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Equivalence of Control Systems on the Pseudo-Orthogonal Group $\mathsf{SO}\,(2,1)_0$

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Abstract

We consider left-invariant control affine systems on the matrix Lie group $\mathsf{SO}\,(2,1)_0$. A classification, under state space equivalence, of all such full-rank control systems is obtained. First, we identify certain subsets on which the group of Lie algebra automorphisms act transitively. We then systematically identify equivalence class representatives (for single-input, two-input and three-input control systems). A brief comparison of these classification results with existing results concludes the paper.

1 Introduction

From a geometric viewpoint, a (smooth) control system is given by a family of (smooth) vector fields parametrized by controls. An admissible trajectory of such a system, associated to a piecewise-constant control, is an integral curve of some vector field of the family or a finite concatenation of such curves. The arbitrary admissible control case can be realized via an approximation by piecewise-constant controls. Invariant control systems are control systems evolving on (real, finite dimensional) Lie groups with dynamics invariant under translations. Such systems were first considered in 1972 by Brockett [12] and by Jurdjevic and Sussmann [17]. For more details about (invariant) control systems see, e.g., [5], [16], [24], [6], [22].

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In order to understand the local geometry of control systems, one needs to introduce some natural equivalence relations. The most natural equivalence relation for control systems is equivalence up to coordinate changes in the state space. This is called state space equivalence (cf. [15], [10]). Two control systems are state space equivalent if they are related by a diffeomorphism (in which case their trajectories, corresponding to the same controls, are also related by that diffeomorphism). This equivalence relation is very strong. Consequently, there are so many equivalence classes that any general classification appears to be very difficult if not impossible. However, there is a chance for some reasonable classification in low dimensions. Another important equivalence relation for control systems is that of feedback equivalence (see, e.g., [23], [15]). Two feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls.

A systematic investigation of state space equivalence and feedback equivalence, in the context of left-invariant control systems, was recently carried out [10]. Incidentally, an appropriate specialization of feedback equivalence, called *detached feedback equivalence*, was also introduced. A classification, under state space equivalence, of invariant control systems evolving on the Euclidean group $\mathsf{SE}(2)$ was obtained in [2]. Classifications, under detached feedback equivalence, of various distinguished subclasses of invariant control systems have also been obtained in recent years (see, e.g., [7], [8], [9], [3], [1], [4]). Furthermore, an investigation of the equivalence of cost-extended control systems has been carried out in [11].

In this paper we consider only left-invariant control affine systems, evolving on a particular group, the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$. We classify, under state space equivalence, all such *full-rank* control systems. Moreover, a representative for each equivalence class is identified in a systematic manner. A tabulation of these results is appended. Several problems related to control systems on $\mathsf{SO}(2,1)_0$ (like controllability, stability, explicit integration by elliptic functions, numerical integration, and the existence of periodic solutions) have been considered in recent years (see [20], [19], [21], [13]).

2 Invariant control systems and equivalence

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) matrix Lie group and the parametrization map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an affine injection (i.e., $B_1, \ldots, B_\ell$ are linearly independent). The admissible controls are piecewise-continuous maps $u(\cdot)$:

$[0,T] \to \mathbb{R}^\ell$ and the trace of the system $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell\rangle$ is an affine subspace of (the Lie algebra) $\mathfrak{g}$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ has *full rank* provided the Lie algebra generated by its trace equals the whole Lie algebra $\mathfrak{g}$. Note that Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1}, \cdot)$. Hence, for a fixed G, we shall specify Σ by simply writing

$$\Sigma \,:\, A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

If the state space G of Σ is a three-dimensional matrix Lie group, then the condition that Σ has full rank can be characterized as follows. No homogeneous single-input system has full rank. An inhomogeneous single-input system has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent, whereas a homogeneous two-input system has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Also, it is clear that any inhomogeneous two-input or (homogeneous) three-input system has full rank. Henceforth we assume that all systems under consideration have full rank.

