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AN OPERATOR EXTENSION OF ČEBYŠEV INEQUALITY

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Abstract

Some operator inequalities for synchronous functions that are related to the čebyšev inequality are given. Among other inequalities for synchronous functions it is shown that

$$\left\|\phi\left(f\left(A\right)g\left(A\right)\right)-\phi\left(f\left(A\right)\right)\phi\left(g\left(A\right)\right)\right\|\le\max\left\{\left\|\phi\left(f^{2}\left(A\right)\right)-\phi^{2}\left(f\left(A\right)\right)\right\|\right.$$
$$\left.,\left\|\phi\left(g^{2}\left(A\right)\right)-\phi^{2}\left(g\left(A\right)\right)\right\|\right\}$$

where A is a self-adjoint and compact operator on $\mathcal{B}\left(\mathscr{H}\right)$, $f,g\in C\left(sp\left(A\right)\right)$ continuous and non-negative functions and $\phi:\mathcal{B}\left(\mathscr{H}\right)\to\mathcal{B}\left(\mathscr{H}\right)$ be a n-normalized bounded positive linear map. In addition, by using the concept of quadruple D-synchronous functions which is generalizes the concept of a pair of synchronous functions, we establish an inequality similar to čebyšev inequality.

1 Introduction and Preliminaries

Let us consider the real sequences $p=\left(p_1,\ldots,p_n\right)$, $a=\left(a_1,\ldots,a_n\right)$ and $b=\left(b_1,\ldots,b_n\right)$. Then the Čebyšev functional is defined by

$$T_n\left(p;a,b\right):=P_n\sum_{i=1}^{n}p_ia_ib_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i,$$

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where $P_n := \sum_{i=1}^{n} p_i$. In 1882-1883, Čebyšev [5, 6], proved that, if a and b are monotonic in the same (opposite) sense and p is non-negative, then

$$T_n(p; a, b) \geq (\leq)\, 0. \tag{1.1}$$

The inequality (1.1) was mentioned by Hardy, Littlewood, and Pólya in their book [8] in 1934 in the more general setting of synchronous sequences, i.e. if $a,\ b$ are synchronous (asynchronous), this means that

$$(a_i - a_j)(b_i - b_j) \geq (\leq)\, 0,$$

for each $i, j \in \{1, \ldots, n\}$, then the inequality (1.1) is valid. For general, real weights, Mitrinović and Pečarić have shown in [16] that the inequality (1.1) holds true if

$$0 \leq P_k \leq P_n,$$

for each $k \in \{1, \ldots, n-1\}$, and $a,\ b$ are monotonic in the same (opposite) sense.

A related notion is synchronicity of functions. We say that the functions $f, g : [a, b] \to \mathbb{R}$ are synchronous (asynchronous) on the interval $[a, b]$ if they have satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0, \tag{1.2}$$

for each $t, s \in [a, b]$.

Dragomir [9] generalized Čebyšev inequality for convex functions on a real inner product and applied this result to show that if $p_1, \ldots, p_n$ is a sequence of non-negative numbers with $\sum_{i=1}^{n} p_i \geq 0$ and two sequences $(v_1, \ldots, v_n)$ and $(u_1, \ldots, u_n)$ in a real inner product space are synchronous, namely, $\langle v_j - v_i, u_j - u_i \rangle \geq 0$ for all $i, j = 1, \ldots, n$, then

$$\sum_{j=1}^{n} p_j \langle v_j, u_j \rangle \geq \left\langle \sum_{j=1}^{n} p_j v_j, \sum_{j=1}^{n} p_j u_j \right\rangle.$$

Recently Dragomir in [10], proved the following theorem.

Theorem 1.1. *Let A be a self-adjoint operator with $sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*

$$\langle f(A)\, g(A)\, x, x \rangle \geq \langle f(A)\, x, x \rangle \langle g(A)\, x, x \rangle, \tag{1.3}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Motivated by the above results, we provide in this paper several operator extensions of the Čebyšev inequality. Some applications for univariate functions of real variable are provided.

