



A novel difference schemes for analyzing the fractional Navier- Stokes equations

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Abstract

In this report, a novel difference scheme is used to analyzing the Navier - Stokes problems of fractional order. Existence and uniqueness of the suggested approach with a Lipschitz condition and Picard theorem are proved. Furthermore, we find a discrete analogue of the derivative and then stability and convergence of our strategy in multi dimensional domain are proved.

1 Introduction

The theory of fractional calculus deals with derivatives and integrals of any arbitrary orders. Fractional calculus have gained importance, mainly due to their demonstrated applications in many area of physics, economics and engineering. The fractional calculus has been occurring in many physical problems such as damping law, perfusion processes and also motion of a large thin plate in a Newtonian fluid. For more details on the scientific applications of fractional calculus, see [1, 2].

Difference schemes [3–5] (grid methods) is explained at numerical approximation of various problems in real world application. Under such an approach the solution of differential equations amounts for solving the systems of algebraic equations. Difference scheme is based on composition of discrete (difference) approximations to equations of mathematics and verifying a priori

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quality characteristics of these approximations, mainly stability, convergence, error of approximation and accuracy of the difference schemes obtained.

We were interested in approximate substitutions of difference operators for differential ones. However, many problems of physics [6, 7] involve not only differential equations, but also the supplementary conditions such as boundary and initial which guide a proper choice of a unique solution from the collection of possible solutions.

This work deals with introducing a novel difference schemes for Navier-Stokes fractional differential equations in multi dimensional domains [8, 9]. A fractional Navier-Stokes equations in cylindrical coordinates is in the following form:

$$D_{0t}^\alpha u(r, t) = p + v(D_r^2 + \frac{1}{r}D_r u), \quad 0 < \alpha \leq 1, \quad (1)$$

where u is the velocity, r is the distance along an x -axis, p is the pressure, v is the kinematics viscosity. Here the fractional derivative (D) is described in the Riemann Liouville form. As we know the fractional Navier - Stokes equations are nonlinear. Therefore there is no known methodology to solve these equations and there are very few cases in which the exact solution of the time fractional Navier - Stokes equations can be achieved.

2 Preliminaries and basic concepts

2.1 Cauchy problem for fractional ordinary differential equation

Consider the following Cauchy problem in the form of fractional order

$$\partial_{0t}^\alpha u = f(u, t), \quad u(0) = u_0, \quad (2)$$

where

$$\partial_{0t}^\alpha u = D_{0t}^\alpha u - \frac{u(0)}{\Gamma(1-\alpha)t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(\tau)d\tau}{(t-\tau)^\alpha}, \quad (3)$$

is a regularized derivative of order $\alpha, 0 < \alpha < 1$. Furthermore, $D_{0t}^\alpha u$ is the Riemann Liouville fractional derivative of order $\alpha, 0 < \alpha < 1$. It is to be noted that

$$(D_{a+}^\alpha u)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u_\tau}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n, \quad (4)$$

and

$$(D_{a+}^\alpha (\tau)^{\beta-1}) = \frac{\Gamma(\beta)}{(\beta-\alpha)} t^{\beta-\alpha-1}, \quad (5)$$

and

$$D_{\alpha+}^{\alpha}(t) = \frac{\Gamma(1)}{\Gamma(1-\alpha)t^{\alpha}} = \frac{u(0)}{\Gamma(1-\alpha)t^{\alpha}}, \text{ if } \beta = 1. \quad (6)$$

In (2), the function $f(u, t)$ is defined on a rectangle $D = \{0 \leq t \leq T, |u - u_0| \leq U\}$ and satisfies the Lipschitz condition. Also, on $0 \leq t \leq T$, we introduce the grid $\bar{\omega}_{\tau} = \{t_j = j\tau, j = 0, \dots, j_0\}$. Denote a grid function by $y_j = y(t_j)$. In calculus, in the study of differential equations, the Picard-Lindelof theorem, Picard's existence theorem or Cauchy-Lipschitz theorem is an fundamental theorem on existence and uniqueness of solutions with given initial and boundary conditions.