State space equivalence is well understood (cf. [5], [15]); it establishes a one-to-one correspondence between the trajectories of equivalent systems. Let G be a fixed connected matrix Lie group and let Σ and Σ' be two (left-invariant control affine) systems on G. We say that Σ and Σ' are *state space equivalent* if there exist a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), u)$ for all $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$.

In this paper we shall refer to state space equivalence, simply, as equivalence. We recall an algebraic characterization of this equivalence.

Proposition 1 ([10])**.** *Systems Σ and Σ' are equivalent if and only if there exists a Lie algebra automorphism $\psi \in d\,\mathsf{Aut}(\mathsf{G})$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in \mathbb{R}^\ell$.*

Here $d\,\mathsf{Aut}(\mathsf{G}) = \{T_{\mathbf{1}}\phi \,:\, \phi \in \mathsf{Aut}(\mathsf{G})\}$ is the subgroup of Lie algebra automorphisms, containing only linearized Lie group automorphisms.

It turns out that a classification of the $(\ell+1)$-input homogeneous systems may be (partially) obtained from a classification of the ℓ-input inhomogeneous systems. Suppose $\{A^i + u_1 B_1^i + \cdots + u_\ell B_\ell^i \,:\, i \in I\}$ is an exhaustive collection of equivalence class representatives for ℓ-input inhomogeneous systems.

Lemma. *If $\Sigma :\, A + u_1 B_1 + \cdots + u_{\ell+1} B_{\ell+1}$ is a $(\ell+1)$-input homogeneous system, then Σ is equivalent to*

$$\widehat{\Sigma}_{i,\boldsymbol{\gamma}} \;:\; \gamma_1 B_1^i + \cdots + \gamma_\ell B_\ell^i + \gamma_{\ell+1} A^i + u_1 B_1^i + \cdots + u_\ell B_\ell^i + u_{\ell+1} A^i$$

for some $i \in I$ and some $\gamma_1, \ldots, \gamma_{\ell+1} \in \mathbb{R}$.

Proof. $\Sigma' : B_{\ell+1} + u_1 B_1 + \cdots + u_\ell B_\ell$ is a ℓ-input inhomogeneous system. Thus (by proposition 1) there exists an automorphism $\psi \in d\,\mathsf{Aut}(\mathsf{G})$ such $\psi \cdot B_{\ell+1} = A^i$ and $\psi \cdot B_j = B^i_j$, $1 \leq j \leq \ell$ for some $i \in I$. Therefore Σ is state space equivalent to $\Sigma'' : \psi \cdot A + u_1 B^i_1 + \cdots + u_\ell B^i_\ell + u_{\ell+1} A^i$. However, as Σ is homogeneous, so is Σ''. Hence $\psi \cdot A$ is a linear combination of $B^i_1, \ldots, B^i_\ell, A^i$, i.e., $\Sigma'' = \widehat{\Sigma}_{i,\gamma}$. □

Accordingly, $\{\widehat{\Sigma}_{i,\gamma} : i \in I, \gamma_1, \ldots, \gamma_{\ell+1} \in \mathbb{R}\}$ is an exhaustive collection of equivalence class representatives for $(\ell+1)$-input inhomogeneous systems. However, some of these systems may be equivalent to one another.

3 The pseudo-orthogonal group $\mathsf{SO}(2,1)_0$

The pseudo-orthogonal group

$$\mathsf{SO}(2,1) = \{g \in \mathbb{R}^{3\times 3} : g^\top J g = J, \det g = 1\}$$

is a three-dimensional simple Lie group. Here $J = \operatorname{diag}(1,1,-1)$. The identity component of $\mathsf{SO}(2,1)$ is $\mathsf{SO}(2,1)_0 = \{g \in \mathsf{SO}(2,1) : g_{33} > 0\}$. Its Lie algebra

