.

Theorem 3.1. *Let $u(x)$ be a weak solution of (BVP) and assumption* (1) *hold. Then there exists the constant $M_0 > 0$ depending only on $\|f\|_{L_p(G)}$, $\|g\|_{L_\alpha(\partial G)}, \|r^{m-\tau}\|_{L_t(G^*)}, \|r^{-\tau}\|_{L_t(G^*)}, n, m, t,$ meas G^*, meas ∂G^*, such that $\|u\|_{L_\infty(G)} \leq M_0$.*

By setting $\eta((|u|-k)_+)\chi_{A(k)} \cdot \mathrm{sign} u$ as test function in *(II)*, where η is defined by Lemma 1.3, $\chi_{A(k)}$ is the characteristic function of the set $A(k)$ and $k \geq k_0$ (without loss of generality we can assume $k_0 \geq 1$) we get:

$$\begin{aligned}
&\int\limits_{A(k)} \{r^\tau |u|^q |\nabla u|^m \eta'(|u|-k) + a_0 r^{\tau-m} |u|^{q+m-1} \eta(|u|-k)\} dx \\
&\quad + \int\limits_{\partial G \cap A(k)} \alpha(x)\gamma(\omega) r^{\tau-m+1} |u|^{q+m-1} \eta(|u|-k) ds \\
&\quad \leq \int\limits_{A(k)} \big\{\mu_0 r^\tau |u|^{q-1} |\nabla u|^m \eta(|u|-k) + |f(x)|\eta(|u|-k)\big\} dx \\
&\quad + \int\limits_{\partial G \cap A(k)} \alpha(x)|g(x)|\eta(|u|-k) ds.
\end{aligned} \tag{13}$$

Now we define the function $w_k(x) := \eta\left(\frac{|u|-k}{m}\right)$. By (4), from Lemma 1.3, we have that

$$\begin{aligned}
\int\limits_{\partial G \cap A(k)} \alpha(x)|g(x)|\eta(|u|-k) ds \leq M &\int\limits_{\partial G \cap A(k+d)} \alpha(x)|g(x)||w_k|^m ds \\
&+ \mathrm{e}^{\varkappa d} \int\limits_{\partial G \cap \{A(k) \setminus A(k+d)\}} \alpha(x)|g(x)| ds \quad \forall d > 0.
\end{aligned} \tag{14}$$

Along similiar lines we can estimate the integral $\int\limits_{A(k)} |f(x)|\eta(|u|-k) dx$.

Now we apply Lemma 1.5. In virtue of the Hölder inequality and (7) we get

$$\int\limits_{\partial G\cap A(k+d)} |g(x)||w_k|^m ds \le \left(\int\limits_{\partial G\cap A(k+d)} |w_k|^{\alpha^*} ds\right)^{\frac{m}{\alpha^*}} \|\alpha(x)g(x)\|_{L_{\frac{n-1}{m-1-\frac{n}{t}}}(\partial G)}$$
$$\le c_4\|\alpha(x)g(x)\|_{L_{\frac{n-1}{m-1-\frac{n}{t}}}(\partial G)} \int\limits_{A(k)} \left(r^\tau|\nabla w_k|^m + r^{\tau-m}|w_k|^m\right) dx.$$

Then, because $|u|^{q-1} \le |u|^q \cdot k_0^{-1}$, from (13) and (14) it follows that

$$\int\limits_{A(k)} r^\tau|u|^q|\nabla u|^m \left(\eta'(|u|-k) - \mu_0 k_0^{-1}\eta(|u|-k)\right) dx$$
$$\le Mc_4\|\alpha(x)g(x)\|_{L_{\frac{n-1}{m-1-\frac{n}{t}}}(\partial G)} \cdot \int\limits_{A(k)} \left(r^\tau|\nabla w_k|^m + r^{\tau-m}|w_k|^m\right) dx$$
$$+ c_5 M \int\limits_{A(k+d)} |f||w_k|^m dx + c_6 \mathrm{e}^{\varkappa d} \left\{ \int\limits_{\{A(k)\setminus A(k+d)\}} |f| dx + \int\limits_{\partial G\cap A(k)} |g| ds \right\}. \tag{15}$$

By the definition of $\eta(x)$ and $w_k(x)$ we get $e^{\varkappa(|u|-k)_+}|\nabla u|^m = \left(\frac{m}{\varkappa}\right)^m |\nabla w_k|^m$ for $\varkappa > 0$. Therefore, by the choice of $\varkappa > m + \frac{2\mu_0}{k_0}$ according to Lemma 1.3, using (3), from (15) we obtain

$$\frac{1}{2}k_0^q\left(\frac{m}{\varkappa}\right)^m \int\limits_{A(k)} r^\tau|\nabla w_k|^m dx$$
$$\le c_4 M\|\alpha(x)g\|_{L_{\frac{n-1}{m-1-\frac{n}{t}}}(\Gamma)} \int\limits_{A(k)} \left(r^\tau|\nabla w_k|^m + r^{\tau-m}|w_k|^m\right) dx$$
$$+ c_5 M \int\limits_{A(k+d)} |f||w_k|^m dx + c_6 \mathrm{e}^{\varkappa d} \left\{ \int\limits_{\{A(k)\setminus A(k+d)\}} |f| dx + \int\limits_{\partial G\cap A(k)} |g| ds \right\}.$$

