DOI: 10.1515/auom-2016-0051
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 24(3),2016, 123–139

Ostrowski Via a Two Functions Pompeiu's Inequality

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Abstract

In this paper, some generalizations of Pompeiu's inequality for two complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type results. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

1 Introduction

In 1946, Pompeiu [6] derived a variant of *Lagrange's mean value theorem*, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1}$$

The following inequality is useful to derive some Ostrowski type inequalities.

Key Words: Ostrowski's inequality, Pompeiu's mean value theorem, Integral inequalities, Schwarz inequality
2010 Mathematics Subject Classification: Primary 26D15; Secondary 26D10
Received: 23.03.2015
Revised: 10.07.2015
Accepted: 21.07.2015

Corollary 1 (Pompeiu's Inequality). *With the assumptions of Theorem 1 and if* $\|f-\ell f'\|_{\infty}=\sup_{t\in(a,b)}|f(t)-tf'(t)|<\infty$ *where* $\ell(t)=t,\ t\in[a,b]$, *then*

$$|tf(x)-xf(t)|\le\|f-\ell f'\|_{\infty}|x-t| \tag{2}$$

for any $t,x\in[a,b]$.

The inequality (2) was stated by the author in [3].

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 2 (Ostrowski, 1938 [4]). *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $|f'(t)|\le M<\infty$ *for all* $t\in(a,b)$. *Then for any* $x\in[a,b]$, *we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a). \tag{3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 3 (Dragomir, 2005 [3]). *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $[a,b]$ *not containing* 0. *Then for any* $x\in[a,b]$, *we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4}$$
$$\le\frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_{\infty},$$

where $\ell(t)=t,\ t\in[a,b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

In [7], E. C. Popa using a mean value theorem obtained a generalization of (4) as follows:

Theorem 4 (Popa, 2007 [7]). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Assume that $\alpha \notin [a, b]$. Then for any $x \in [a, b]$, we have the inequality*

$$\left|\left(\frac{a+b}{2}-\alpha\right) f(x)+\frac{\alpha-x}{b-a} \int_a^b f(t)\, dt\right| \leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f-\ell_\alpha f'\right\|_{\infty}, \tag{5}$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \leq p \leq \infty$ which for $p = \infty$ gives Dragomir's result.

Theorem 5 (Pečarić & Ungar, 2006 [5]). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality*

$$\left|\frac{a+b}{2} \cdot \frac{f(x)}{x}-\frac{1}{b-a} \int_a^b f(t)\, dt\right| \leq PU(x, p)\left\|f-\ell f'\right\|_p, \tag{6}$$

for $x \in [a, b]$, where

$$\begin{aligned} PU(x, p) \quad:\quad =&\ (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q} x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right. \\ & \left.+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q} x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right]. \end{aligned}$$

In the cases $(p, q) = (1, \infty), (\infty, 1)$ and $(2, 2)$ the quantity $PU(x, p)$ has to be taken as the limit as $p \to 1, \infty$ and 2, respectively.

For other inequalities in terms of the p-norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$ and $\alpha \notin [a, b]$ see [1] and [2].

In this paper, some new Pompeiu's type inequalities for two complex-valued absolutely continuous functions are provided. They are applied to obtain some Ostrowski type inequalities. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

2 A General Pompeiu's Inequality

We start with the following generalization of (2).

Theorem 6. *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t) \neq 0$ for all $t \in [a,b]$. Then for any $t, x \in [a,b]$ we have*

$$\left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| \tag{7}$$
$$\leq \begin{cases} \|f'g - fg'\|_{\infty} \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_{\infty}[a,b], \\ \\ \|f'g - fg'\|_{p} \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \\ \|f'g - fg'\|_{1} \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} & \end{cases}$$

or, equivalently

$$|g(t) f(x) - f(t) g(x)| \tag{8}$$
$$\leq \begin{cases} \|f'g - fg'\|_{\infty} |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_{\infty}[a,b], \\ \\ \|f'g - fg'\|_{p} |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \\ \|f'g - fg'\|_{1} |g(t) g(x)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\}. & \end{cases}$$

Proof. If f and g are absolutely continuous and $g(t) \neq 0$ for all $t \in [a,b]$, then f/g is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}$$

for any $t, x \in [a,b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds,$$

then we get the following identity

$$\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} = \int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds \tag{9}$$

for any $t, x \in [a, b]$.