$$\mathfrak{so}(2,1) = \{A \in \mathbb{R}^{3\times 3} : A^\top J + JA = 0\}$$

has an ordered basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutation operation is given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = -E_3$. The group $\mathsf{Aut}(\mathfrak{so}(2,1))$ of automorphisms of $\mathfrak{so}(2,1)$ is exactly $\mathsf{SO}(2,1)$. Also, the group $\mathsf{Inn}(\mathfrak{so}(2,1))$ of inner automorphisms of $\mathfrak{so}(2,1)$ is exactly $\mathsf{SO}(2,1)_0$ (cf. [18]). (Here each automorphism ψ is identified with its corresponding matrix g with respect to the chosen basis.) We have that $\mathsf{SO}(2,1)$ is generated by

$$\rho_2(t) = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix} \qquad \rho_3(t) = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\eta(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 & t & \frac{1}{2}t^2 \\ -t & 1 & t \\ -\frac{1}{2}t^2 & t & 1 + \frac{1}{2}t^2 \end{bmatrix} \qquad \varsigma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Remark. $\rho_2(t) = \exp(tE_2)$, $\rho_3(t) = \exp(tE_3)$, and $\eta(t) = \exp(t(E_1 + E_3))$. Also, $\rho_2(t), \rho_3(t), \eta(t) \in \mathsf{Inn}(\mathfrak{so}(2,1))$, whereas $\varsigma \notin \mathsf{Inn}(\mathfrak{so}(2,1))$.

Proposition 2. *The map* $d : \mathsf{Aut}(\mathsf{SO}(2,1)_0) \to \mathsf{Aut}(\mathfrak{so}(2,1))$, $\phi \mapsto T_{\mathbf{1}}\phi$ *is bijective.*

Proof. As $\mathsf{SO}(2,1)_0$ is connected, d is injective (see, e.g., [14]). Furthermore, as $\rho_2(t), \rho_3(t), \eta(t) \in \mathsf{Inn}(\mathfrak{so}(2,1))$ and the elements $\rho_2(t)$, $\rho_3(t)$, $\eta(t)$, and ς generate $\mathsf{SO}(2,1) = \mathsf{Aut}(\mathfrak{so}(2,1))$, it suffices to show that $\varsigma \in d\,\mathsf{Aut}(\mathsf{SO}(2,1)_0)$. Let $\phi : \mathsf{SO}(2,1)_0 \to \mathsf{SO}(2,1)_0$, $g \mapsto \varsigma g \varsigma$. We claim that ϕ is a Lie group automorphism such that $T_{\mathbf{1}}\phi = \varsigma$. Let $g \in \mathsf{SO}(2,1)_0$. Now $(\varsigma g \varsigma)^\top J (\varsigma g \varsigma) = \varsigma g^\top J g \varsigma = J$ and $\det(\varsigma g \varsigma) = \det \varsigma^2 \det g = 1$. Thus $\phi(g) \in \mathsf{SO}(2,1)$. Furthermore, the entry of the third column, third row of g is fixed by ϕ. Thus $\phi(g) \in \mathsf{SO}(2,1)_0$. As $\phi \circ \phi$ is the identity map on $\mathsf{SO}(2,1)_0$, it follows that ϕ is bijective. Also, $\phi(gh) = \varsigma gh \varsigma = \varsigma g \varsigma \varsigma h \varsigma = \phi(g)\phi(h)$. Finally, a simple calculation shows that $\phi(\exp(tE_1)) = \exp(\varsigma \cdot tE_1)$, $\phi(\exp(tE_2)) = \exp(\varsigma \cdot tE_2)$, and $\phi(\exp(tE_3)) = \exp(\varsigma \cdot tE_3)$. Thus $T_{\mathbf{1}}\phi = \varsigma$. □

The (Lorentzian) product $\odot$ on $\mathfrak{so}(2,1)$ is given by $A \odot B = a_1 b_1 + a_2 b_2 - a_3 b_3$. Here $A = \sum_{i=1}^3 a_i E_i$ and $B = \sum_{i=1}^3 b_i E_i$. Any automorphism ψ preserves $\odot$, i.e., $(\psi \cdot A) \odot (\psi \cdot B) = A \odot B$. Consider the level sets $\mathcal{H}_\alpha = \{A \in \mathfrak{so}(2,1) : A \odot A = \alpha,\ A \neq 0\}$. $\mathcal{H}_\alpha$ is a hyperboloid of two sheets when $\alpha < 0$, a hyperboloid of one sheet when $\alpha > 0$, and a (punctured) cone when $\alpha = 0$. As $\odot$ is preserved by automorphisms, each level set $\mathcal{H}_\alpha$ is also preserved. Moreover,

Proposition 3. *The group* $\mathsf{Aut}(\mathfrak{so}(2,1))$ *acts transitively on each level set* $\mathcal{H}_\alpha$.