Now, by (7), the inequality above gives

$$(k_0^q - c_7) \int\limits_{A(k)} r^\tau |\nabla w_k|^m dx$$

$$\leq c_8 \int\limits_{A(k+d)} |f(x)||w_k|^m dx + c_9 \left\{ \int\limits_{A(k)} |f(x)|dx + \int\limits_{\partial G \cap A(k)} |g(x)|ds \right\}, \quad (16)$$

where $c_7 = 2\left(\frac{\varkappa}{m}\right)^m (1+c_1)Mc_4 \|\alpha(x)g(x)\|_{L_{\frac{n-1}{m-1-\frac{n}{t}}}(\partial G)}$, $c_8 = 2\left(\frac{\varkappa}{m}\right)^m Mc_5$, $c_9 = 2\left(\frac{\varkappa}{m}\right)^m c_6 \mathrm{e}^{\ \varkappa d}$.

Using the Hölder inequality with exponents p and p' for the first integral on the right hand side, we obtain:

$$\int\limits_{A(k+d)} |f(x)||w_k|^m dx \leq \|f(x)\|_{L_p(G)} \left(\int\limits_{A(k)} |w_k|^{mp'} dx \right)^{\frac{1}{p'}}. \quad (17)$$

From the inequality $\frac{1}{p} < \frac{m}{n} - \frac{1}{t}$ it follows that $mp' < m^\#$, where $m^\#$ is defined by (6). Let j be a real number such that $mp' < j < m^\#$. In virtue of the interpolation inequality we get

$$\left(\int\limits_{A(k)} |w_k|^{mp'} dx \right)^{\frac{1}{p'}} \leq \left(\int\limits_{A(k)} |w_k|^m dx \right)^\theta \cdot \left(\int\limits_{A(k)} |w_k|^j dx \right)^{\frac{(1-\theta)m}{j}}$$

with $\theta \in (0,1)$, which is defined by the equality $\frac{1}{mp'} = \frac{\theta}{m} + \frac{1-\theta}{j}$. On strength of the Hölder inequality with exponents $\frac{m^\#}{j}$ and $\frac{m^\#}{m^\# - j}$, from (17) we have :

$$\int\limits_{A(k+d)} |f(x)||w_k|^m dx \leq c_{10} \left(\int\limits_{A(k)} |w_k|^m dx \right)^\theta \cdot \left(\int\limits_{A(k)} |w_k|^{m^\#} dx \right)^{\frac{(1-\theta)m}{m^\#}},$$

where $c_{10} = \|f(x)\|_{L_p(G)} (measA(k))^{\frac{m(m^\# - j)(1-\theta)}{jm^\#}}$. Using the Young inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, we obtain

$$\int\limits_{A(k+d)} |f||w_k|^m dx \leq \frac{c_{11}}{\varepsilon^{\frac{1}{\theta}}} \int\limits_{A(k)} |w_k|^m dx + \varepsilon^{\frac{1}{1-\theta}} (1-\theta) \left(\int\limits_{A(k)} |w_k|^{m^\#} dx \right)^{\frac{m}{m^\#}} \quad (18)$$

for all $\varepsilon > 0$, where $c_{11} = \theta \|f(x)\|^{\frac{1}{\theta}}_{L_p(G)} (measA(k))^{\frac{m(m^{\#}-j)(1-\theta)}{jm^{\#}}}$.
Now (18) and (16) yield:

$$(k_0^q - c_7) \int\limits_{A(k)} r^\tau |\nabla w_k|^m dx$$

$$\leq c_{12}\varepsilon^{-\frac{1}{\theta}} \int\limits_{A(k)} |w_k|^m dx + c_9 \left\{ \int\limits_{A(k)} |f(x)| dx + \int\limits_{\partial G \cap A(k)} |g(x)| ds \right\}$$

$$+ c_{13}\varepsilon^{\frac{1}{1-\theta}} \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}}, \ \forall \varepsilon > 0.$$

Further, by (7), we obtain:

$$\int\limits_{A(k)} |w_k|^m dx = \int\limits_{A(k)} r^{\tau-m} |w_k|^m r^{m-\tau} dx \leq c_1 (\mathrm{diam} A(k)) \int\limits_{A(k)} r^\tau |\nabla w_k|^m dx.$$

By the two inequalities above, we have

$$(k_0^q - c_7 - c_{14}\varepsilon^{-\frac{1}{\theta}}) \int\limits_{A(k)} r^\tau |\nabla w_k|^m dx \leq c_9 \left\{ \int\limits_{A(k)} |f(x)| dx + \int\limits_{\partial G \cap A(k)} |g(x)| ds \right\}$$

$$+ c_{13}\varepsilon^{\frac{1}{1-\theta}} \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}}, \ \forall \varepsilon > 0, \ \forall k \geq k_0.$$

Let us choose now

$$\begin{cases} c_{14}\varepsilon^{-\frac{1}{\theta}} = \frac{1}{2} k_0^q \Rightarrow \varepsilon = (2c_{14})^\theta k_0^{-q\theta}, \\ k_0^q \geq 4c_7. \end{cases} \tag{19}$$