2.2 Picard-Lindelof theorem

Theorem 1. *Consider the following initial value problem*

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0 - \epsilon, t_0 + \epsilon]. \quad (7)$$

If f be a uniformly Lipschitz continuous in y and also ontinuous in t . Then, for some value $\epsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

Theorem 2. *Let $U \subset R^n$ be an open and connected set. Also, assume $(0, b) \subset R^+$ and define $D = (0, b) \times U$. Furthermore, supposed that $f(x, y)$ be a real valued function on $\bar{D}$. Now consider the following equation:*

$$y^{\alpha} = f(x, y), \quad 0 < \alpha \leq 1, \quad (8)$$

under the conditions,

1. *f is continuous in $\bar{D}$, ($\bar{D}$ is closure of D),*
2. *f satisfies a Lipschitz condition in $\bar{D}$,*

then $\forall (x_0, y_0) \in D$, a positive number β can be found such that the closed interval $I = [x_0 - \beta, x_0 + \beta]$ is contained in $(0, b)$ and there exists a unique continuous function $y : I \rightarrow U$ such that

$$y^{\alpha} = f(x, y), \quad \forall x \in I, \quad y(x_0) = y_0. \quad (9)$$

Proposition 1. *Theorem 2, implies that the Cauchy problem (2), has a unique solution.*

3 Discretization process of differential equations of fractional order

Hereunder, we find a discrete methodology of the fractional derivative of order α . It is assumed that the solutions have the required smoothness. Thus,

we have

$$\begin{aligned} \partial_{0t}^\alpha u|_{t=t_j} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} \frac{\dot{u}(\eta) d\eta}{(t_j - \eta)^\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} ((j-s+1)\tau)^{1-\alpha} - ((j-s)\tau)^{1-\alpha} \sum_{s=1}^j u_{\bar{t},s} + O(\tau)^2, \end{aligned} \quad (10)$$

where ξ is an intermediate point between η and $t_{s-\frac{1}{2}}$ and $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$, $\dot{u} = \frac{\partial u}{\partial t}$ and $u_{\bar{t},s} = \frac{u(t_{s+1}) - u(t_s)}{\tau}$ and big O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity.

If $t_j = t_0 + j\tau$, $t_{s-1} = (s-1)\tau$, $t_s = s\tau$, and $u_i = u(t_i)$, then by using Taylor series expansion we will have

$$\begin{aligned} u(t_s) &= u(t_{s-\frac{1}{2}}) + \dot{u}(t_{s-\frac{1}{2}})(t_s - t_{s-\frac{1}{2}}) + \dots, \\ u(t_{s+1}) &= u(t_{s-\frac{1}{2}}) + \dot{u}(t_{s+1-\frac{1}{2}})(t_s - t_{s-\frac{1}{2}}) + \dots, \\ u(t_{s+1}) - u(t_s) &= \dot{u}(t_{s+1-\frac{1}{2}})(t_s - t_{s-\frac{1}{2}}) + O(\tau^2). \end{aligned} \quad (11)$$

We say that the expression

$$\dot{u}(t_{s-\frac{1}{2}}) = \frac{u_{s+1} - u_s}{\tau} = \frac{u(t_{s+1}) - u(t_s)}{\tau} + O(\tau), \quad (12)$$

approximates the first derivative $\dot{u} = \frac{du}{dt}$. Now, to find an upper bound on the second sum in expression (10) one will set

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^j \int_{t_{s-1}}^{t_s} \frac{\ddot{u}(\xi)(\eta - t_{s-\frac{1}{2}})}{(t_j - \eta)^\alpha} d\eta \\ &\leq \frac{M\tau}{\Gamma(1-\alpha)} \sum_{s=1}^j \int_{t_{s-1}}^{t_s} (t_j - \eta)^\alpha d\eta \leq \frac{M\tau}{\Gamma(1-\alpha)} \sum_{s=1}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) + O(\tau). \end{aligned} \quad (13)$$