Hence, for every $A \in \mathfrak{so}(2,1)$, there exists $\psi \in \mathsf{Aut}(\mathfrak{so}(2,1))$ such that $\psi \cdot A$ equals αE_2, αE_3, or $E_1 + E_3$ for some $\alpha > 0$. We now consider the subgroups of automorphisms fixing these respective vectors.

Theorem 1.

(i) The subgroup of $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* E_2 *is* $\{\rho_2(t), \varsigma \circ \rho_2(t) : t \in \mathbb{R}\}$.

(ii) The subgroup of $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* E_3 *is* $\{\rho_3(t) : t \in \mathbb{R}\}$.

(iii) The subgroup of $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* $E_1 + E_3$ *is* $\{\eta(t) : t \in \mathbb{R}\}$.

Proof. Let $\psi \in \mathsf{Aut}(\mathfrak{so}(2,1))$ and let

$$\psi = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Suppose $\psi \cdot E_2 = E_2$. Then $a_2 = c_2 = 0$ and $b_2 = 1$. The conditions $\psi^\top J \psi = J$ and $\det \psi = 1$ then yield $b_1 = b_3 = 0$ and $\begin{bmatrix} a_1 & a_3 \\ c_1 & c_3 \end{bmatrix} \in \mathsf{SO}(1,1)$.

Therefore $\psi = \rho_2(t)$ or $\psi = \varsigma \circ \rho_2(t)$ for some $t \in \mathbb{R}$. Clearly $(\varsigma \circ \rho_2(t)) \cdot E_2 = E_2$ and $\rho_2(t) \cdot E_2 = E_2$ for every $t \in \mathbb{R}$.

Suppose $\psi \cdot E_3 = E_3$. Then $a_3 = b_3 = 0$ and $c_3 = 1$. The conditions $\psi^\top J \psi = J$ and $\det \psi = 1$ then yield $c_1 = c_2 = 0$ and $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \in \mathsf{SO}(2)$. Therefore $\psi = \rho_3(t)$ for some $t \in \mathbb{R}$. Clearly $\rho_3(t) \cdot E_3 = E_3$ for every $t \in \mathbb{R}$.

Suppose $\psi \cdot (E_1 + E_3) = E_1 + E_3$. Then $a_3 = 1 - a_1$, $b_3 = -b_1$ and $c_3 = 1 - c_1$. Again we impose the conditions $\psi^\top J \psi = J$ and $\det \psi = 1$. A straightforward but tedious calculation then shows that $\psi = \eta(t)$ for some $t \in \mathbb{R}$. It is easy to verify that $\eta(t) \cdot (E_1 + E_3) = E_1 + E_3$. $\square$

Remark. The ordered basis for $\mathfrak{so}(2,1)$ has been chosen so that $\mathsf{Aut}(\mathfrak{so}(2,1)) = \mathsf{SO}(2,1)$. Indeed, with respect to this choice of basis, we have that the linear map $\operatorname{ad} A = [A, \cdot]$ has matrix $\varsigma A \varsigma$. This accounts for the convenient situation that the subgroup of automorphisms fixing E_2, E_3, and $E_1 + E_3$, respectively, are exactly $\exp(\mathbb{R}E_2) \cup (\varsigma \exp(\mathbb{R}E_2))$, $\exp(\mathbb{R}E_3)$, and $\exp(\mathbb{R}(E_1 + E_3))$, respectively.