As is customary, we reserve M, m for scalars. Other capital letters are used to denote general elements of the C^*-algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. An operator $A \in \mathcal{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathscr{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map establishes an isometrically $*$-isomorphism Φ between the set $C(sp(A))$ of all continuous functions on the spectrum of A, denoted $sp(A)$, and the C^*-algebra generated by A and I (see for instance [18, p. 15]). For any $f, g \in C(sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(I) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(II) $\Phi(fg) = \Phi(f)\Phi(g)$;

(III) $\|\Phi(f)\| = \|f\| := \sup_{t \in sp(A)} |f(t)|$;

(IV) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in sp(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(sp(A))$ and we call it the continuous functional calculus for a self-adjoint operator A. It is well known that, if A is a self-adjoint operator and $f \in C(sp(A))$, then $f(t) \geq 0$ for any $t \in sp(A)$ implies that $f(A) \geq 0$. It is extendible for two real valued functions on $sp(A)$. A linear map ϕ is positive if $\phi(A) \geq 0$ whenever $A \geq 0$. It said to be normalized if $\phi(I) = I$. For more studies in this direction, we refer to [4].

2 Main Results

2.1 Inequalities for Synchronous Functions

First of all, we state a generalization of Theorem 1.1 for normalized positive linear map as follows:

Theorem 2.1. *Let A be a self-adjoint operator and $f, g \in C(sp(A))$ are continuous and synchronous (asynchronous) functions, and let ϕ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$, then*

$$\langle\phi(f(A)g(A))x, x\rangle \geq (\leq) \langle\phi(f(A))x, x\rangle \langle\phi(g(A))x, x\rangle, \tag{2.1}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. We give a proof only in the first case. Since f, g are synchronous functions, from (1.2) we have for any $s, t \in [a, b]$ that

$$f(t)\,g(t) + f(s)\,g(s) \geq f(t)\,g(s) + f(s)\,g(t)\,.$$

If we fix $s \in [a, b]$, and apply the functional calculus for the above inequality we get

$$f(A)\,g(A) + f(s)\,g(s)\,1_{\mathscr{H}} \geq f(A)\,g(s) + f(s)\,g(A)$$

and since ϕ is normalized positive linear map we can write

$$\phi(f(A)\,g(A)) + f(s)\,g(s)\,1_{\mathscr{H}} \geq g(s)\,\phi(f(A)) + f(s)\,\phi(f(A))$$

or

$$\langle \phi(f(A)\,g(A))\,x, x\rangle + f(s)\,g(s) \geq g(s)\,\langle \phi(f(A))\,x, x\rangle + f(s)\,\langle \phi(g(A))\,x, x\rangle\,, \tag{2.2}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Apply again functional calculus to obtain

$$\begin{aligned}&\langle \phi(f(A)\,g(A))\,x, x\rangle\,1_{\mathscr{H}} + f(A)\,g(A)\\ &\quad\geq \langle \phi(f(A))\,x, x\rangle\,g(A) + \langle \phi(g(A))\,x, x\rangle\,f(A)\,.\end{aligned}$$

Again, since ϕ is normalized positive linear map we get

$$\begin{aligned}&\langle \phi(f(A)\,g(A))\,x, x\rangle + \phi(f(A)\,g(A))\\ &\quad\geq \langle \phi(f(A))\,x, x\rangle\,\phi(g(A)) + \langle \phi(g(A))\,x, x\rangle\,\phi(f(A))\end{aligned}$$

or

$$\begin{aligned}&\langle \phi(f(A)\,g(A))\,x, x\rangle + \langle \phi(f(A)\,g(A))\,y, y\rangle\\ &\geq \langle \phi(f(A))\,x, x\rangle\,\langle \phi(g(A))\,y, y\rangle + \langle \phi(g(A))\,x, x\rangle\,\langle \phi(f(A))\,y, y\rangle\,,\end{aligned} \tag{2.3}$$

for each $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$.

Finally, on making $y = x$ in (2.3), we deduce the desired result (2.1). □

The case of norm operator may be of interest and is embodied in the following remark.