Assume that $|\ddot{u}| \leq M$ and

$$D_{0t_j}^\alpha u = \Delta_{0t_j}^\alpha u + O(\tau). \quad (14)$$

Consequently, we have

$$\Delta_{0t_j}^\alpha u = \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) u_{\bar{t},s}. \quad (15)$$

which is a discrete methodology of the fractional derivative. Hence

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_{\bar{t},s} = f(y_j, t_j), \quad (16)$$

where

$$y(0) = y_0 = u_0, \quad y_{\bar{t},s} = \frac{(y_{s+1} - y_s)}{\tau} = \frac{y(t_{s+1}) - y(t_s)}{\tau}. \quad (17)$$

Now, for the error $z = y - u$, we have

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) z_{\bar{t},s} = \frac{\partial \bar{f}}{\partial u} z_j + \Psi_j, \quad (18)$$

where

$$z(0) = z_0 = 0, \quad \Psi_j = O(\tau). \quad (19)$$

It is to be noted that $\frac{\partial \bar{f}}{\partial u}$ denotes the derivative at ξ and $u_j \leq \xi \leq u_j + z_j$.

Let $y = z + u$, consequently

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j ((t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha})(z_{\bar{t},s} + u_{t,s})) = f(z_j + u_j, t_j), \quad (20)$$

Let $A = (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha})$, hence

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A(z_{\bar{t},s}) + \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A u_{t,s} = f(z_j + u_j, t_j). \quad (21)$$

In the other words

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A(z_{\bar{t},s}) + f(z_j + u_j, t_j) - \frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A u_{t,s} = f(z_j + u_j, t_j). \quad (22)$$

Consequently

$$D_{0t_j}^\alpha u = \Delta_{0t_j}^\alpha u + O(\tau), \quad (23)$$

and

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A z_{\bar{t}} = f(z_j + u_j, t_j) - f(u_j, t_j) + O(\tau). \quad (24)$$

As we know

$$\exists \xi \in (u_j, u_j + z_j) \text{ s.t. } f(u_j + z_j, t_j) - f(u_j, t_j) = \frac{\partial f}{\partial u}(\xi, t_j) z_j = \frac{\partial \bar{f}}{\partial u} z_j. \quad (25)$$

Consequently

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=1}^j A z_{\bar{t},s} = \frac{\partial \bar{f}}{\partial u} z_j + O(\tau). \quad (26)$$

Now, consider the following equation

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) \frac{z_{s+1} - z_s}{\tau} = f'(\xi, t) z_j + \Psi_j, \quad (27)$$

where

$$z_{\bar{t},s} = \frac{z_{s+1} - z_s}{\tau}. \quad (28)$$

As a direct consequence of (27) we obtain

$$\begin{aligned} z_{j+1} = & \left[(1 - (t_2^{1-\alpha} - t_1^{1-\alpha})) \tau^{\alpha-1} + \tau^\alpha \Gamma(2-\alpha) f'(\xi, t) \right] z_j \\ & + \tau^{\alpha-1} \left[(t_{j+1}^{1-\alpha} - t_j^{1-\alpha}) z_0 + (-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}) z_1 + \dots \right. \\ & \left. + (-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}) z_{j-1} \right] + \tau^\alpha \Gamma(2-\alpha) \Psi_j. \end{aligned} \quad (29)$$

Assume that

$$\forall x \geq 1, \quad 1 \leq \frac{2x^{1-\alpha}}{(x+1)^{1-\alpha} + (x-1)^{1-\alpha}} \leq 2^\alpha. \quad (30)$$

Let us first show the validity of the inequality

$$A_j = t_{j+1}^{1-\alpha} - 2t_j^{1-\alpha} + t_{j-1}^{1-\alpha} < 0, \quad \forall j \geq 1, \quad (31)$$

or

$$2j^{1-\alpha} > (j+1)^{1-\alpha} + (j-1)^{1-\alpha}, \quad \forall j \geq 1. \quad (32)$$

Define the function

$$f(t) = \frac{2t^{1-\alpha}}{(t+1)^{1-\alpha} + (t-1)^{1-\alpha}}, \quad \forall t \geq 1. \quad (33)$$

