DOI: 10.1515/auom-2017-0014
An. Şt. Univ. Ovidius Constanţa **Vol. 25(1),2017, 177–194**

ON BI-ALGEBRAS

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Abstract

In this paper, we introduce a new algebra, called a BI-algebra, which is a generalization of a (dual) implication algebra and we discuss the basic properties of BI-algebras, and investigate ideals and congruence relations.

1 Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim ([19]) introduced the notion of d-algebras, which is another useful generalization of BCK-algebras and investigated several relations between d-algebras and BCK-algebras, and then investigated other relations between oriented digraphs and d-algebras.

It is known that several generalizations of a B-algebra were extensively investigated by many researchers and properties have been considered systematically. The notion of B-algebras was introduced by J. Neggers and H. S. Kim ([17]). They defined a *B-algebra* as an algebra $(X, *, 0)$ of type (2,0) (i.e., a non-empty set with a binary operation "$*$" and a constant 0) satisfying the following axioms:

$(B1)$ $x * x = 0$,

$(B2)$ $x * 0 = x$,

Key Words: BI-algebra, (right, left) distributive, congruence relation.
2010 Mathematics Subject Classification: 06F35; 20N02.
Received: 15.12.2015
Accepted: 30.04.2016

(B) $(x * y) * z = x * [z * (0 * y)]$

for any $x, y, z \in X$.

C. B. Kim and H. S. Kim ([12]) defined a BG-algebra, which is a generalization of B-algebra. An algebra $(X, *, 0)$ of type (2,0) is called a *BG-algebra* if it satisfies $(B1), (B2)$, and

(BG) $x = (x * y) * (0 * y)$

for any $x, y \in X$.

Y. B. Jun, E. H. Roh and H. S. Kim ([9]) introduced the notion of a BH-algebra which is a generalization of $BCK/BCI/BCH$-algebras. An algebra $(X, *, 0)$ of type (2,0) is called a *BH-algebra* if it satisfies $(B1), (B2)$, and

(BH) $x * y = y * x = 0$ implies $x = y$

for any $x, y \in X$.

Moreover, A. Walendziak ([21]) introduced the notion of $BF/BF_1/BF_2$-algebras. An algebra $(X, *, 0)$ of type (2,0) is called a *BF-algebra* if it satisfies $(B1), (B2)$ and

(BF) $0 * (x * y) = y * x$

for any $x, y \in X$.

A BF-algebra is called a *BF_1-algebra* (resp., a *BF_2-algebra*) if it satisfies (BG) (resp., (BH)).

In this paper, we introduce a new algebra, called a BI-algebra, which is a generalization of a (dual) implication algebra, and we discuss the basic properties of BI-algebras, and investigate ideals and congruence relations.

2 Preliminaries

In what follows we summarize several axioms for construct several generalizations of $BCK/BCI/B$-algebras. Let $(X; *, 0)$ be an algebra of type $(2, 0)$. We provide several axioms which were discussed in general algebraic structures as follows: for any $x, y, z \in X$,

$(B1)$ $x * x = 0$,

$(B2)$ $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$,

(BG) $x = (x * y) * (0 * y)$,

(BM) $(z * x) * (z * y) = y * x$,

(BH) $x * y = 0$ and $y * x = 0$ implies $x = y$,

(BF) $0 * (x * y) = y * x$,

(BN) $(x * y) * z = (0 * z) * (y * x)$,

(BO) $x * (y * z) = (x * y) * (0 * z)$,

(BP) $x * (x * y) = y$,

(Q) $(x * y) * z = (x * z) * y$,

(CO) $(x * y) * z = x * (y * z)$,

(BZ) $((x * z) * (y * z)) * (x * y) = 0$,

(K) $0 * x = 0$.

These axioms played important roles for researchers to construct algebraic structures and investigate several properties. For details, we refer to [1-23].