Then, by virtue of inequality (8), we have

$$\left(\frac{k_0^q}{4c_2} - c_{13}\varepsilon^{\frac{1}{1-\theta}}\right) \cdot \left\{ \left(\int\limits_{A(k)} |w_k|^{m^\#} dx \right)^{\frac{m}{m^\#}} + \left(\int\limits_{\partial G \cap A(k)} |w_k|^{\alpha^*} ds \right)^{\frac{m}{\alpha^*}} \right\}$$
$$\leq c_9 \left\{ \int\limits_{A(k)} |f(x)| dx + \int\limits_{\partial G \cap A(k)} |g(x)| ds \right\}, \ \forall k \geq k_0. \quad (20)$$

Now, if $\frac{k_0^q}{8c_2} \geq c_{13}\varepsilon^{\frac{1}{1-\theta}}$, so by (19) we choose

$$k_0 \geq \max\left\{1; (8c_2c_{13})^{\frac{1-\theta}{q}} (2c_{14})^{\frac{\theta}{q}}; (4c_7)^{\frac{1}{q}}\right\},$$

then from (20) we obtain

$$\left(\int\limits_{A(k)} |w_k|^{m^\#} dx \right)^{\frac{m}{m^\#}} + \left(\int\limits_{\partial G \cap A(k)} |w_k|^{\alpha^*} ds \right)^{\frac{m}{\alpha^*}}$$
$$\leq c_{15} \left\{ \int\limits_{A(k)} |f(x)| dx + \int\limits_{\partial G \cap A(k)} |g(x)| ds \right\}. \quad (21)$$

Let now $l > k > k_0$. By inequality (5) and because $|w_k| \geq \frac{1}{m}(|u| - k)_+$, we have $\int\limits_{A(l)} |w_k|^{m^\#} dx \geq \left(\frac{l-k}{m}\right)^{m^\#} \text{meas} A(l)$, $\int\limits_{\partial G \cap A(l)} |w_k|^{\alpha^*} ds \geq \left(\frac{l-k}{m}\right)^{\alpha^*}$ $\cdot \, \text{meas}\,(\partial G \cap A(l))$. We also observe that

$$\int\limits_{A(k)} |f(x)| dx \leq \|f\|_{L_p(G)} \text{meas}^{1-\frac{1}{p}} A(k);$$
$$\int\limits_{\partial G \cap A(k)} |g(x)| ds \leq \|g\|_{L_\alpha(\partial G)} \text{meas}^{\frac{1}{\alpha'}} (\partial G \cap A(k)), \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

From (21), by the Jensen inequality, we get

$$\begin{aligned}
&\text{meas}A(l) + \text{meas}^{\frac{m^{\#}}{\alpha^*}}(\partial G \cap A(l)) \\
&\quad \le \left(\frac{m}{l-k}\right)^{m^{\#}} \left\{ \int\limits_{A(k)} |w_k|^{m^{\#}} dx + \left(\int\limits_{\partial G \cap A(k)} |w_k|^{\alpha^*} ds \right)^{\frac{m}{\alpha^*}} \right\} \\
&\quad \le 2^{\frac{m^{\#}}{m}-1} \left(\frac{m}{l-k}\right)^{m^{\#}} c_{15}^{\frac{m^{\#}}{m}} \left(\|f\|_{L_p(G)} + \|g\|_{L_\alpha(\partial G)} \right)^{\frac{m^{\#}}{m}} \\
&\qquad \cdot \left(\text{meas}^{\frac{m^{\#}}{m}\left(1-\frac{1}{p}\right)} A(k) + \text{meas}^{\frac{m^{\#}}{m\alpha'}}(\partial G \cap A(k)) \right).
\end{aligned}$$

Setting $\psi(t) = \text{meas}A(t) + \text{meas}^{\frac{m^{\#}}{\alpha^*}}(\partial G \cap A(t))$ from the above inequality it follows that

$$\psi(l) \le c_{16} \left(\frac{m}{l-k}\right)^{m^{\#}} \left\{ [\psi(k)]^{\frac{m^{\#}}{m}\left(1-\frac{1}{p}\right)} + [\psi(k)]^{\frac{\alpha^*}{m\alpha'}} \right\}. \tag{22}$$

Notice that from assumption (1), definition of $m^{\#}$ and α^*, we conclude that $\gamma = \min\left\{ \frac{m^{\#}}{m}\left(1-\frac{1}{p}\right); \frac{\alpha^*}{m\alpha'} \right\} > 1$. In this way (22) yields

$$\psi(l) \le \frac{c_{17}}{(l-k)^{m^{\#}}} \psi^{\gamma}(k) \quad \forall l > k \ge k_0$$

and therefore we have, because of the Stampacchia Lemma, that $\psi(k_0+\delta) = 0$ with δ depending only on quantities in the formulation of Theorem 3.1. This fact means that $|u(x)| < k_0 + \delta$ for almost all $x \in G^*$. By the definition of R^* it is obvious that $|u(x)| < k_0 + \delta$ for almost all $x \in G$. Theorem 3.1 is proved.