Corollary 1. *The only automorphism fixing at least two of E_1, E_2, E_3, and $E_1 + E_3$ is the identity automorphism.*

The subgroups of automorphisms fixing E_2, E_3, and $E_1 + E_3$, respectively, preserve certain affine subspaces. Moreover, these subgroups are transitive on certain subsets of these affine subspaces. Let $A \in \mathfrak{so}(2,1)$, $A \neq 0$, $A = a_1 E_1 + a_2 E_2 + a_3 E_3$ and let

$$\begin{aligned}
\Gamma_2 &= a_2 E_2 + \langle E_3, E_1 \rangle, && a_1^2 - a_3^2 \neq 0 \\
\Gamma_2' &= (a_2 E_2 + \langle a_1 E_1 + a_3 E_3 \rangle) \backslash \{a_2 E_2\}, && a_1^2 - a_3^2 = 0 \\
\Gamma_3 &= a_3 E_3 + \langle E_1, E_2 \rangle, && a_1^2 + a_2^2 \neq 0 \\
\Gamma_{13} &= (a_1 - a_3) E_1 + \langle E_2, E_1 + E_3 \rangle, && a_1 \neq a_3 \\
\Gamma_{13}' &= a_2 E_2 + \langle E_1 + E_3 \rangle, && a_1 = a_3 \text{ and } a_2 \neq 0.
\end{aligned}$$

(These sets are generated by considering the orthogonal compliments, with respect to $\odot$, of $\langle E_2 \rangle$, $\langle E_3 \rangle$, and $\langle E_1 + E_3 \rangle$.) If $A \notin \langle E_2 \rangle$, then $A \in \Gamma_2$ or $A \in \Gamma_2'$. If $A \notin \langle E_3 \rangle$, then $A \in \Gamma_3$. If $A \notin \langle E_1 + E_3 \rangle$, then $A \in \Gamma_{13}$ or $A \in \Gamma_{13}'$.

Proposition 4. *Any automorphism $\rho_2(t)$ or $\varsigma \circ \rho_2(t)$ leaves Γ_2 and Γ_2' invariant. Any automorphism $\rho_3(t)$ leaves Γ_3 invariant. Any automorphism $\eta(t)$ leaves Γ_{13} and Γ_{13}' invariant.*

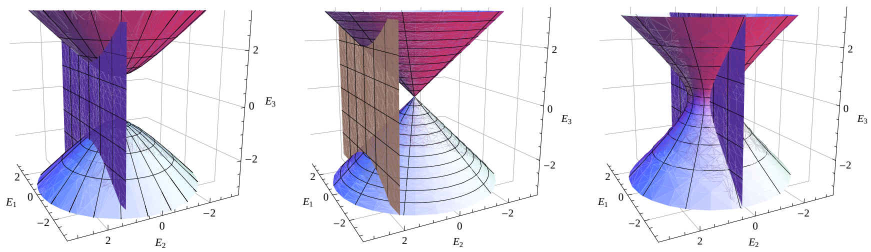
Theorem 2.

(i) *The subgroup of* $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* E_2 *acts transitively on* $\Gamma_2 \cap \mathcal{H}_{A\odot A}$ *and* $\Gamma_2' \cap \mathcal{H}_{A\odot A}$.

(ii) *The subgroup of* $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* E_3 *acts transitively on* $\Gamma_3 \cap \mathcal{H}_{A\odot A}$.

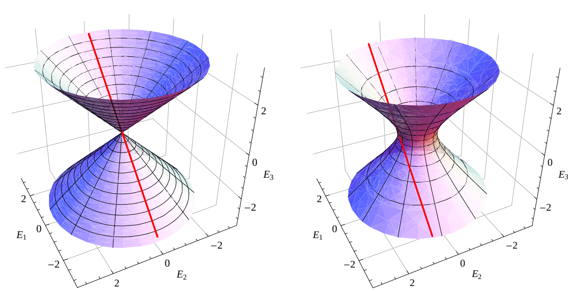
(iii) *The subgroup of* $\mathsf{Aut}(\mathfrak{so}(2,1))$ *fixing* $E_1 + E_3$ *acts transitively on* $\Gamma_{13} \cap \mathcal{H}_{A\odot A}$ *and* $\Gamma_{13}' \cap \mathcal{H}_{A\odot A}$.