Definition 2.1. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a

- *BCI-algebra* if satisfies in $(B2)$, (BH) and $((x * y) * (x * z)) * (z * y) = 0$ for all $x, y, z \in X$ ([7]).
- *BCK-algebra* if it is a BCI-algebra and satisfies in (K) ([22]).
- *BCH-algebra* if satisfies in $(B1)$, (BH) and (Q) ([6]).
- *BH-algebra* if satisfies in $(B1)$, $(B2)$ and (BH) ([9]).
- *BZ-algebra* if satisfies in $(B2)$, (BH) and (BZ) ([23]).
- *d-algebra* if satisfies in $(B1)$, (K) and (BH) ([19]).
- *Q-algebra* if satisfies in $(B1)$, $(B2)$ and (Q) ([20]).
- *B-algebra* if satisfies in $(B1)$, $(B2)$ and (B) ([17]).
- *BM-algebra* if satisfies in $(B2)$ and (BM) ([11]).
- *BO-algebra* if satisfies in $(B1)$, $(B2)$ and (BO) ([13]).
- *BG-algebra* if satisfies in $(B1)$, $(B2)$ and (BG) ([12]).

- BP-*algebra* if satisfies in $(B1)$, $(BP1)$ and $(BP2)$ ([3]).
- BN-*algebra* if satisfies in $(B1)$, $(B2)$ and (BN) ([10]).
- BF-*algebra* if satisfies in $(B1)$, $(B2)$ and (BF) ([21]).
- *Coxeter algebra* if satisfies in $(B1)$, $(B2)$ and (CO) ([15]).

Definition 2.2. A groupoid $(X;*)$ is called an *implication algebra* ([1]) if it satisfies the following identities

(I1) $(x * y) * x = x$,

(I2) $(x * y) * y = (y * x) * x$,

(I3) $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

Definition 2.3. Let $(X;*)$ be an implication algebra and let a binary operation " $\circ$ " on X be defined by

$$x * y := y \circ x.$$

Then $(X;\circ)$ is said to be a *dual implication algebra.* In fact, the axioms of that are as follows:

(DI1) $x \circ (y \circ x) = x$,

(DI2) $x \circ (x \circ y) = y \circ (y \circ x)$,

(DI3) $(x \circ y) \circ z = (x \circ z) \circ y$,

for all $x, y, z \in X$. W. Y. Chen and J. S. Oliveira ([4]) proved that in any implication algebra $(X;*)$ the identity $x * x = y * y$ holds for all $x, y \in X$. We denote the identity $x * x = y * y$ by the constant 0. The notion of BI-algebras comes from the (dual) implication algebra.

3 BI–algebras

Definition 3.1. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BI-*algebra* if

(B1) $x * x = 0$,

(BI) $x * (y * x) = x$

for all $x, y \in X$.

Let $(X, *, 0)$ be a BI-algebra. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$. We note that "$\leq$" is not a partially order set, but it is only reflexive.

Example 3.2. (i). Every implicative BCK-algebra is a BI-algebra.
(ii). Let $X := \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then it is easy to see that $(X; *, 0)$ is a BI-algebra, but it is not implicative BCK-algebra, since

$$(c * (c * a)) * a = (c * b) * a = c * a = b \neq 0.$$

(iii). Let X be a set with $0 \in X$. Define a binary operation "$*$" on X by

$$x * y = \begin{cases} 0 & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}$$

Then $(X; *, 0)$ is an implicative BCK-algebra ([22]), and hence a BI-algebra.

Note that in Example 3.2(ii), we can see that it is not a B-algebra, since

$$(c * a) * b = b * b = 0 \neq c * (b * (0 * a)) = c * (b * 0) = c * b = c.$$

It is not a BG-algebra, since

$$c \neq (c * a) * (0 * a) = b * 0 = b.$$

It is not a BM-algebra, since

$$(b * a) * (b * c) = b * b = 0 \neq c * a = b.$$

It is not a BF-algebra, since

$$0 * (a * b) = 0 \neq b * a = b.$$

It is not a BN-algebra, since

$$(c * b) * a = c * a = b \neq (0 * a) * (b * c) = 0.$$

It is not a BO-algebra, since

$$c * (a * a) = c * 0 = c \neq (c * a) * (0 * a) = b * 0 = b.$$

It is not a BP–algebra, since

$$c * (c * b) = c * c = 0 \neq b.$$

It is not a Q-algebra, since

$$(c * b) * a = c * a = b \neq (c * a) * b = b * b = 0.$$

It is not a Coxeter algebra, since

$$(c * a) * b = b * b = 0 \neq c * (a * b) = c * a = b.$$

It is not a BZ-algebra, since

$$((a * c) * (0 * c)) * (a * 0) = (b * 0) * a = b \neq 0.$$

Also, we consider the following example.