Remark 2.1. *Let A be a positive operator in $\mathcal{B}(\mathscr{H})$ and $f, g \in C(sp(A))$ asynchronous and non-negative functions, and let ϕ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$. By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we obtain*

$$\|\phi(f(A)\,g(A))\| \leq \|\phi(f(A))\|\,\|\phi(g(A))\|\,.$$

Corollary 2.1. *Let A be a self-adjoint operator and $f, g \in C(sp(A))$ be synchronous functions. If we take $\phi(A) = A$, then we have the inequality* (1.3).

The following result follows from Davis-Choi-Jensen's inequality (see for instance [7, Theorem 1.20]).

Corollary 2.2. *All as in Theorem 2.1, and f, g are non-negative and operator convex. Then by Davis-Choi-Jensen's inequality we get*

$$\begin{aligned}\langle \phi(f(A)g(A))x, x\rangle &\geq \langle \phi(f(A))x, x\rangle \langle \phi(g(A))x, x\rangle \\ &\geq f(\langle \phi(A)x, x\rangle) g(\langle \phi(A)x, x\rangle),\end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

As a special case of Corollary 2.2, we have the following Kadison's inequality:

Corollary 2.3. *If we take $f(t) = g(t) = t$, we obtain*

$$\left\langle \phi\left(A^2\right)x, x\right\rangle \geq \langle \phi(A)x, x\rangle^2,$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

The following lemma is known as the McCarty inequality.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathscr{H}), A \geq 0$ and let $x \in \mathscr{H}$ be any unit vector. Then*

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r, \ 0 < r \leq 1. \tag{2.4}$$

Corollary 2.4. *If we put $f(t) = t^p$, $g(t) = t^q$ with $p, q \geq 0$ and $\phi(A) = A$, by* (2.4) *we get*

$$\langle Ax, x\rangle^{p+q} \geq \langle A^p x, x\rangle \langle A^q x, x\rangle,$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Remark 2.2. *A few remarks are in order.*

- *It is well-known that $\phi(A) = X^*AX$ where X is an operator in $\mathcal{B}(\mathscr{H})$ with $X^*X = I$, is a normalized positive linear map. According to this fact from* (2.1), *we have the following*

$$\langle X^* f(A) g(A) Xx, x\rangle \geq \langle X^* f(A) Xx, x\rangle \langle X^* g(A) Xx, x\rangle, \tag{2.5}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

- *Let $f(t) = g(t) = t^r$ where $r \geq 0$, in (2.5), then*

$$\left\langle X^* A^{2r} X x, x \right\rangle \geq \left\langle X^* A^r X x, x \right\rangle^2, \tag{2.6}$$

  for any $x \in \mathscr{H}$ with $\|x\| = 1$.

- *Let X be a unitary and $0 < r \leq 1$ in (2.6), Then*

$$\left\langle X^* A^{2r} X x, x \right\rangle \geq \left\langle X^* A X x, x \right\rangle^{2r},$$

  for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Remark 2.3. *Let $A \in \mathscr{M}_n(\mathbb{C})$ be a Hermitian matrix. Define $\phi : \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$ where $\phi(A) = \frac{1}{n} tr(A) 1_{\mathscr{H}}$. Then from inequality (2.1), we have*

$$\frac{1}{n} tr(f(A)) \, tr(g(A)) \leq tr(f(A) g(A)).$$

Furthermore, if we choose $f(t) = g(t) = t$, we get

$$\frac{1}{n} tr^2(A) \leq tr\left(A^2\right).$$

We remark that, if $A \in \mathscr{M}_n(\mathbb{C})$ be a positive semi definite matrix then

$$\frac{1}{n} tr^2(A) \leq tr\left(A^2\right) \leq tr^2(A).$$

The following general result for two operators also holds:

Proposition 2.1. *Let A, B be a self-adjoint operators and $f, g \in C(sp(A))$ and $f, g \in C(sp(B))$ are continuous and synchronous functions, and let ϕ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$, then*

$$\begin{aligned} &\langle \phi(f(A) g(A)) x, x \rangle + \langle \phi(f(B) g(B)) y, y \rangle \\ &\quad \geq \langle \phi(f(A)) x, x \rangle \langle \phi(g(B)) y, y \rangle + \langle \phi(g(A)) x, x \rangle \langle \phi(f(B)) y, y \rangle, \end{aligned} \tag{2.7}$$

for any $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$.