Taking the modulus in (9) we have

$$\left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| = \left| \int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds \right| \tag{10}$$
$$\leq \left| \int_t^x \frac{|f'(s) g(s) - f(s) g'(s)|}{|g(s)|^2} ds \right| := I$$

and utilizing Hölder's integral inequality we deduce

$$I \leq \begin{cases} \sup_{s \in [t,x]([x,t])} |f'(s) g(s) - f(s) g'(s)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \\ \left| \int_t^x |f'(s) g(s) - f(s) g'(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \\ \left| \int_t^x |f'(s) g(s) - f(s) g'(s)| ds \right| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\}, \end{cases}$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \\ \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \\ \|f'g - fg'\|_1 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \end{cases}$$

and the inequality (7) is proved. □

The following particular case extends Pompeiu's inequality to other p-norms than $p = \infty$ obtained in (2).

Corollary 2. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $t, x \in [a, b]$ we have*

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|$$

$$\leq\begin{cases}\|f-\ell f'\|_{\infty}\left|\frac{1}{t}-\frac{1}{x}\right| & \textit{if } f-\ell f'\in L_{\infty}[a,b],\\ \frac{1}{(2q-1)^{1/q}}\|f-\ell f'\|_{p}\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} & \begin{array}{c}\textit{if } f-\ell f'\in L_{p}[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \|f-\ell f'\|_{1}\frac{1}{\min\{t^2,x^2\}}\end{cases}\tag{11}$$

or, equivalently

$$|tf(x)-xf(t)|$$

$$\leq\begin{cases}\|f-\ell f'\|_{\infty}|x-t| & \textit{if } f-\ell f'\in L_{\infty}[a,b],\\ \frac{1}{(2q-1)^{1/q}}\|f-\ell f'\|_{p}\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{l}\textit{if } f-\ell f'\in L_{p}[a,b]\\ p>1, \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \|f-\ell f'\|_{1}\frac{\max\{t,x\}}{\min\{t,x\}},\end{cases}\tag{12}$$

where $\ell(t)=t, t\in[a,b]$.

The proof follows by (7) for $g(t)=\ell(t)=t$, $t\in[a,b]$.
The general case for power functions is as follows.

Corollary 3. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b>a>0$. *If* $r\in\mathbb{R}$, $r\neq0$, *then for any* $t,x\in[a,b]$ *we have*

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right|$$

$$\leq\begin{cases}\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\left|\frac{1}{x^r}-\frac{1}{t^r}\right|,\ \text{if } f'\ell-rf\in L_\infty[a,b],\\[2ex] \left\|f'\ell-rf\right\|_p\\ \quad\times\begin{cases}\frac{1}{|1-q(r+1)|^{1/q}}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|^{1/q},\ \text{for } r\neq-\frac{1}{p}\\[2ex] \left|\ln x-\ln t\right|^{1/q},\ \text{for } r=-\frac{1}{p}\end{cases}\\ \quad\text{if } f'\ell-rf\in L_p[a,b],\\[2ex] \left\|f'\ell-rf\right\|_1\frac{1}{\min\{x^{r+1},t^{r+1}\}},\end{cases}\tag{13}$$

or, equivalently

$$\left|t^rf(x)-x^rf(t)\right|$$

$$\leq\begin{cases}\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\left|t^r-x^r\right|,\text{if } f'\ell-rf\in L_\infty[a,b],\\[2ex] \left\|f'\ell-rf\right\|_p\\ \quad\times\begin{cases}\frac{t^rx^r}{|1-q(r+1)|^{1/q}}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|^{1/q},\ \text{for } r\neq-\frac{1}{p}\\[2ex] t^rx^r\left|\ln x-\ln t\right|^{1/q},\ \text{for } r=-\frac{1}{p}\end{cases}\\ \quad\text{if } f'\ell-rf\in L_p[a,b],\\[2ex] \left\|f'\ell-rf\right\|_1\frac{t^rx^r}{\min\{x^{r+1},t^{r+1}\}},\end{cases}\tag{14}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

The proof follows by (7) for $g(t)=t^r$, $t\in[a,b]$. The details for calculations are omitted.

We have the following result for exponential.