Example 3.3. Let $X := \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

Then $(X; *, 0)$ is a BI-algebra, but not a $BH/BCI/BCK$-algebra, since

$$a * b = 0 \text{ and } b * a = 0, \text{ while } a \neq b.$$

Proposition 3.4. *If $(L; \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice, then $(L; *, 0)$ is a BI-algebra, where "$*$" is defined by $x * y = \neg y \wedge x$, for all $x, y \in L$.*

Proposition 3.5. *Any dual implication algebra is a BI-algebra.*

Note that the converse of Proposition 3.5 does not hold in general. See the following example.

Example 3.6. Let $X := \{0, a, b\}$ be a set with the following table.

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X;*,0)$ is a BI-algebra, but it is not a dual implication algebra, since

$$a*(a*c)=a*b=a, \text{ while } c*(c*a)=c*b=c.$$

Proposition 3.7. *Let X be a BI-algebra. Then*

(i) $x*0=x$,

(ii) $0*x=0$,

(iii) $x*y=(x*y)*y$,

(iv) if $y*x=x$, $\forall x,y\in X$, *then* $X=\{0\}$,

(v) if $x*(y*z)=y*(x*z)$, $\forall x,y\in X$, *then* $X=\{0\}$,

(vi) if $x*y=z$, *then* $z*y=z$ *and* $y*z=y$,

(vii) if $(x*y)*(z*u)=(x*z)*(y*u)$, *then* $X=\{0\}$,

for all $x,y,z,u\in X$.

Proof. (i). Using (BI) and $(B1)$ we have $x=x*(x*x)=x*0$.
(ii). By (BI) and (i) we have $0=0*(x*0)=0*x$.
(iii). Given $x,y\in X$, we have

$$x*y=(x*y)*(y*(x*y))=(x*y)*y.$$

(iv). For $x\in X$, we have

$$x=x*(y*x)=x*x=0.$$

Hence $X=\{0\}$.
(v). Given $x\in X$, we have

$$0=0*(x*0)=x*(0*0)=x*0=x,$$

Hence $X=\{0\}$.
(vi). If $x*y=z$, then by (iii) we have

$$z*y=(x*y)*y=x*y=z.$$

Also, $y*z=y*(x*y)=y$.
(vii). If $x\in X$, then we have

$$x=x*0=(x*0)*(x*x)=(x*x)*(0*x)=0*(0*x)=0*0=0.$$

Hence $X = \{0\}$. $\square$

Definition 3.8. A BI-algebra X is said to be *right distributive* (or *left distributive*, resp.) if

$$(x * y) * z = (x * z) * (y * z), \ (z * (x * y) = (z * x) * (z * y), \ resp.)$$

for all $x, y, z \in X$.

Proposition 3.9. *If BI-algebra X is a left distributive, then $X = \{0\}$.*

Proof. Let $x \in X$. Then by (BI) and $(B1)$ we have

$$x = x * (x * x) = (x * x) * (x * x) = 0 * 0 = 0.$$

$\square$

Example 3.10. (i). Let $X := \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X; *, 0)$ is a right distributive BI-algebra.
(ii). Example 3.2(ii) is not right distributive, since

$$(c * a) * b = b * b = 0 \neq (c * b) * (a * b) = c * a = b.$$

Proposition 3.11. *Let $(X; *)$ be a groupoid with $0 \in X$. If the following axioms holds:*

(i) $x * x = 0$,

(ii) $x * y = x$, *for all* $x \neq y$,

then $(X; *, 0)$ is a right distributive BI-algebra.

Proposition 3.12. *Let X be a right distributive BI-algebra. Then*

(i) $y * x \leq y$,

(ii) $(y * x) * x \leq y$,

(iii) $(x * z) * (y * z) \leq x * y$,

(iv) *if* $x \leq y$, *then* $x * z \leq y * z$,

(v) $(x * y) * z \leq x * (y * z)$,

(vi) *if* $x * y = z * y$, *then* $(x * z) * y = 0$,

for all $x, y, z \in X$.