Since

$$f(1) = 2^\alpha, \quad \lim_{t \rightarrow \infty} f(t) = 1, \quad f'(t) < 0, \quad (34)$$

we have

$$1 < f(t) \leq 2^\alpha, \quad x \geq 1, \quad t \in [1, \infty), \quad -t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha} > 0, \quad \forall t \geq 1. \quad (35)$$

Assume that $\bar{f}(\xi, t) < -a$ when $a < 0$. Consequently (29) implies the estimate:

$$\begin{aligned} |z_{j+1}| &\leq 1 - (2^\alpha - 1) - \tau^\alpha \Gamma(2 - \alpha) a \max_{0 \leq s \leq j} |z_s| \\ &\quad + (2^{1-\alpha} - 1) \max_{0 \leq s \leq j} |z_s| + \tau^\alpha \Gamma(2 - \alpha) |\Psi_j|, \end{aligned} \quad (36)$$

or

$$|z_{j+1}| \leq 1 - \tau^\alpha \Gamma(2 - \alpha) a \max_{0 \leq s \leq j} |z_s| + \tau^\alpha \Gamma(2 - \alpha) |\Psi_j|. \quad (37)$$

Whence, (37) implies that

$$|z_{j+1}| \leq \Gamma(2 - \alpha) \sum_{j'=0}^j \tau^\alpha \max_{0 \leq s \leq j'} |\Psi_j|. \quad (38)$$

In the other words, expression (38) implies that scheme (16), (17) converges at a rate of $O(\tau^\alpha)$.

4 Navier - Stokes flow with measures as initial vorticity

As we know, (1) can be written in the following form [10]

$$D_{0t}^\alpha u(x, t) = q(x, t)u(x, t) - k(x, t)u(x, t) + f(x, t), \quad (39)$$

where

$$\begin{aligned} u(0, t) &= u(l, t) = 0, \\ D_{0t}^{\alpha-1} u|_{t=0} &= u_0(x), \\ k(x, t) &\geq c_1 > 0, \quad q(x, t) \geq 0. \end{aligned}$$

Now, in the cylinder $Q_T = G \times [0 < t \leq T]$, consider the following problem

$$\partial_{0t}^\alpha u = Lu + f(x, t), \quad (x, t) \in Q_T, \quad (40)$$

where

$$\begin{aligned} u(x, t)|_\Gamma &= \mu(x, t), \quad t \geq 0, \\ u(x, 0) &= u_0(x), \quad x \in \bar{G}, \\ Lu &= \sum_{k=1}^p L_k u, \quad L_k u = \frac{\partial^2 u}{\partial x_k^2}, \quad k = 1, 2, \dots, p. \end{aligned}$$

Furthermore, $\partial_{0t}^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\alpha} d\tau$, $0 < \alpha < 1$, is the Riemann Liouville fractional derivative of order α .

Introduce a grid $\bar{w}_h$ that is uniform in each direction x_k :

$$\bar{w}_h = \{x_i = (i_1 h_1, \dots, i_p h_p) \in G, i_k = 0, 1, \dots, N_k, h_k = \frac{l_k}{N_k}\}, \quad (41)$$

$$\bar{w}_\tau = \{t_j = j\tau, j = 1, 2, \dots, j_0\},$$

where γ_h is boundary nodes.

Problem (40) is approximated by the difference scheme:

$$\Delta_{0t}^\alpha y = \Lambda y^{(\sigma)} + \varphi, \quad x \in w_h, \quad t \in \bar{w}_\tau, \quad (42)$$

$$y|_{\gamma_h} = \mu(x, t), \quad x \in \gamma_h, \quad t \in \bar{w}, \quad (43)$$

$$y(x, 0) = u_0(x), \quad x \in \bar{w}_h, \quad u_{\bar{x}x} = \Lambda. \quad (44)$$

Hence one will set:

$$\Lambda y = \sum_{k=1}^p (\Lambda_k) y, \quad \Lambda_k = y_{\bar{x}_k x_k}, \quad k = 1, 2, \dots, p. \quad (45)$$

Now, we get a one-parameter family of difference schemes

$$y_{\bar{x}x} = y_{\bar{x}x,i} = (y_{i+1} - 2y_i - y_{i-1})/h^2, \quad (46)$$

$$y^{(\sigma)} = \sigma \hat{y} + (1 - \sigma)y, \quad 0 \leq \sigma \leq 1. \quad (47)$$