Corollary 4. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the* *interval* $[a,b]$ *and* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Then for any* $t, x \in [a,b]$ *we have*

$$\left|\frac{f(x)}{\exp(i\alpha x)} - \frac{f(t)}{\exp(i\alpha t)}\right| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x-t| & \text{if } f' - i\alpha f \in L_{\infty}[a,b], \\ \|f' - i\alpha f\|_{p} |x-t|^{1/q} & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \|f' - i\alpha f\|_{1} \end{cases} \tag{15}$$

or, equivalently

$$|\exp(i\alpha t) f(x) - f(t)\exp(i\alpha x)| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x-t| & \text{if } f' - i\alpha f \in L_{\infty}[a,b], \\ \|f' - i\alpha f\|_{p} |x-t|^{1/q} & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \|f' - i\alpha f\|_{1}. \end{cases} \tag{16}$$

3 An Inequality Generalizing Ostrowski's

The following result holds:

Theorem 7. *Let* $f, g : [a,b] \to \mathbb{C}$ *be absolutely continuous functions on the interval* $[a,b]$. *If* $0 < m \leq |g(t)| \leq M < \infty$ *for any* $t \in [a,b]$, *then*

$$\left|f(x)\int_a^b g(t)\,dt - g(x)\int_a^b f(t)\,dt\right|$$

$$\leq \left(\frac{M}{m}\right)^2 \begin{cases} \|f'g-fg'\|_\infty (b-a)^2\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] & \textit{if } f'g-fg'\in L_\infty[a,b], \\ \\ \|f'g-fg'\|_p\left[\frac{(b-x)^{1+1/q}+(x-a)^{1+1/q}}{1+1/q}\right] & \begin{array}{c}\textit{if } f'g-fg'\in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1,\end{array} \\ \\ \|f'g-fg'\|_1 (b-a) \end{cases} \tag{17}$$

for any $x\in[a,b]$.

Proof. Utilizing (8) we have

$$\left|f(x)\int_a^b g(t)\,dt - g(x)\int_a^b f(t)\,dt\right|$$

$$\leq \int_a^b |g(t)f(x)-f(t)g(x)|\,dt$$

$$\leq \begin{cases} \|f'g-fg'\|_\infty |g(x)|\int_a^b\left(|g(t)|\left|\int_t^x \frac{1}{|g(s)|^2}ds\right|\right)dt, \\ \\ \|f'g-fg'\|_p |g(x)|\int_a^b\left(|g(t)|\left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{1/q}\right)dt, \\ \\ \|f'g-fg'\|_1 |g(x)|\int_a^b\left(|g(t)|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}\right)dt \end{cases} \tag{18}$$

for any $x\in[a,b]$, which is of interest in itself.

Since $0 < m \le |g(t)| \le M < \infty$ for any $t \in [a,b]$, then

$$|g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt \le \left(\frac{M}{m} \right)^2 \int_a^b |x-t|\, dt$$
$$= \left(\frac{M}{m} \right)^2 \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right],$$

$$|g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt$$
$$\le \left(\frac{M}{m} \right)^2 \int_a^b |x-t|^{1/q}\, dt = \left(\frac{M}{m} \right)^2 \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}$$

and

$$|g(x)| \int_a^b \left(|g(t)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \le \left(\frac{M}{m} \right)^2 \int_a^b dt = \left(\frac{M}{m} \right)^2 (b-a)$$

for any $x \in [a,b]$ and by (18) we get the desired result (17). □

Remark 1. If we take $g(t) = 1, t \in [a,b]$ in the first inequality (17) we recapture Ostrowski's inequality.

Corollary 5. *With the assumptions in Theorem 7 we have the midpoint inequalities*

$$\left| f\left(\frac{a+b}{2} \right) \int_a^b g(t)\, dt - g\left(\frac{a+b}{2} \right) \int_a^b f(t)\, dt \right|$$
$$\le \left(\frac{M}{m} \right)^2 \begin{cases} \frac{1}{4}(b-a)^2 \|f'g - fg'\|_\infty & \text{if } f'g - fg' \in L_\infty[a,b], \\ \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f'g - fg'\|_p & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases} \tag{19}$$

The following result also holds:

Theorem 8. *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$, $g(x) \neq 0$ for $x \in [a, b]$ and $g^{-2} \in L_\infty [a, b]$. Then*

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t)\, dt - \int_a^b f(t)\, dt \right| \tag{20}$$
$$\leq \left\| g^{-2} \right\|_\infty \times \begin{cases} \| f'g - fg' \|_\infty \int_a^b |g(t)| \, |x - t| \, dt, & \text{if } f'g - fg' \in L_\infty [a, b], \\ \| f'g - fg' \|_p \int_a^b |g(t)| \, |x - t|^{1/q} \, dt & \begin{array}{c} \text{if } f'g - fg' \in L_p [a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \| f'g - fg' \|_1 \int_a^b |g(t)| \, dt \end{cases}$$

for any $x \in [a, b]$.