4 Construction of the barrier function

In this section, for an $n-$ dimensional unbounded domain we consider the following elliptic problem for the model equation:

$$\begin{cases}
-\frac{d}{dx_i}\left(r^{\tau}|w|^q|\nabla w|^{m-2}w_{x_i}\right) + a_0 r^{\tau-m} w|w|^{q+m-2} \\
\qquad\qquad -\mu r^{\tau} w|w|^{q-2}|\nabla w|^m = 0, \quad x \in G_d; \\
\alpha(x)|w|^q|\nabla w|^{m-2}\frac{\partial w}{\partial \vec{n}} + \gamma r^{1-m} w|w|^{q+m-2} = 0, \quad x \in \partial G_d; \\
\lim\limits_{|x|\to\infty} w(x) = 0,
\end{cases} \tag{MVP}$$

where $\gamma = \begin{cases} \gamma_+ < \gamma(\frac{\omega_0}{2}), & x \in \Gamma_d^+, \\ \gamma_- < \gamma(-\frac{\omega_0}{2}), & x \in \Gamma_d^-. \end{cases}$ $\quad a_0 \geq 0,\ \mu > \mu_0,\ q \geq 0,\ m > 1,\ \tau \leq m-2.$

We shall seek a solution of the problem (MVP) in the form

$$w(x) = r^{\lambda_-}\Theta(\omega), \quad \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right], \quad \lambda_- < 0.$$

Substituting $w(x)$ in (MVP) and calculating in the cylindrical coordinates, we get the following Sturm-Liouville boundary problem for the function $\Theta(\omega)$:

$$\begin{cases} \frac{d}{d\omega}\left[\left(\lambda_-^2\Theta^2 + \Theta'^2\right)^{\frac{m-2}{2}} |\Theta|^q \Theta'\right] \\ +\lambda_-[\lambda_-(q+m-1) - m + 2 + \tau]\Theta|\Theta|^q \left(\lambda_-^2\Theta^2 + \Theta'^2\right)^{\frac{m-2}{2}} \\ = a_0\Theta|\Theta|^{q+m-2} - \mu\Theta|\Theta|^{q-2}\left(\lambda_-^2\Theta^2 + \Theta'^2\right)^{\frac{m}{2}}, \quad \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right), \\ \\ \pm\alpha(x)|\Theta|^q\left(\lambda_-^2\Theta^2 + \Theta'^2\right)^{\frac{m-2}{2}}\Theta'\Big|_{\omega=\pm\frac{\omega_0}{2}} + \gamma_\pm\Theta|\Theta|^{q+m-2}\Big|_{\omega=\pm\frac{\omega_0}{2}} = 0. \end{cases}$$

Performing the function change $w = z|z|^{\zeta-1}$ with $\zeta = \frac{m-1}{q+m-1}$ we get the problem for *the barrier function* z:

$$\begin{cases} -\frac{d}{dx_i}\left(r^\tau|\nabla z|^{m-2}z_{x_i}\right) + \overline{a}_0 r^{\tau-m}z|z|^{m-2} \\ \qquad\qquad -\overline{\mu}r^\tau z^{-1}|\nabla z|^m = 0, \qquad x \in G_d; \\ \\ \alpha(x)\zeta^{m-1}|\nabla z|^{m-2}\frac{\partial z}{\partial \vec{n}} + \gamma r^{1-m}z|z|^{m-2} = 0, \quad x \in \partial G_d; \\ \\ \lim\limits_{|x|\to\infty} z(x) = 0, \end{cases} \qquad (\overline{MVP})$$

where $\overline{a}_0 = a_0\zeta^{1-m}, \overline{\mu} = \mu\zeta$.

For the solution z of the problem $(\overline{MVP})$ we have the representation

$$z(x) = r^{\overline{\lambda}_-}\psi(\omega),\ \psi(\omega) = \Theta(\omega)|\Theta(\omega)|^{\frac{1-\zeta}{\zeta}},\ \ \overline{\lambda}_- = \frac{\lambda_-}{\zeta},\ \frac{\psi'(\omega)}{\psi(\omega)} = \frac{\Theta'(\omega)}{\zeta\Theta(\omega)}. \quad (23)$$

Substituting the function (23) in $(\overline{MVP})$ and calculating in the polar coordinates, we get the following Sturm-Liouville boundary value problem for the function $\psi(\omega)$:

$$\begin{cases} \frac{d}{d\omega}\left[\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-2}{2}}\psi'\right] \\ +\overline{\lambda}_-[\overline{\lambda}_-(m-1)-m+2+\tau]\psi\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-2}{2}} \\ =\overline{a}_0\psi|\psi|^{m-2}-\overline{\mu}\psi^{-1}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m}{2}}, \quad \omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right), \\ \\ \pm\alpha(x)\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-2}{2}}\psi'\Big|_{\omega=\pm\frac{\omega_0}{2}}+\gamma_\pm\psi|\psi|^{m-2}\Big|_{\omega=\pm\frac{\omega_0}{2}}=0. \end{cases} \tag{$\overline{StL}$}$$

The problem is a nonlinear eigenvalue problem. Now we want to study properties of eigenfunctions $\psi(\omega)$.

Lemma 4.1. *Let $\lambda_-<0$ be an eigenvalue and $\psi(\omega)$ be the associated eigenfunction of $(\overline{StL})$. Suppose that following inequalities are satisfied*

$$|\lambda_-|\geq\left(\frac{a_0}{\mu+\frac{m-2}{4\zeta}}\right)^{\frac{1}{m}};\ \mu>\frac{2-m}{4\zeta};\ \zeta=\frac{m-1}{q+m-1},\ \textit{if } 1<m<2; \tag{24}$$

$$\begin{cases} |\lambda_-|>\left(\frac{a_0}{\mu}\right)^{\frac{1}{m}}\ \textit{for } \mu>0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } m\geq 2 \\ (q+m-1)|\lambda_-|^m+(m-2-\tau)|\lambda_-|^{m-1}\geq a_0\ \textit{for } \mu=0. \end{cases} \tag{25}$$

Then $\psi\psi''\leq 0$.