We illustrate some of the typical cases in figures 1, 2, 3, and 4.



(a) $A \odot A < 0$ (b) $A \odot A = 0$ (c) $A \odot A > 0$

Figure 1: Typical cases of $\Gamma_2 \cap \mathcal{H}_{A\odot A}$



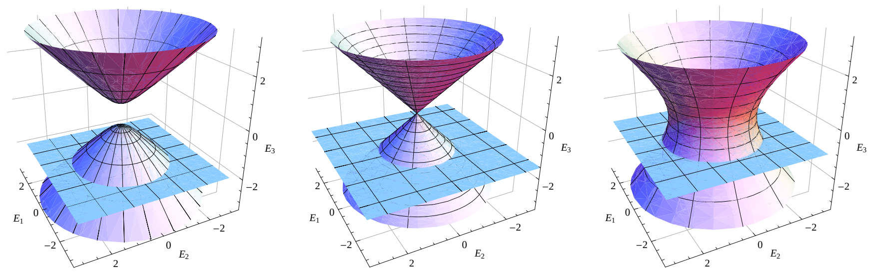
(a) $A \odot A = 0$ (b) $A \odot A > 0$

Figure 2: Typical cases of $\Gamma_2' \cap \mathcal{H}_{A\odot A}$ and $\Gamma_{13}' \cap \mathcal{H}_{A\odot A}$

Proof. (i) By proposition 1, any automorphism ψ fixing E_2 is of the form

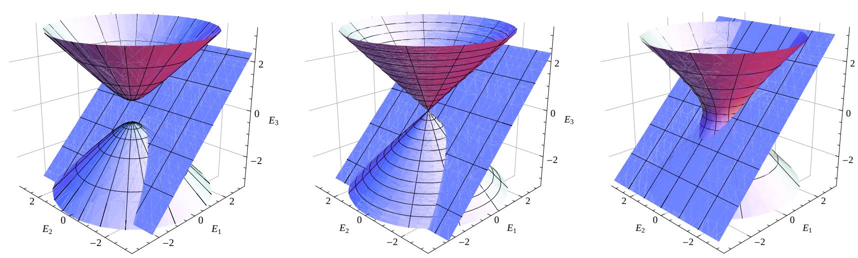
$$\psi = \begin{bmatrix} k\cosh t & 0 & k\sinh t \\ 0 & 1 & 0 \\ k\sinh t & 0 & k\cosh t \end{bmatrix}$$

where $t \in \mathbb{R}$ and $k \in \{-1, 1\}$.



(a) $A \odot A < 0$ (b) $A \odot A = 0$ (c) $A \odot A > 0$

Figure 3: Typical cases of $\Gamma_3 \cap \mathcal{H}_{A \odot A}$



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(a) $A \odot A < 0$ (b) $A \odot A = 0$ (c) $A \odot A > 0$

Figure 4: Typical cases of $\Gamma_{13} \cap \mathcal{H}_{A \odot A}$

Suppose $a_1^2 - a_3^2 \neq 0$. Let $xE_1 + yE_2 + zE_3 \in \Gamma_2 \cap \mathcal{H}_{A \odot A}$. Then $y = a_2$ and $x^2 - z^2 = a_1^2 - a_3^2$. It suffices to show that there exists and automorphism ψ fixing E_2 such that $\psi \cdot A = xE_1 + yE_2 + zE_3$. Now $\psi \cdot A = k\,(a_3 \sinh t + a_1 \cosh t)\,E_1 + a_2E_2 + k\,(a_1 \sinh t + a_3 \cosh t)\,E_3$. Thus $\psi \cdot A = xE_1 + yE_2 + zE_3$ only if there exists $k \in \{-1, 1\}$ and $t \in \mathbb{R}$ such that

$$\begin{bmatrix} a_1 & a_3 \\ a_3 & a_1 \end{bmatrix} \begin{bmatrix} k \cosh t \\ k \sinh t \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix}.$$

Let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_3 & a_1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ z \end{bmatrix}$ and let $k = \operatorname{sgn} v_1$. A simple calculation shows that $v_1^2 - v_2^2 = \frac{x^2 - z^2}{a_1^2 - a_3^2} = 1$ (and so $v_1 \neq 0$). There exists $t \in \mathbb{R}$ such that $k \sinh t = v_2$. Therefore $v_1^2 = 1 - \sinh^2 t = \cosh^2 t$. Hence, as $k = \operatorname{sgn} v_1$, it follows that $v_1 = k \cosh t$.