Proof. Utilizing (8) we have

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t)\, dt - \int_a^b f(t)\, dt \right|$$
$$\leq \begin{cases} \| f'g - fg' \|_\infty \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \| f'g - fg' \|_p \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \| f'g - fg' \|_1 \int_a^b \left(|g(t)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases} \tag{21}$$

for any $x \in [a, b]$.

Since

$$\left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \leq \left\| g^{-2} \right\|_\infty |x - t|,$$
$$\left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \leq \left\| g^{-2} \right\|_\infty |x - t|^{1/q}$$

and

$$\sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \leq \left\| g^{-2} \right\|_\infty$$

for any $x, t \in [a, b]$, then on making use of (21) we get the desired result (20). $\square$

We have the midpoint inequalities:

Corollary 6. *With the assumptions of Theorem 8 we have*

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\int_a^b g(t)\,dt - \int_a^b f(t)\,dt\right|$$

$$\leq \left\|g^{-2}\right\|_\infty \times \begin{cases} \left\|f'g - fg'\right\|_\infty \int_a^b |g(t)| \left|\frac{a+b}{2} - t\right| dt, & \text{if } f'g - fg' \in L_\infty[a,b], \\ \\ \left\|f'g - fg'\right\|_p \int_a^b |g(t)| \left|\frac{a+b}{2} - t\right|^{1/q} dt & \begin{array}{c}\text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1.\end{array} \end{cases} \tag{22}$$

We have the following exponential version of Ostrowski's inequality as well:

Theorem 9. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then for any $x \in [a, b]$ we have*

$$\left|\frac{\exp(i\alpha(b-x)) - \exp(-i\alpha(x-a))}{i\alpha} f(x) - \int_a^b f(t)\,dt\right|$$

$$\leq \begin{cases} \left\|f' - i\alpha f\right\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right], & \text{if } f' - i\alpha f \in L_\infty[a,b], \\ \\ \left\|f' - i\alpha f\right\|_p \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}, & \begin{array}{c}\text{if } f' - i\alpha f \\ \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1,\end{array} \\ \\ \left\|f' - i\alpha f\right\|_1. \end{cases} \tag{23}$$

Proof. If we write the inequality (18) for $g(t) = \exp(i\alpha t)$, $t \in [a,b]$, then we get

$$\left| f(x) \int_a^b \exp(i\alpha t)\, dt - \exp(i\alpha x) \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty \int_a^b |x-t|\, dt, & \text{if } f' - i\alpha f \in L_\infty[a,b] \\ \\ \|f' - i\alpha f\|_p |g(x)| \int_a^b |x-t|^{1/q}\, dt, & \begin{array}{c} \text{if } f' - i\alpha f \\ \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \\ \|f' - i\alpha f\|_1, \end{cases}$$

which, after simple calculation, is equivalent with (23).

The details are omitted. □

Corollary 7. *With the assumptions of Theorem 9 we have the midpoint inequalities*

$$\left| \frac{\exp\left(i\alpha\left(\frac{b-a}{2}\right)\right) - \exp\left(-i\alpha\left(\frac{b-a}{2}\right)\right)}{i\alpha} f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \frac{1}{4} \|f' - i\alpha f\|_\infty (b-a)^2, & \textit{if } f' - i\alpha f \in L_\infty[a,b], \\ \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f' - i\alpha f\|_p, & \begin{array}{l} \textit{if } f' - i\alpha f \in L_p[a,b] \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases} \tag{24}$$

or, equivalently

$$\left|\frac{2\sin\left(\alpha\left(\frac{b-a}{2}\right)\right)}{\alpha}f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|$$

$$\leq\begin{cases}\frac{1}{4}\left\|f'-i\alpha f\right\|_\infty(b-a)^2, & \text{if } f'-i\alpha f\in L_\infty[a,b],\\ \\ \frac{1}{2^{1/q}(1+1/q)}(b-a)^{1+1/q}\left\|f'-i\alpha f\right\|_p, & \begin{array}{l}\text{if } f'-i\alpha f\in L_p[a,b]\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases}\tag{25}$$

4 An Application for CBS-Inequality

The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$\left|\int_a^b f(t)\,g(t)\,dt\right|^2\leq\int_a^b|f(t)|^2\,dt\int_a^b|g(t)|^2\,dt,\tag{26}$$

provided that $f,g\in L_2[a,b]$.