Sometimes, scheme (47) will be treated as a scheme with weights. We denote by y_i^j the value at the node (x_i, t_j) of the grid function y given on $\bar{w}_{j\tau}$. Now, consider the following equation

$$\hat{y} = y^{j+1}, \quad y = y^j, \quad \varphi = f(x, \bar{t}), \quad \bar{t} = t_{j+\frac{1}{2}}, \quad (48)$$

$$\Delta_{0t_j}^\alpha y = \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^j (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_t^s. \quad (49)$$

we define sum $y = \overset{\circ}{\nu} + \overset{*}{\nu}$ and then estimate the solution of difference problem (42), (43), $\overset{\circ}{\nu}$ is the solution to problem (42), (43) with $\varphi = 0$ and $\overset{*}{\nu}$ is the solution to the same problem with $\mu = 0$ and $u_0(x) = 0$.

First, we rewrite $\overset{\circ}{\nu}$ to the canonical form (see [3, 4])

$$\left(\frac{1}{\Gamma(2-\alpha)\tau^\alpha} + \sum_{k=1}^p \frac{\sigma}{h_k^2} \right) \overset{\circ}{\nu}_{i_k}^{j+1} = \sum_{k=1}^p \frac{\sigma}{h_k^2} (\overset{\circ}{\nu}_{i_k+1}^{j+1} + \overset{\circ}{\nu}_{i_k-1}^{j+1}) + \quad (50)$$

$$\begin{aligned} & \left(\frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha} - \sum_{k=1}^p \frac{2(1-\sigma)}{h_k^2} \right) \nu_{i_k}^j + \sum_{k=1}^p \frac{2(1-\sigma)}{h_k^2} (\nu_{i_k+1}^j + \nu_{i_k-1}^j) \\ & + \frac{1}{\Gamma(2-\alpha)} \left(\frac{t_{j+1}^{1-\alpha} - t_j^{1-\alpha}}{\tau} \nu_{i_k}^0 + \frac{-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}}{\tau} \nu_{i_k}^1 + \dots + \right. \\ & \quad \left. \frac{-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}}{\tau} \nu_{i_k}^{j-1} \right). \end{aligned}$$

Since

$$A(p) = \frac{1}{\Gamma(2-\alpha)\tau^\alpha} + \sum_{k=1}^p \frac{2\sigma}{h_k^2} > 0, \quad (51)$$

and $B(P, Q) > 0$ if

$$\tau^\alpha < \frac{2-2^{1-\alpha}}{2\Gamma(2-\alpha)(1-\sigma)} \left(\sum_{k=1}^p \frac{1}{h_k^2} \right)^{-1}, \quad 0 < \sigma < 1, \quad (52)$$

$$-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha} > 0, \quad \forall j \geq 1, \quad (53)$$

by using maximum principle we have (see [3])

$$\|\nu^0\|_C \leq \|\tilde{\mu}\|_{C^*}, \quad (54)$$

where

$$\tilde{\mu} = \begin{cases} \mu, & x \in \gamma_h, \quad t \in w_\tau, \\ u_0(x), & x \in w_h, \quad t = 0, \end{cases} \quad (55)$$

and

$$\|\tilde{\mu}\|_{C^*} = \max_{x \in \gamma_h, t \in \bar{w}_h} |\tilde{\mu}(x, t)|. \quad (56)$$