Proof. We rewrite the $(\overline{StL})$ equation in the form

$$\begin{aligned} &-\psi\psi''\left\{\overline{\lambda}_-^2\psi^2+(m-1)\psi'^2\right\}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-4}{2}} \\ &=\overline{\mu}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m}{2}}-\overline{a}_0|\psi|^m+(m-2)\overline{\lambda}_-^2\psi^2\psi'^2\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-4}{2}} \\ &+\overline{\lambda}_-[\overline{\lambda}_-(m-1)+2+\tau-m]\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m-2}{2}}\psi^2\equiv j(\psi). \end{aligned} \tag{26}$$

At first we consider the case when $1<m<2$. By the Cauchy inequality

$$-\overline{\lambda}_-|\psi||\psi'|\leq\frac{1}{2}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)\Rightarrow(m-2)\overline{\lambda}_-^2\psi^2\psi'^2\geq\frac{m-2}{4}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^2.$$

Simple observation gives

$$\overline{a}_0|\psi|^m=\frac{\overline{a}_0}{|\overline{\lambda}_-|^m}\left(\overline{\lambda}_-^2\psi^2\right)^{\frac{m}{2}}\leq\frac{\overline{a}_0}{|\overline{\lambda}_-|^m}\left(\overline{\lambda}_-^2\psi^2+\psi'^2\right)^{\frac{m}{2}}. \tag{27}$$

Hence $j(\psi) \geq \left(\overline{\mu} + \frac{m-2}{4} - \frac{\overline{a}_0}{|\overline{\lambda}_-|^m}\right)\left(\overline{\lambda}_-^2\psi^2 + \psi'^2\right)^{\frac{m}{2}}$, in virtue of $2+\tau-m \leq 0$. If (24) holds, then the required statement follows.

Let us consider the case when $m \geq 2$. If $\mu > 0$, then from (26)-(27) we obtain

$$\begin{aligned} &-\psi\psi''\left\{\overline{\lambda}_-^2\psi^2 + (m-1)\psi'^2\right\} \\ &\geq \left(\overline{\mu} - \frac{\overline{a}_0}{|\overline{\lambda}_-|^m}\right)\left(\overline{\lambda}_-^2\psi^2 + \psi'^2\right)^2 + (m-2)\overline{\lambda}_-^2\psi^2\psi'^2 \\ &+ \overline{\lambda}_-[\overline{\lambda}_-(m-1) + 2 + \tau - m]\left(\overline{\lambda}_-^2\psi^2 + \psi'^2\right)\psi^2 \geq 0 \end{aligned}$$

by (25). Finally, if $\mu = 0$, then

$$j(\psi) \geq \left\{(m-1)|\overline{\lambda}_-|^m + (m-2-\tau)|\overline{\lambda}_-|^{m-1} - \overline{a}_0\right\}|\psi|^m \geq 0$$

by (25). □

Now, note that solutions of $(\overline{StL})$ are determined uniquely up to a scalar multiple. Therefore we can consider the solution $\psi(\omega)$ normed by the condition

$$\psi\left(-\frac{\omega_0}{2}\right) = 1. \tag{28}$$

Lemma 4.2. *Let $\psi(\omega)$ be an eigenfunction of the Sturm-Liouville boundary problem $(\overline{StL})$. Suppose, in addition, that $m \geq 2$ and the inequality*

$$|\lambda_-|^m(q + m - 1 + \mu) + |\lambda_-|^{m-1}(m - \tau - 2) > a_0 \tag{29}$$

holds. Then $\psi(\omega) > 0$ in $\left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$ and $\psi''(\omega) < 0$ in $\left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right)$.

Proof. We rewrite the $(\overline{StL})$ equation in the form (26). By setting $\overline{y}(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$, we arrive at the problem for $\overline{y}(\omega), \overline{\lambda}_-$:

$$\begin{cases} \left\{(m-1)\overline{y}^2 + \overline{\lambda}_-^2\right\}\left(\overline{\lambda}_-^2 + \overline{y}^2\right)^{\frac{m-4}{2}}\overline{y}' + (m-1+\overline{\mu})\left(\overline{\lambda}_-^2 + \overline{y}^2\right)^{\frac{m}{2}} \\ + \overline{\lambda}_-(2+\tau-m)\left(\overline{\lambda}_-^2 + \overline{y}^2\right)^{\frac{m-2}{2}} = \overline{a}_0, \quad \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right); \\ \\ \alpha(x)\left(\overline{\lambda}_-^2 + \overline{y}^2\right)^{\frac{m-2}{2}}\overline{y}(\omega)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \mp\gamma_\pm. \end{cases} \tag{30}$$