We have the following result concerning some reverses of the CBS-inequality:

Theorem 10. *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t)\neq0$ for all $t\in[a,b]$. Then*

$$0\leq\int_a^b|g(t)|^2\,dt\int_a^b|f(t)|^2\,dt-\left|\int_a^b f(t)\,g(t)\,dt\right|^2$$

$$\leq\frac{1}{2}\times\begin{cases}\left\|f'\overline{g}-f\overline{g}'\right\|_\infty^2\left(\int_a^b|g(t)|^2\,dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^2}dt\right)^2, & \text{if }\begin{array}{c}f'\overline{g}-f\overline{g}'\in L_\infty[a,b],\\ \frac{1}{|g|^2}\in L[a,b]\end{array}\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_p^2\left(\int_a^b|g(t)|^2\,dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^{2q}}dt\right)^{2/q}, & \text{if }\begin{array}{c}f'\overline{g}-f\overline{g}'\in L_p[a,b],\\ \frac{1}{|g|^{2q}}\in L[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_1^2\left(\int_a^b|g(t)|^2\,dt\right)^2 ess\sup_{t\in[a,b]}\left\{\frac{1}{|g(t)|^4}\right\}, & \text{if } \frac{1}{|g|}\in L_\infty[a,b].\end{cases}\tag{27}$$

Proof. Utilising the inequality (8) we have

$$\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|$$

$$\leq\begin{cases}\left\|f'\overline{g}-f\overline{g}'\right\|_{\infty}\left|g(t)g(x)\right|\left|\int_{t}^{x}\frac{1}{|g(s)|^{2}}ds\right| & \text{if } f'\overline{g}-f\overline{g}' \in L_{\infty}[a,b],\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_{p}\left|g(t)g(x)\right|\left|\int_{t}^{x}\frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \text{if } f'\overline{g}-f\overline{g}' \in L_{p}[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_{1}\left|g(t)g(x)\right|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^{2}}\right\}.\end{cases}\tag{28}$$

for any $t,x\in[a,b]$.

Taking the square in (28) and integrating over $(t,x)\in[a,b]^{2}$ we have

$$\int_{a}^{b}\int_{a}^{b}\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|^{2}dtdx$$

$$\leq\begin{cases}\left\|f'\overline{g}-f\overline{g}'\right\|_{\infty}^{2}\int_{a}^{b}\int_{a}^{b}\left|g(t)g(x)\right|^{2}\left|\int_{t}^{x}\frac{1}{|g(s)|^{2}}ds\right|^{2}dtdx,\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_{p}^{2}\int_{a}^{b}\int_{a}^{b}\left|g(t)g(x)\right|^{2}\left|\int_{t}^{x}\frac{1}{|g(s)|^{2q}}ds\right|^{2/q}dtdx,\\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_{1}^{2}\int_{a}^{b}\int_{a}^{b}\left|g(t)g(x)\right|^{2}\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^{4}}\right\}dtdx.\end{cases}\tag{29}$$

Observe that

$$\int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 dtdx$$
$$= \int_a^b \int_a^b \left(|g(t)|^2 |f(x)|^2 - 2Re \left[\overline{g(t)} f(x) \overline{f(t) \overline{g(x)}} \right] + |g(x)|^2 |f(t)|^2 \right) dtdx$$
$$= \int_a^b |g(t)|^2 dt \int_a^b |f(x)|^2 dx - 2Re \left[\int_a^b \overline{f(t) g(t)} dt \int_a^b f(x) g(x) dx \right]$$
$$+ \int_a^b |g(x)|^2 dx \int_a^b |f(t)|^2 dt$$
$$= 2 \left[\int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \right],$$

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 \right] dtdx$$
$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^2} dt \right)^2,$$

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dtdx$$
$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}$$

and

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dtdx$$
$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 ess \sup_{t \in [a,b]} \left\{ \frac{1}{|g(t)|^4} \right\},$$

then by (29) we get the desired result (27). □

Acknowledgement. The authors would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

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