Proof. For any $x, y \in X$, we have
(i).
$$(y * x) * y = (y * y) * (x * y) = 0 * (x * y) = 0,$$
which shows that $y * x \leq y$.
(ii).
$$\begin{aligned} ((y * x) * x) * y &= ((y * x) * y) * (x * y) \\ &= ((y * y) * (x * y)) * (x * y) \\ &= (0 * (x * y)) * (x * y) \\ &= 0 * (x * y) = 0, \end{aligned}$$
which shows that $(y * x) * x \leq y$.
(iii).
$$\begin{aligned} ((x * z) * (y * z)) * (x * y) &= ((x * y) * z) * (x * y) \\ &= ((x * y) * (x * y)) * (z * (x * y)) \\ &= 0 * (z * (x * y)) = 0, \end{aligned}$$
proving that $(x * z) * (y * z) \leq x * y$.
(iv). If $x \leq y$, then $x * y = 0$ and hence
$$(x * z) * (y * z) = (x * y) * z = 0 * z = 0,$$
proving that $x * z \leq y * z$.
(v). By (i), we have $x * z \leq x$. It follows from (iv) that $(x * z) * (y * z) \leq x * (y * z)$. Using the right distributivity, we obtain $(x * y) * z \leq x * (y * z)$.
(vi). Let $x * y = z * y$. Since X is right distributive, we obtain
$$(x * z) * y = (x * y) * (z * y) = (x * y) * (x * y) = 0.$$
□

It is easy to see that, if $x \leq y$, we does not conclude that $z * x \leq z * y$ in general, since, in Example 3.10(i), $a \leq c$ but
$$b * a = b \not\leq b * c = 0.$$

Proposition 3.13. *Let X have the condition: $(z * x) * (z * y) = y * x$ for all $x, y, z \in X$. If $x \leq y$, then $z * y \leq z * x$.*

Proof. If $x \leq y$, then $x * y = 0$. It follows that $(z * y) * (z * x) = x * y = 0$. Hence $z * y \leq z * x$. ☐

An algebra $(X; *)$ is said to have an *inclusion condition* if $(x * y) * x = 0$ for all $x, y \in X$. Every right distributive BI-algebra has the inclusion condition by Proposition 3.12(i). If X is a right distributive BI-algebra, then X is a *quasi-associative algebra* by Proposition 3.12(v).

Proposition 3.14. *Let X be a right distributive BI-algebra. Then induced relation " $\leq$ " is a transitive relation.*

Proof. If $x \leq y$ and $y \leq z$, then we obtain by Proposition 3.7(i)

$$\begin{aligned} x * z &= (x * z) * 0 \\ &= (x * z) * (y * z) \\ &= (x * y) * z \\ &= 0 * z \\ &= 0. \end{aligned}$$

Therefore $x \leq z$. ☐

4 Ideals in BI-algebras

In what follows, let X denote a BI-algebra unless otherwise specified.

Definition 4.1. A subset I of X is called an *ideal* of X if

(I1) $0 \in I$,

(I2) $y \in I$ and $x * y \in I$ imply $x \in I$ for any $x, y \in X$.

Obviously, $\{0\}$ and X are ideals of X. We shall call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I is said to be *proper* if $I \neq X$.

Example 4.2. In Example 3.2(ii), $I_1 = \{0, a, c\}$ is an ideal of X, while $I_2 = \{0, a, b\}$ is not an ideal of X, since $c * a = b \in I_2$ and $a \in I_2$, but $c \notin I_2$.

We denote the set of all ideals of X by $I(X)$.

Lemma 4.3. *If $\{I_i\}_{i\in\Lambda}$ is a family of ideals of X, then $\bigcap_{i\in\Lambda} I_i$ is an ideal of X.*

Proof. Straightforward. □

Since the set $I(X)$ is closed under arbitrary intersections, we have the following theorem.