Proof. Follows from proof of Theorem 2.1 by applying functional calculus for self-adjoint operator B in (2.2). However, the details are not given here. □

Remark 2.4. *We provide now some particular inequalities of interest that can be derived from Proposition 2.1.*

- *By replacing B with A^{-1} in (2.7), we get*

$$\begin{aligned} &\langle \phi\left(f\left(A\right)g\left(A\right)\right)x,x\rangle + \left\langle \phi\left(f\left(A^{-1}\right)g\left(A^{-1}\right)\right)y,y\right\rangle \\ &\le \left\langle \phi\left(g\left(A^{-1}\right)\right)y,y\right\rangle \langle \phi\left(f\left(A\right)\right)x,x\rangle + \left\langle \phi\left(f\left(A^{-1}\right)\right)y,y\right\rangle \langle \phi\left(g\left(A\right)\right)x,x\rangle, \end{aligned} \tag{2.8}$$

  for any $x,y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$.

- *By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, and $y \in \mathscr{H}$ with $\|y\| = 1$ in (2.8) respectively, we obtain*

$$\begin{aligned} &\|\phi\left(f\left(A\right)g\left(A\right)\right)\| + \left\|\phi\left(f\left(A^{-1}\right)g\left(A^{-1}\right)\right)\right\| \\ &\quad \le \left\|\phi\left(g\left(A^{-1}\right)\right)\right\| \|\phi\left(f\left(A\right)\right)\| + \left\|\phi\left(f\left(A^{-1}\right)\right)\right\| \|\phi\left(g\left(A\right)\right)\|. \end{aligned} \tag{2.9}$$

- *If we put in (2.9), $\phi\left(A\right) = A$ and $f\left(t\right) = t^p, g\left(t\right) = t^q$ where $p, q \le 0$, we get*

$$\left\|A^{p+q}\right\| + \left\|A^{-p-q}\right\| \le \|A^p\| \left\|A^{-q}\right\| + \left\|A^{-p}\right\| \|A^q\|.$$

The following multiple operator version of Theorem 2.1 holds:

Proposition 2.2. *Let $A_i \in \mathcal{B}\left(\mathscr{H}\right)$ be self-adjoint operators and ϕ_i normalized positive linear maps $(i = 1, \dots, n)$. If $f, g \in C\left(sp\left(A_i\right)\right)$ are continuous and synchronous, then*

$$\sum_{i=1}^{n} \langle \phi_i\left(f\left(A_i\right)g\left(A_i\right)\right)x_i, x_i\rangle \ge \sum_{i=1}^{n} \langle \phi_i\left(f\left(A_i\right)\right)x_i, x_i\rangle \sum_{i=1}^{n} \langle \phi_i\left(g\left(A_i\right)\right)x_i, x_i\rangle,$$

for each $x_i \in \mathscr{H}, i \in \{1, \dots, n\}$ with $\sum\limits_{i=1}^{n} \|x_i\|^2 = 1$.

Proposition 2.3. *Let $A_i \in \mathcal{B}\left(\mathscr{H}\right)$ be self-adjoint operators and ϕ_i normalized positive linear maps $(i = 1, \dots, n)$. Let $\omega_1, \dots, \omega_n \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum\limits_{i=1}^{n} \omega_i = 1$. If $f, g \in C\left(sp\left(A_i\right)\right)$ are continuous and synchronous, then*

$$\begin{aligned} &\left\langle \sum_{i=1}^{n} \omega_i \phi_i\left(f\left(A_i\right)g\left(A_i\right)\right)x, x\right\rangle \\ &\quad \ge \left\langle \sum_{i=1}^{n} \omega_i \phi_i\left(f\left(A_i\right)\right)x, x\right\rangle \left\langle \sum_{i=1}^{n} \omega_i \phi_i\left(g\left(A_i\right)\right)x, x\right\rangle, \end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

The following useful lemma can be found in [14, Lemma 1] and [2, Theorem I.1].