Suppose $a_3 = a_1 \neq 0$. Let $xE_1 + yE_2 + zE_3 \in \Gamma_2' \cap \mathcal{H}_{A \odot A}$. Then $y = a_2$ and $x = z \neq 0$. Now $\psi \cdot A = ke^t a_1 E_1 + a_2 E_2 + ke^t a_1 E_3$. Hence there exists $k \in \{-1, 1\}$ and $t \in \mathbb{R}$ such that $\psi \cdot A = xE_1 + yE_2 + zE_3$.

Suppose $a_3 = -a_1 \neq 0$. Let $xE_1 + yE_2 + zE_3 \in \Gamma_2' \cap \mathcal{H}_{A \odot A}$. Then $y = a_2$ and $x = -z \neq 0$. Now $\psi \cdot A = ke^{-t} a_1 E_1 + a_2 E_2 - ke^{-t} a_1 E_3$. Hence there exists $k \in \{-1, 1\}$ and $t \in \mathbb{R}$ such that $\psi \cdot A = xE_1 + yE_2 + zE_3$.

If $a_1 = a_3 = 0$, then $\Gamma_2' = \emptyset$.

(ii) By proposition 1, any automorphism ψ fixing E_3 is of the form $\psi = \rho_3(t)$ for some $t \in \mathbb{R}$. Let $xE_1 + yE_2 + zE_3 \in \Gamma_3 \cap \mathcal{H}_{A \odot A}$. Then $z = a_3$, $x^2 + y^2 = a_1^2 + a_2^2 \neq 0$. Now $\rho_3(t) \cdot A = (a_2 \sin t + a_1 \cos t)E_1 + (a_2 \cos t - a_1 \sin t)E_2 + a_3 E_3$. Thus $\rho_3(t) \cdot A = xE_1 + yE_2 + zE_3$ only if there exists $t \in \mathbb{R}$ such that

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$. Then $v_1^2 + v_2^2 = \frac{x^2 + y^2}{a_1^2 + a_2^2} = 1$. Thus there does indeed exist a $t \in \mathbb{R}$ satisfying the above equation.

(iii) Again by proposition 1, any automorphism ψ fixing $E_1 + E_3$ is of the form $\psi = \eta(t)$ for some $t \in \mathbb{R}$. Now

$$\begin{aligned} \eta(t) \cdot A = {} & (a_1 + a_2 t + \tfrac{1}{2}\,(a_3 - a_1)\,t^2)E_1 + (a_2 + (a_3 - a_1)\,t)E_2 \\ & + (a_3 + a_2 t + \tfrac{1}{2}\,(a_3 - a_1)\,t^2)E_3. \end{aligned}$$

Suppose $a_1 \neq a_3$ and let $xE_1 + yE_2 + zE_3 \in \Gamma_{13} \cap \mathcal{H}_{A \odot A}$. Then $x = a_1 - a_3 + z$ and $y^2 = a_1^2 + a_2^2 - a_3^2 + z^2 - x^2$. Thus $\rho_3(t) \cdot A = xE_1 + yE_2 + zE_3$

only if there exists $t \in \mathbb{R}$ such that

$$\begin{bmatrix} a_1 & a_2 & \frac{1}{2}(a_3 - a_1) \\ a_2 & a_3 - a_1 & 0 \\ a_3 & a_2 & \frac{1}{2}(a_3 - a_1) \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The determinant of the above matrix equals $\frac{1}{2}(a_1 - a_3)^3$ and so is nonzero. We have