To estimate ν^* rewriting problem for ν

$$\left(\frac{1}{\Gamma(2-\alpha)\tau^\alpha} + \sum_{k=1}^p \frac{2\sigma}{h_k^2} \right) \nu_{i_k}^{*j+1} = \sum_{k=1}^p \frac{\sigma}{h_k^2} (\nu_{i_k+1}^{*j+1} + \nu_{i_k-1}^{*j+1}) + \Phi(P_{(j+1)}), \quad (57)$$

where

$$\begin{aligned} \Phi(P_{(j+1)}) &= \left(\frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha} - \sum_{k=1}^p \frac{2(1-\sigma)}{h_k^2} \right) \nu_{i_k}^* + \sum_{k=1}^p \frac{1-\sigma}{h_k^2} (\nu_{i_k+1}^* + \nu_{i_k-1}^*) \quad (58) \\ &+ \frac{1}{\Gamma(2-\alpha)} \left(\frac{t_{j+1}^{1-\alpha} - t_j^{1-\alpha}}{\tau} \nu_{i_k}^{*0} + \frac{-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}}{\tau} \nu_{i_k}^{*1} + \dots \right. \end{aligned}$$

$$+ \frac{-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}}{\tau} \nu_{i_k}^{*j-1} \Big) + \varphi_{i_k}^j.$$

Consequently

$$\Phi(P_{(j+1)}) = \sum_{Q \in S_j''} B(P, Q) \nu^*(Q) + F(P_{(j+1)}), \quad P_{(j+1)} = P(x, t_{j+1}). \quad (59)$$

Thus, canonical form (57) can be rewritten to yield the equation

$$A(P_{(j+1)}) \nu^*(P_{(j+1)}) = \sum_{Q \in S_{j+1}'} B(P, Q) \nu^*(Q) + \phi(P_{(j+1)}), \quad (60)$$

where S_{j+1}' is the set of nodes $Q(\xi, t_{j+1}) \in S'(P(x, t_{j+1}))$ and S_j'' is the set of nodes $Q(\xi, t_j), \dots, Q(\xi, t_1) \in S''(P(x, t_{j+1}))$.

Introduce the notation

$$D'(P_{(j+1)}) = A(P_{(j+1)}) + \sum_{Q \in S_{j+1}'} B(P_{(j+1)}, Q). \quad (61)$$

Note that

$$D'(P_{(j+1)}) = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}, \quad A(P_{(j+1)}) > 0, \quad (62)$$

$$B(P_{(j+1)}, Q) = \sum_{k=1}^p \frac{\sigma}{h_k^2} > 0, \quad 0 < \sigma < 1, \quad (63)$$

for all $Q \in S_j''$ and $Q \in S_{j+1}'$ if

$$\tau^\alpha < \frac{2-2^{1-\alpha}}{2\Gamma(1-\alpha)(1-\sigma)} \left(\sum_{k=1}^p \frac{1}{h_k^2} \right)^{-1}, \quad (64)$$

$$\frac{1}{D'(P_{(j+1)})} \sum_{Q \in S_j''} B(P_{(j+1)}, Q) = \tau^\alpha \Gamma(2-\alpha) \sum_{Q \in S''} B(P_{(j+1)}, Q) = 1. \quad (65)$$

Consequently

$$\|\nu^{*j+1}\|_{C_h} \leq \|y^{*0}\|_{C_h} + \Gamma(2-\alpha) \sum_{j'=0}^j \tau^\alpha \max_{0 \leq s \leq j'} \|\varphi^s\|_{C_h}. \quad (66)$$

Combining (57) and (66) gives

$$\|y^{j+1}\|_{C_h} \leq \|y^0\|_{C_h} + \max_{0 < k \leq j+1} |\mu(t_k)| + \Gamma(2-\alpha) \sum_{j'=0}^j \tau^\alpha \max_{0 \leq s \leq j'} \|\varphi^s\|_{C_h}. \quad (67)$$

The error $z = y - u$ satisfies the estimate

$$\|z^{j+1}\|_{C_h} \leq \Gamma(2 - \alpha) \sum_{j'=0}^j \tau^\alpha \max_{0 \leq s \leq j'} \|\psi^s\|_{C_h}. \quad (68)$$

Since $\psi = O(|h|^2 + \epsilon)$, where $|h|^2 = h_1^2 + \dots + h_p^2$, it follows from (68) that

$$\|z^{j+1}\|_{C_h} = O(\tau^\alpha + \frac{|h^2|}{\tau^{1-\alpha}}). \quad (69)$$

For $\alpha \rightarrow 1$, (69) implies the well-known result

$$\|z^{j+1}\|_{C_h} = O(|h^2| + \tau). \quad (70)$$

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