Lemma 2.2. *Let A, B and C be operators in $\mathcal{B}(\mathscr{H})$, where A and B are positive. Then $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathcal{B}(\mathscr{H} \oplus \mathscr{H})$ if and only if $|\langle Cx, y\rangle|^2 \le |\langle Ax, x\rangle| \, |\langle By, y\rangle|$ for all x, y in $\mathscr{H}$.*

Theorem 2.2. *Let A be a self-adjoint operator on $\mathcal{B}(\mathscr{H})$ and $f, g \in C(sp(A))$ continuous and non-negative functions, and let $\phi : \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$ be a n-normalized positive linear map. Then*

$$\begin{aligned} &|\langle \phi(g(A) f(A)) - \phi(g(A)) \phi(f(A)) x, y\rangle|^2 \\ &\quad \le \left|\left\langle \phi\left(f^2(A)\right) - \phi^2(f(A)) x, x\right\rangle\right| \left|\left\langle \phi\left(g^2(A)\right) - \phi^2(g(A)) y, y\right\rangle\right| \end{aligned} \tag{2.10}$$

for all x, y in $\mathscr{H}$.

Proof. Let $f(A)$ and $g(A)$ be two self-adjoint operators in $\mathcal{B}(\mathscr{H})$. We have

$$\begin{aligned} 0 &\le \begin{bmatrix} f(A) \\ g(A) \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} f(A) & g(A) & I & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} f^2(A) & f(A) g(A) & f(A) & 0 & \cdots & 0 \\ g(A) f(A) & g^2(A) & g(A) & 0 & \cdots & 0 \\ f(A) & g(A) & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Since ϕ is n-normalized positive linear map, we get

$$0 \le \begin{bmatrix} \phi\left(f^2(A)\right) & \phi(f(A) g(A)) & \phi(f(A)) & 0 & \cdots & 0 \\ \phi(g(A) f(A)) & \phi\left(g^2(A)\right) & \phi(g(A)) & 0 & \cdots & 0 \\ \phi(f(A)) & \phi(g(A)) & \phi(I) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

From the above argument, we have

$$0 \le \begin{bmatrix} \phi\left(f^2(A)\right) & \phi(f(A) g(A)) & \phi(f(A)) \\ \phi(g(A) f(A)) & \phi\left(g^2(A)\right) & \phi(g(A)) \\ \phi(f(A)) & \phi(g(A)) & I \end{bmatrix}.$$

It is known that the matrix $\begin{bmatrix} R & T \\ T^* & S \end{bmatrix}$ is positive if and only if R, S are positive and $R \geq TS^{-1}T^*$, where S^{-1} denoted the inverse of S. Based on this fact we have

$$\begin{bmatrix} \phi\left(f^2(A)\right) & \phi(f(A)\,g(A)) \\ \phi(g(A)\,f(A)) & \phi\left(g^2(A)\right) \end{bmatrix} \geq \begin{bmatrix} \phi(f(A)) \\ \phi(g(A)) \end{bmatrix} I^{-1} \begin{bmatrix} \phi(f(A)) & \phi(g(A)) \end{bmatrix}$$
$$= \begin{bmatrix} \phi^2(f(A)) & \phi(f(A))\,\phi(g(A)) \\ \phi(g(A))\,\phi(f(A)) & \phi^2(g(A)) \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \phi\left(f^2(A)\right) - \phi^2(f(A)) & \phi(f(A)\,g(A)) - \phi(f(A))\,\phi(g(A)) \\ \phi(g(A)\,f(A)) - \phi(g(A))\,\phi(f(A)) & \phi\left(g^2(A)\right) - \phi^2(g(A)) \end{bmatrix} \geq 0. \tag{2.11}$$