From the equation of (30) we get:

$$
\begin{aligned}
&-\left\{(m-1)\overline{y}^2+\overline{\lambda}_-^2\right\}\left(\overline{\lambda}_-^2+\overline{y}^2\right)^{\frac{m-4}{2}}\overline{y}' \\
&\geq \left(\overline{\lambda}_-^2+\overline{y}^2\right)^{\frac{m-2}{2}}\left[\overline{\lambda}_-^2(m-1+\overline{\mu})+\overline{\lambda}_-(2+\tau-m)\right]-\overline{a}_0 \\
&\geq |\overline{\lambda}_-|^m(m-1+\overline{\mu})+|\overline{\lambda}_-|^{m-1}(m-\tau-2)-\overline{a}_0>0
\end{aligned}
$$

by virtue of (29). Thus, it is proved that $\overline{y}'(\omega)<0,\ \omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right)$. Therefore $\overline{y}(\omega)$ decreases on the interval $\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right)$. Therefore

$$
\overline{y}\left(-\frac{\omega_0}{2}\right)>\overline{y}(\omega)>\overline{y}\left(\frac{\omega_0}{2}\right),\quad \omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right) \tag{31}
$$

and hence it follows that $\psi(\omega)\neq 0$ for all $\omega\in\left[-\frac{\omega_0}{2},\frac{\omega_0}{2}\right]$. By (28) $\psi(\omega)>0$ in $\left[-\frac{\omega_0}{2},\frac{\omega_0}{2}\right]$. Finally, from Lemma 4.1, $\psi''(\omega)<0$ in $\left[-\frac{\omega_0}{2},\frac{\omega_0}{2}\right]$. □

Now our aim is to estimate $\psi(\omega)$.

Lemma 4.3. *Let $\psi(\omega)$ be an eigenfunction of the Sturm-Liouville boundary problem $(\overline{StL})$ and $m\geq 2$. Then*

$$
\psi(\omega)\geq \exp\left(-\omega_0\gamma_+^{\frac{1}{m-1}}\right). \tag{32}
$$

Proof. First of all, by the definition of $\overline{y}(\omega)$, we have

$$
\psi(\omega)=\exp\left(\int_{-\frac{\omega_0}{2}}^{\omega}\overline{y}(\xi)d\xi\right),\quad \omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right). \tag{33}
$$

By the boundary condition of (30), we have

$$
\begin{aligned}
\gamma_\pm=\left|\overline{y}\left(\pm\frac{\omega_0}{2}\right)\right|\cdot\left[\overline{\lambda}_-^2+\overline{y}^2\left(\pm\frac{\omega_0}{2}\right)\right]^{\frac{m-2}{2}}\geq\left|\overline{y}\left(\pm\frac{\omega_0}{2}\right)\right|^{m-1} \\
\Rightarrow\left|\overline{y}\left(\pm\frac{\omega_0}{2}\right)\right|\leq\gamma_\pm^{\frac{1}{m-1}}.
\end{aligned}
$$

Hence, again by the boundary conditions, it follows that:

$$
-\gamma_+^{\frac{1}{m-1}}\leq-\left|\overline{y}\left(\frac{\omega_0}{2}\right)\right|=\overline{y}\left(\frac{\omega_0}{2}\right)<0\quad\text{and}\quad 0<\overline{y}\left(-\frac{\omega_0}{2}\right)\leq\gamma_-^{\frac{1}{m-1}}. \tag{34}
$$

Moreover, in virtue of the decrease of $\overline{y}(\omega)$,

$$
-\gamma_+^{\frac{1}{m-1}}\leq\overline{y}(\omega)\leq\gamma_-^{\frac{1}{m-1}},\quad\omega\in\left[-\frac{\omega_0}{2},\frac{\omega_0}{2}\right]. \tag{35}
$$

From (33) - (35) the required (32) follows. □

By integrating (30) with regard to $\overline{\lambda_-} = \frac{\lambda_-}{\zeta}$, $\overline{a}_0 = a_0\zeta^{1-m}$, $\overline{\mu} = \mu\zeta$ and performing the change of the variable $\overline{y}(\omega) = \frac{y(\omega)}{\zeta}$, we find that the eigenvalue λ satisfies the system

$$\begin{cases} \int\limits_{y(\frac{\omega_0}{2})}^{y(-\frac{\omega_0}{2})} \frac{\left\{(m-1)y^2+\lambda_-^2\right\}\left(\lambda_-^2+y^2\right)^{\frac{m-4}{2}}dy}{(m-1+q+\mu)\left(\lambda_-^2+y^2\right)^{\frac{m}{2}}+\lambda_-(2+\tau-m)\left(\lambda_-^2+y^2\right)^{\frac{m-2}{2}}-a_0} = \omega_0; \\ \left(\lambda_-^2+y^2\right)^{\frac{m-2}{2}} y(\omega)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \mp\gamma_\pm, \end{cases} \tag{Λ}$$

where $a_0 \geq 0,\ \mu > 0,\ q \geq 0,\ m > 1,\ \tau \leq m-2,\ \gamma_\pm > 0$. We may integrate system (Λ) for $m = 2$ or $a_0 = 0$. In fact, by integrating of this system we obtain the following.

m = 2

In this case we have the system:

$$\begin{cases} \int\limits_{y(\frac{\omega_0}{2})}^{y(-\frac{\omega_0}{2})} \frac{dy}{(1+q+\mu)\left(\lambda_-^2+y^2\right)+\tau\lambda_--a_0} = \omega_0; \\ y(\omega)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \mp\gamma_\pm. \end{cases}$$