Theorem 4.4. *$(I(X), \subseteq)$ is a complete lattice.*

Proposition 4.5. *Let I be an ideal of X. If $y \in I$ and $x \leq y$, then $x \in I$.*

Proof. If $y \in I$ and $x \leq y$, then $x * y = 0 \in I$. Since $y \in I$ and I is an ideal, we obtain $x \in I$. □

For any $x, y \in X$, define $A(x, y) := \{t \in X : (t * x) * y = 0\}$. It is easy to see that $0, x \in A(x, y)$. In Example 3.2(ii), $A(a, b) = \{0, a, b, c\}$ and $A(b, a) = \{0, a, b\}$. Hence $A(a, b) \neq A(b, a)$. We note that

$$\begin{aligned} A(a,0) &= \{t \in X : (t * a) * 0 = 0\} \\ &= \{t \in X : t * a = 0\} \\ &= \{t \in X : (t * 0) * a = 0\} \\ &= A(0, a). \end{aligned}$$

Theorem 4.6. *If X is a right distributive BI-algebra, then $A(x, y)$ is an ideal of X where $x, y \in X$.*

Proof. Let $x * y \in A(a, b)$, $y \in A(a, b)$. Then $((x * y) * a) * b = 0$ and $(y * a) * b = 0$. By the right distributivity we have

$$\begin{aligned} 0 = ((x * y) * a) * b &= ((x * a) * (y * a)) * b \\ &= ((x * a) * b) * ((y * a) * b) \\ &= ((x * a) * b) * 0 \\ &= (x * a) * b, \end{aligned}$$

whence $x \in A(a, b)$. This proves that $A(a, b)$ is an ideal of X. □

Proposition 4.7. *Let X be a BI-algebra. Then*

(i) $A(0, x) \subseteq A(x, y)$, for all $x, y \in X$,

(ii) if $A(0, y)$ is an ideal and $x \in A(0, y)$, then $A(x, y) \subseteq A(0, y)$.

Proof. (i). Let $z \in A(0,x)$. Then $z * x = (z * 0) * x = 0$. Hence $(z * x) * y = 0 * y = 0$. Thus $z \in A(x,y)$ and so $A(0,x) \subseteq A(x,y)$.

(ii). Let $A(0,y)$ be an ideal and $x \in A(0,y)$. If $z \in A(x,y)$, then $(z*x)*y = 0$. Hence $((z * x) * 0) * y = 0$. Therefore $z * x \in A(0,y)$. Now, since $A(0,y)$ is an ideal and $x \in A(0,y)$, $z \in A(0,y)$. Thus $A(x,y) \subseteq A(0,y)$. □

Proposition 4.8. *Let X be a BI-algebra. Then*

$$A(0,x) = \bigcap_{y \in X} A(x,y).$$

for all $x, y \in X$.

Proof. By Proposition 4.7(i), we have $A(0,x) \subseteq \bigcap_{y \in X} A(x,y)$. If $z \in \bigcap_{y \in X} A(x,y)$, then $z \in A(x,y)$, for all $y \in X$. It follows that $z \in A(0,x)$. Hence $\bigcap_{y \in X} A(x,y) \subseteq A(0,x)$. □

Theorem 4.9. *Let I be a non-empty subset of X. Then I is an ideal of X if and only if $A(x,y) \subseteq I$ for all $x, y \in I$.*

Proof. Assume that I is an ideal of X and $x, y \in I$. If $z \in A(x,y)$, then $(z * x) * y = 0 \in I$. Since I is an ideal and $x, y \in I$, we have $z \in I$. Hence $A(x,y) \subseteq I$.

Conversely, suppose that $A(x,y) \subseteq I$ for all $x, y \in I$. Since $(0 * x) * y = 0$, $0 \in A(x,y) \subseteq I$. Let $a * b$ and $b \in I$. Since $(a * b) * (a * b) = 0$, we have $a \in A(b, a * b) \subseteq I$, i.e., $a \in I$. Thus I is an ideal of X. □

Proposition 4.10. *If I is an ideal X, then*

$$I = \bigcup_{x,y \in I} A(x,y).$$

Proof. Let I be an ideal of X and $z \in I$. Since $(z * 0) * z = z * z = 0$, we have $z \in A(0,z)$. Hence

$$I \subseteq \bigcup_{z \in I} A(0,z) \subseteq \bigcup_{x,y \in I} A(x,y)$$

If $z \in \bigcup_{x,y \in I} A(x,y)$, then there exist $a, b \in I$ such that $z \in A(a,b)$. It follows from Theorem 4.9 that $z \in I$, i.e., $\bigcup_{x,y \in I} A(x,y) \subseteq I$. □