From Lemma 2.2, the inequality (2.11) implies that

$$\begin{aligned} &|\langle \phi(g(A)\,f(A)) - \phi(g(A))\,\phi(f(A))\,x, y\rangle|^2 \\ &\quad \leq \left|\left\langle \phi\left(f^2(A)\right) - \phi^2(f(A))\,x, x\right\rangle\right| \left|\left\langle \phi\left(g^2(A)\right) - \phi^2(g(A))\,y, y\right\rangle\right|, \end{aligned} \tag{2.12}$$

for all x, y in $\mathscr{H}$. □

Remark 2.5. *If $y = x$ is a unit vector in* (2.12)*, by taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ we get*

$$\begin{aligned} \|\phi(f(A)\,g(A)) - \phi(f(A))\,\phi(g(A))\| \leq \max\big\{ &\left\|\phi\left(f^2(A)\right) - \phi^2(f(A))\right\| \\ &, \left\|\phi\left(g^2(A)\right) - \phi^2(g(A))\right\|\big\} \end{aligned} \tag{2.13}$$

An application of Theorem 2.2 can be seen in the following result. This result presents a refinement of the inequality (2.13).

As a matter of fact, if in (2.11), A be a compact operator and ϕ be a n-normalized bounded positive linear map, from [1, Theorem 2.1] we have

$$\begin{aligned} s_j(\phi(g(A)\,A(A)) - \phi(g(A))\,\phi(f(A))) \leq s_j\big(&\phi\left(f^2(A)\right) - \phi^2(f(A)) \\ &\oplus \phi\left(g^2(A)\right) - \phi^2(g(A))\big), \end{aligned} \tag{2.14}$$

for $j = 1, 2, \cdots, n$.

Since every unitarily invariant norm is a monotone function of the singular values of an operator, from the inequality (2.14) we can write

$$\begin{aligned} |||\phi(g(A)\,A(A)) - \phi(g(A))\,\phi(f(A))\,||| \leq |||&\phi\left(f^2(A)\right) - \phi^2(f(A)) \\ &\oplus \phi\left(g^2(A)\right) - \phi^2(g(A))\,|||. \end{aligned} \tag{2.15}$$

One of the most famous examples of unitarily invariant norms is the usual operator norm $\|\cdot\|$. Therefore from (2.15), we have

$$\begin{aligned}&\|\phi(f(A)g(A)) - \phi(f(A))\phi(g(A))\| \\ &\le \max\left\{\left\|\phi\left(f^2(A)\right) - \phi^2(f(A))\right\|, \left\|\phi\left(g^2(A)\right) - \phi^2(g(A))\right\|\right\}.\end{aligned}$$

This result based on the following fact

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\},$$

where the direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ defined on $\mathscr{H} \oplus \mathscr{H}$.

2.2 D-Synchronous Functions

The quadruple (f, g, h, k) is called D-Synchronous
(D-Asynchronous) on I if

$$\det\begin{pmatrix} f(s) & f(t) \\ g(s) & g(t) \end{pmatrix} \det\begin{pmatrix} h(s) & h(t) \\ k(s) & k(t) \end{pmatrix} \ge (\le)\, 0,$$

for each $s, t \in I$. This concept is generalization of synchronous functions, since for $g = 1, k = 1$ the quadruple (f, g, h, k) is D-Synchronous if and only if (f, g) is synchronous on I (see [11]). We observe that

$$\begin{aligned}&\det\begin{pmatrix} f(s) & f(t) \\ g(s) & g(t) \end{pmatrix} \det\begin{pmatrix} h(s) & h(t) \\ k(s) & k(t) \end{pmatrix} \\ &= (f(s)g(t) - g(s)f(t))(h(s)k(t) - k(s)h(t)),\end{aligned}$$

for each $s, t \in I$. For D-Synchronous (D-Asynchronous) functions, the reader is referred to [11].

Theorem 2.3. *Let A be a self-adjoint operator and $f, g, h, k \in C(sp(A))$ are continuous and D-synchronous functions, and let ϕ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$, then*

$$det\begin{pmatrix} \langle\phi(f(A)h(A))x, x\rangle & \langle\phi(f(A)k(A))x, x\rangle \\ \langle\phi(g(A)h(A))x, x\rangle & \langle\phi(g(A)k(A))x, x\rangle \end{pmatrix} \ge 0. \tag{2.16}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. Since the quadruple (f, g, h, k) is D-synchronous, then

$$\begin{aligned}0 &\le (f(s)g(t) - g(s)f(t))(h(x)k(t) - k(s)h(t)) \\ &= f(s)h(s)g(t)k(t) + g(s)k(s)f(t)h(t) \\ &\quad - f(s)k(s)g(t)h(t) - g(s)h(s)f(t)k(t)\end{aligned}$$

this is equivalent to

$$\begin{aligned} &f(s)\,h(s)\,g(t)\,k(t)+g(s)\,k(s)\,f(t)\,h(t) \\ &\quad \geq f(s)\,k(s)\,g(t)\,h(t)+g(s)\,h(s)\,f(t)\,k(t)\,. \end{aligned} \tag{2.17}$$