Let us define the value $\Upsilon_- = \sqrt{\lambda_-^2 + \frac{\tau\lambda_- - a_0}{1+q+\mu}}$. Then λ_- is the largest negative root of the system

$$\begin{cases} \frac{\gamma_-+\gamma_+}{\zeta\Upsilon_-} = \tan\{\omega_0(1+q+\mu)\Upsilon_-\}; \\ \lambda_- < \frac{-\tau-\sqrt{\tau^2+4a_0(1+q+\mu)}}{2(1+q+\mu)}. \end{cases}$$

$a_0 = 0$

Let us define the value

$$\Lambda_- = \sqrt{\lambda_-^2 + \frac{\lambda_-(2+\tau-m)}{m-1+q+\mu}}. \tag{36}$$

Then our system (Λ) takes the form

$$\begin{cases} \frac{2-m}{2-m+\tau}\left\{\arctan\frac{y(-\frac{\omega_0}{2})}{\lambda_-}-\arctan\frac{y(\frac{\omega_0}{2})}{\lambda_-}\right\} \\ +\frac{\frac{m-1}{m-1+q+\mu}-\frac{\lambda_-(2-m)}{2-m+\tau}}{\Lambda_-}\left\{\arctan\frac{y(-\frac{\omega_0}{2})}{\Lambda_-}-\arctan\frac{y(\frac{\omega_0}{2})}{\Lambda_-}\right\}=\omega_0; \\ \\ \left(\lambda_-^2+y^2\right)^{\frac{m-2}{2}}y(\omega)\Big|_{\omega=\pm\frac{\omega_0}{2}}=\mp\gamma_\pm. \end{cases}$$

Lemma 4.4. *Let $\psi(\omega)$ be an eigenfunction of the Sturm-Liouville boundary problem $(\overline{StL})$ and $m\geq 2$. Then*

$$\psi(\omega)\leq\exp\left\{\frac{q+m-1}{m-1}\cdot\omega_0\Lambda_-\tan\left[(m-1+q+\mu)\omega_0\Lambda_-\right]\right\}. \tag{37}$$

Proof. Because $m\geq 2,\ a_0\geq 0$ and (36), from system (Λ) it follows that

$$\begin{aligned}\omega_0 &\geq \int\limits_{y(\frac{\omega_0}{2})}^{y(-\frac{\omega_0}{2})}\frac{dy}{(m-1+q+\mu)\left(\lambda_-^2+y^2\right)+\lambda_-(2+\tau-m)} \\ &=\frac{1}{m-1+q+\mu}\int\limits_{y(\frac{\omega_0}{2})}^{y(-\frac{\omega_0}{2})}\frac{dy}{y^2+\Lambda_-^2} \\ &=\frac{1}{(m-1+q+\mu)\Lambda_-}\left\{\arctan\frac{y(-\frac{\omega_0}{2})}{\Lambda_-}-\arctan\frac{y(\frac{\omega_0}{2})}{\Lambda_-}\right\} \\ &\geq\frac{1}{(m-1+q+\mu)\Lambda_-}\arctan\frac{y(-\frac{\omega_0}{2})}{\Lambda_-}.\end{aligned}$$

Hence we get

$$y\left(-\frac{\omega_0}{2}\right)\leq\Lambda_-\cdot\tan[(m-1+q+\mu)\omega_0\Lambda_-]. \tag{38}$$

Now, from (31), (33) and (38) we derive the required (37). □

Corollary 4.5. *Let $\psi(\omega)$ be an eigenfunction of the Sturm-Liouville boundary problem $(\overline{StL})$ and $m\geq 2$. Then*

$$\begin{aligned}\psi_0\equiv\exp\left(-\omega_0\gamma_+^{\frac{1}{m-1}}\right)&\leq\psi(\omega) \\ &\leq\exp\left\{\frac{q+m-1}{m-1}\cdot\omega_0\Lambda_-\tan\left[(m-1+q+\mu)\omega_0\Lambda_-\right]\right\}\equiv\Psi_0.\end{aligned}$$

Lemma 4.6. *Let $\psi(\omega)$ be an eigenfunction of the Sturm-Liouville boundary problem $(\overline{StL})$ and $m \geq 2$. Then for $\omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$:*

$$-\zeta^{-1}\gamma_+^{\frac{1}{m-1}}\Psi_0 \leq \psi'(\omega) \leq \zeta^{-1}\Lambda_- \cdot \tan[(m-1+q+\mu)\omega_0\Lambda_-]\Psi_0.$$

Proof. In fact, because $\psi'(\omega) = \overline{y}(\omega)\psi(\omega),\ \psi(\omega) > 0,\ \overline{y}(\omega) = \zeta^{-1}y(\omega)$ and (31) we have

$$\overline{y}\left(\frac{\omega_0}{2}\right)\psi(\omega) \leq \psi'(\omega) \leq \zeta^{-1}y\left(-\frac{\omega_0}{2}\right)\psi(\omega),\ \zeta = \frac{m-1}{q+m-1},\ \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right].$$

Next we use inequalities (34), (38) and Corollary 4.5. Thus we obtain the required statement. □

5 Estimates of the (BVP) solution modulus

Eventually we can estimate $|u(x)|$ for the problem (BVP) near the infinity.