Theorem 4.11. *If I is an ideal of X, then*

$$I = \bigcup_{x \in I} A(0, x).$$

Proof. Let I be an ideal of X and $z \in I$. Since $(z * 0) * z = z * z = 0$, we have $z \in A(0, z)$. Hence

$$I \subseteq \bigcup_{z \in I} A(0, z).$$

If $z \in \bigcup_{x \in I} A(0, z)$, then there exists $a \in I$ such that $z \in A(0, a)$, which means that $z * a = (z * 0) * a = 0 \in I$. Since I is an ideal of X and $a \in I$, we obtain $z \in I$. This means that $\bigcup_{x \in I} A(0, x) \subseteq F$. $\square$

Let X be a right distributive BI-algebra and let I be an ideal of X and $a \in X$. Define

$$I_a^l := \{x \in X : \; x * a \in I\}.$$

Theorem 4.12. *If X is a right distributive BI-algebra, then I_a^l is the least ideal of X containing I and a.*

Proof. By $(B1)$ we have $a * a = 0$, for all $a \in X$, i.e. $a \in I_a^l$ and so $I_a \neq \emptyset$. Assume that $x * y \in I_a^l$ and $y \in I_a^l$. Then $(x * y) * a \in I$ and $y * a \in I$. By the right distributivity, we have $(x * a) * (y * a) \in I$. Since $y * a \in I$, we have $x * a \in I$ and so $x \in I_a^l$. Therefore I_a^l is an ideal of X.

Let $x \in I$. Since $(x * a) * x = (x * x) * (a * x) = 0 * (a * x) = 0 \in I$ and I is an ideal of X, we obtain $x * a \in I$. Hence $x \in I_a$. Thus $I \subseteq I_a^l$.

Now, let J be an ideal of X containing I and a. Let $x \in I_a^l$. Then $x * a \in I \subseteq J$. Since $a \in J$ and J is an ideal of X, we have $x \in J$. Therefore $I_a^l \subseteq J$. $\square$

The following example shows that the condition, right distributivity, is very necessary.

Example 4.13. In Example 3.2(ii), $(X; *, 0)$ is a BI-algebra, but not right distributive, since

$$(c * a) * b = b * b = 0 \neq (c * b) * (a * b) = c * a = b.$$

We can see that $I = \{0, a\}$ is an ideal of X, but $I_b^l = \{0, a, b\}$ is not an ideal of X.

Note. Let I be an ideal of X and $a \in X$. If we denote

$$I_a^r := \{x \in X : \ a * x \in I\}$$

Then I_a^r is not an ideal of X in general.

Example 3.14. In Example 3.10(i), $I = \{0, b\}$ is an ideal of X but $I_c^r = \{a, c\}$ is not an ideal of X, because $0 \notin I_c^r$.

Let A be a non-empty subset of X. The set $\bigcap\{I \in I(X)| \ A \subseteq I\}$ is called an *ideal generated by* A, written $< A >$. If $A = \{a\}$, we will denote $< \{a\} >$, briefly by $< a >$, and we call it a *principal ideal* of X. For $I \in I(X)$ and $a \in X$, we denote by $[I \cup \{a\})$ the ideal generated by $I \cup \{a\}$. For convenience, we denote $[\emptyset) = \{0\}$.

Proposition 4.15. *Let A and B be two subsets of X. Then the following statements hold:*

(i) $[0) = \{0\}, \ [X) = X$,

(ii) $A \subseteq B$ *implies* $[A) \subseteq [B)$,

(iii) *if* $I \in I(X)$, *then* $[I) = I$.

5 Congruence relations in BI-algebras

Let I be a non-empty set of X. Define a binary relation " $\sim_I$ " by

$$x \sim_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

The set $\{y : x \sim_I y\}$ will be denoted by $[x]_I$.

Theorem 5.1. *Let I be an ideal of a right distributive BI-algebra X. Then " $\sim_I$ " is an equivalence relation on X.*

Proof. Since I is an ideal of X, we have $x * x = 0 \in I$. Thus $x \sim_I x$. So, $\sim_I$ is reflexive. It is obvious that $\sim_I$ is symmetric. Now, let $x \sim_I y$ and $y \sim_I z$. Then $x * y, \ y * x \in I$ and $y * z, \ z * y \in I$. By Proposition 3.12(iii), we have $(x * z) * (y * z) \leq x * y$. Since I is an ideal and $x * y \in I$, we have $(x*z)*(y*z) \in X$ and so $x*z \in I$. Similarly, we obtain $z*x \in I$. Thus $x \sim_I z$ and so $\sim_I$ is a transitive relation. Therefore $\sim_I$ is an equivalence relation on X. ☐

Recall that a binary relation "θ" on an algebra $(X; *)$ is said to be

(i) a *right compatible relation* if $x\theta y$ and $u \in X$, then $(x * u)\theta(y * u)$,

(ii) a *left compatible relation* if $x\theta y$ and $v \in X$, then $(v * x)\theta(v * y)$,

(iii) a *compatible relation* if $x\theta y$ and $u\theta v$, then $(x * u)\theta(y * v)$.