Fix $s \in [a,b]$, and apply the functional calculus for the operator A in (2.17), we deduce

$$\begin{aligned} &f(s)\,h(s)\,g(A)\,k(A)+g(s)\,k(s)\,f(A)\,h(A) \\ &\quad \geq f(s)\,k(s)\,g(A)\,h(A)+g(s)\,h(s)\,f(A)\,k(A)\,. \end{aligned}$$

Since ϕ is normalized positive linear map we get

$$\begin{aligned} &f(s)\,h(s)\,\phi(g(A)\,k(A))+g(s)\,k(s)\,\phi(f(A)\,h(A)) \\ &\quad \geq f(s)\,k(s)\,\phi(g(A)\,h(A))+g(s)\,h(s)\,\phi(f(A)\,k(A))\,, \end{aligned}$$

which is clearly equivalent with

$$\begin{aligned} &f(s)\,h(s)\,\langle\phi(g(A)\,k(A))\,x,x\rangle+g(s)\,k(s)\,\langle\phi(f(A)\,h(A))\,x,x\rangle \\ &\quad \geq f(s)\,k(s)\,\langle\phi(g(A)\,h(A))\,x,x\rangle+g(s)\,h(s)\,\langle\phi(f(A)\,k(A))\,x,x\rangle\,, \end{aligned}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Apply again functional calculus we obtain

$$\begin{aligned} &f(A)\,h(A)\,\langle\phi(g(A)\,k(A))\,x,x\rangle+g(A)\,k(A)\,\langle\phi(f(A)\,h(A))\,x,x\rangle \\ &\quad \geq f(A)\,k(A)\,\langle\phi(g(A)\,h(A))\,x,x\rangle+g(A)\,h(A)\,\langle\phi(f(A)\,k(A))\,x,x\rangle\,. \end{aligned}$$

Again, since ϕ is normalized positive linear map we get

$$\begin{aligned} &\phi(f(A)\,h(A))\,\langle\phi(g(A)\,k(A))\,x,x\rangle+\phi(g(A)\,k(A))\,\langle\phi(f(A)\,h(A))\,x,x\rangle \\ &\quad \geq \phi(f(A)\,k(A))\,\langle\phi(g(A)\,h(A))\,x,x\rangle+\phi(g(A)\,h(A))\,\langle\phi(f(A)\,k(A))\,x,x\rangle \end{aligned}$$

or

$$\begin{aligned} &\langle\phi(f(A)\,h(A))\,y,y\rangle\,\langle\phi(g(A)\,k(A))\,x,x\rangle \\ &\qquad +\langle\phi(g(A)\,k(A))\,y,y\rangle\,\langle\phi(f(A)\,h(A))\,x,x\rangle \\ &\quad \geq \langle\phi(f(A))\,k(A)\,y,y\rangle\,\langle\phi(g(A))\,h(A)\,x,x\rangle \\ &\qquad +\langle\phi(g(A))\,h(A)\,y,y\rangle\,\langle\phi(f(A))\,k(A)\,x,x\rangle\,, \end{aligned} \tag{2.18}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Finally, on making $y = x$ in (2.18) we deduce the desired result (2.16). □

Remark 2.6. *If we take $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$, $k(t) = t^s$ whre $p,q,r,s \geq 0$ and $\phi(A) = A$ in (2.16), then*

$$\left\langle A^{p+r}x,x\right\rangle\left\langle A^{q+s}x,x\right\rangle \geq \left\langle A^{p+s}x,x\right\rangle\left\langle A^{q+r}x,x\right\rangle,$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. The proof is similar to the proof of Theorem 2.3. The details are omitted. □

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