Theorem 5.1. *Let u be a weak solution of the problem (BVP) with $\gamma(\pm\frac{\omega_0}{2}) > \gamma_\pm > 0$. Let assumptions (1)-(2) with $m \geq 2$ be satisfied and λ_- be the largest negative number satisfying the system (Λ). Then there exist $d \gg 1$ and a constant C_0 independent of u such that*

$$|u(x)| \leq C_0 r^{\lambda_-}, \quad \forall x \in \overline{G}_d. \tag{39}$$

Proof. At first, we perform the function change (2) and will consider the function $v(x)$. For the proof we use the above constructed barrier function $z(x)$ (a solution of $(\overline{MVP})$) and apply the comparison principle to $v(x)$ and $z(x)$. We consider the operator Q,which is defined by (9).
We will show that $Q(Az,\eta) \geq 0 = Q(v,\eta)$ for all non-negative $\eta(x) \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d)$ and some $A > 0$. Integrating by parts in the first integral, from the $(\overline{MVP})$ equations, we have

$$\begin{aligned} Q(Az,\eta) = &\int\limits_{G_d} \left(A^{m-1}(\mu-\mu_0)\zeta^m r^\tau z^{-1}|\nabla z|^m - f(x)\right)\eta(x)dx \\ &+ \int\limits_{\Gamma_d} \alpha(x)\left\{A^{m-1}(\gamma(\omega)-\gamma)r^{\tau-m+1}z|z|^{m-2} - g(x)\right\}\eta(x)ds. \end{aligned} \tag{40}$$

Further, by (23) and Corollary 4.5, we obtain

$$r^{\overline{\lambda}_-}\psi_0 \leq r^{\overline{\lambda}_-}\psi(\omega) \leq r^{\overline{\lambda}_-}\Psi_0; \tag{41}$$

$$|\nabla z|^2 = z_r^2 + r^{-2}z_\omega^2 = r^{2\overline{\lambda}_- - 2}(\overline{\lambda}_-^2\psi^2 + \psi'^2) \Longrightarrow \left|\frac{\nabla z}{z}\right| \geq -\overline{\lambda}_- r^{-1}.$$

Therefore from (40) according to the above inequalities and because of our assumptions it follows that

$$\begin{aligned}
Q(Az,\eta) \geq & \int\limits_{G_d} \left(A^{m-1}(\mu-\mu_0)\zeta^m r^\tau z^{-1}|\nabla z|^m - f_1 r^{\tau-m+(m-1)\overline{\lambda}_-} \right) \eta(x)dx \\
& + \int\limits_{\Gamma_d} \left(A^{m-1}(\gamma(\omega)-\gamma) r^{\tau-m+1} z^{m-1} - g_1 r^{\tau-m+1+(m-1)\overline{\lambda}_-} \right) \eta(x)ds \\
& \geq \left(|\overline{\lambda}_-|^m (A\psi_0)^{m-1}(\mu-\mu_0)\zeta^m - f_1 \right) \int\limits_{G_d} r^{\tau-m+(m-1)\overline{\lambda}_-}\eta(x)dx \\
& + \left((A\psi_0)^{m-1}(\gamma(\omega)-\gamma) - g_1 \right) \int\limits_{\Gamma_d} r^{\tau-m+1+(m-1)\overline{\lambda}_-}\eta(x)ds.
\end{aligned}$$

Now, taking into consideration that $\mu > \mu_0,\ \gamma(\pm\frac{\omega_0}{2}) > \gamma$ and choosing $A > 0$ sufficiently large: $A \geq \frac{1}{\psi_0} \max\left\{ \left(\frac{f_1}{|\lambda_-|^m(\mu-\mu_0)} \right)^{\frac{1}{m-1}},\ \left(\frac{g_1}{\gamma(\pm\frac{\omega_0}{2})-\gamma_\pm} \right)^{\frac{1}{m-1}} \right\}$, we provide the required inequality $Q(Az,\eta) \geq 0 = Q(v,\eta)$.

Now, by the continuity of $v(x)$ and by Theorem 3.1, we have $v(x)\Big|_{\Omega_d} \leq M_0^{1/\zeta} = \max\limits_{\overline{G}} |v(x)|$. On the other hand, by virtue of (41), $Az\Big|_{\Omega_d} \geq Ad^{\overline{\lambda}_-}\psi_0 \geq M_0^{1/\zeta} \geq v\Big|_{\Omega_d}$ provided that $A \geq (d^{\overline{\lambda}_-}\psi_0)^{-1} \cdot M_0^{1/\zeta}$. Thus from above we get

$$\begin{cases} Q(Az,\eta) \geq Q(v,\eta), & \forall \eta \in C^0(\overline{G_d}) \cap \mathfrak{N}^1_{m,0,\tau}(G_d); \\ Az\Big|_{\Omega_d} \geq v\Big|_{\Omega_d}. \end{cases}$$

All conditions of the comparison principle are fulfiled, so by this we get

$$v(x) \leq Az(r,\omega), \quad \forall x \in \overline{G}_d.$$

Similarly, $v(x) \geq -Az(r,\omega)$ for all $x \in \overline{G}_d$. Thus, finally, we obtain

$$|v(x)| \leq Az(r,\omega) \leq c_0 r^{\overline{\lambda}_-}, \quad \forall x \in \overline{G}_d; \quad c_0 = A\Psi_0,$$

by (41). On returning to the old variables, by virtue of (2), we get the desired estimate (39). □

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