A compatible equivalence relation on X is called a *congruence relation* on X.

Theorem 5.2. *The equivalence relation "$\sim_I$" in Theorem 5.1 is a right congruence relation on X.*

Proof. If $x \sim_I y$ and $u \in X$, then $x * y$ and $y * x \in I$. By Proposition 3.12(iii), we have $((x * u) * (y * u)) * (x * y) = 0 \in I$. Since I is an ideal and $x * y \in I$, we have $(x * u) * (y * u) \in I$. Similarly we obtain $(y * u) * (x * u) \in I$. Therefore $(x * u) \sim_I (y * u)$. □

Example 5.3. In Example 3.10(i), $I = \{0, a\}$ is an ideal of X and

$$\sim_I := \{(0,0),(a,a),(0,a),(a,0),(0,b),(b,0),(b,b),(c,b),(b,c),(c,0),(0,c),(c,c)\}$$

is a right congruence relation on X and

$$[0]_I = [a]_I = \{0, a\} \text{ and } [b]_I = [c]_I = \{0, a, b, c\}.$$

Proposition 5.4. *Let I be a subset of X with $0 \in I$. If I has the condition: if $x * y \in I$, then $(z * x) * (z * y) \in I$. Then $X = I$.*

Proof. Let $x := 0$ and $y := z$. Then $0 * z = 0 \in I$ imply $(z * 0) * (z * z) = z * 0 = z \in I$. Therefore $X \subseteq I$ and so $I = X$. □

Proposition 5.5. *Let X be a right distributive BI-algebra and let $I, J \subseteq X$.*

(i) If $I \subseteq J$, then $\sim_I \subseteq \sim_J$,

(ii) If $\sim_{I_i}$ for all $i \in \Lambda$ are right congruence relations on X, then $\sim_{\cap I_i}$ is also a right congruence relation on X.

Lemma 5.6. *If $\sim_I$ is a left congruence relation on a right distributive BI-algebra X, then $[0]_I$ is an ideal of X.*

Proof. Obviously, $0 \in [0]_I$. If y and $x * y$ are in $[0]_I$, then $x * y \sim_I 0$ and $y \sim_I 0$. It follows that $x = x * 0 \sim_I x * y \sim_I 0$. Therefore $x \in [0]_I$. □

Proposition 5.7. *Let X be a right distributive BI-algebra. Then*

$$\phi_x := \{(a, b) \in X \times X : x * a = x * b\}$$

is a right congruence relation on X.

Proof. Straightforward. □

Example 5.8. In Example 3.10(i),

$$\phi_b = \{(0,0),(0,a),(a,0),(a,a),(b,b),(c,c),(b,c),(c,b)\}$$

is a right congruence relation on X.

Proposition 5.9. *Let X be a BI-algebra. Then*

(i) $\phi_0 = X \times X$,

(ii) $\phi_x \subseteq \phi_0$,

(iii) if X is right distributive, then $\phi_x \cap \phi_y \subseteq \phi_{x*y}$,

for all $x, y \in X$.

6 Conclusion and future work

Recently, researchers proposed several kinds of algebraic structures related to some axioms in many-valued logic and several papers have been published in this field.

In this paper, we introduced a new algebra which is a generalization of a (dual) implication algebra, and we discussed the basic properties of BI-algebras, and investigated ideals and congruence relations. We hope the results can be a foundation for future works.

As future works, we shall define commutative BI-algebras and discuss on some relationships between other several algebraic structures. Also, we intend to study other kinds of ideals, and apply vague sets, soft sets, fuzzy structures to BI-algebras.

Acknowledgments: The authors wish to thank the reviewers for their excellent suggestions that have been incorporated into this paper.

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