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} a_1 & a_2 & \frac{1}{2}(a_3 - a_1) \\ a_2 & a_3 - a_1 & 0 \\ a_3 & a_2 & \frac{1}{2}(a_3 - a_1) \end{bmatrix}^{-1} \begin{bmatrix} a_1 - a_3 + z \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{a_2 - y}{a_1 - a_3} \\ \frac{2(a_2(-y+a_2)+(z-a_3)(-a_1+a_3))}{(a_1-a_3)^2} \end{bmatrix}. \end{aligned}$$

Let $t = v_2$. It is then a simple matter to verify (using the identity $y^2 = a_1^2 + a_2^2 - a_3^2 + z^2 - x^2$) that $v_3 = t^2$. Therefore $\eta(t) \cdot A = xE_1 + yE_2 + zE_3$.

Suppose $a_1 = a_3$ and $a_2 \neq 0$. Let $xE_1 + yE_2 + zE_3 \in \Gamma'_{13} \cap \mathcal{H}_{A \odot A}$. Then $y = a_2$ and $x = z$. Now $\eta(t) \cdot A = (a_1 + a_2 t)E_1 + a_2 E_2 + (a_1 + a_2 t)E_3$. So if $t = \frac{x - a_1}{a_2}$, then $\eta(t) \cdot A = xE_1 + yE_2 + zE_3$. □

We shall find it useful to restate this result by identifying a typical point for each intersection. (This allows for easier application to classifying systems.)

Corollary 2.

1. *Suppose $A \notin \langle E_2 \rangle$.*
   (a) *If $a_1^2 - a_3^2 \neq 0$, then there exists $t \in \mathbb{R}$ such that $\rho_2(t) \cdot A$ or $(\varsigma \circ \rho_2(t)) \cdot A$ equals $(\beta + \frac{1}{4})E_1 + a_2 E_2 + (\beta - \frac{1}{4})E_3$, where $\beta = a_1^2 - a_3^2$.*
   (b) *If $a_1^2 - a_3^2 = 0$, then there exists $t \in \mathbb{R}$ such that $\rho_2(t) \cdot A$ or $(\varsigma \circ \rho_2(t)) \cdot A$ equals $E_1 + a_2 E_2 + kE_3$, where $k = \frac{a_3}{a_1} = \pm 1$.*
2. *Suppose $A \notin \langle E_3 \rangle$. Then there exists $t \in \mathbb{R}$ such that $\rho_3(t) \cdot A = \alpha E_1 + a_3 E_3$, where $\alpha = \sqrt{a_1^2 + a_2^2} > 0$.*
3. *Suppose $A \notin \langle E_1 + E_3 \rangle$.*
   (a) *If $a_1 \neq a_3$, then there exists $t \in \mathbb{R}$ such that $\eta(t) \cdot A = (\gamma + \beta)E_1 + \gamma E_3$, where $\gamma = a_3 + \frac{a_2^2}{2a_1 - 2a_3}$ and $\beta = a_1 - a_3$.*
   (b) *If $a_1 = a_3$, then there exists $t \in \mathbb{R}$ such that $\eta(t) \cdot A = E_1 + \beta E_2 + E_3$, where $\beta = a_2 \neq 0$.*

4 Classification

We now proceed to classify, under state space equivalence, all (full-rank) left-invariant control affine systems on $\mathsf{SO}(2,1)_0$. This reduces (by propositions 1 and 2) to an algebraic classification of the corresponding affine parametrization maps. More precisely, Σ and Σ' are equivalent if an only if there exists $\psi \in d\,\mathsf{Aut}(\mathsf{SO}(2,1)_0) = \mathsf{Aut}(\mathfrak{so}(2,1))$ such that $\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$. We outline the approach to be used in classifying these systems. First, we distinguish between the number of controls involved and the homogeneity of the systems; this yields four types of systems. For each of these types, we simplify an arbitrary system by successively applying automorphisms. This simply involves applying proposition 3 and corollary 2. Finally, we verify that all the candidates for class representatives are distinct and not equivalent. Families of these representatives are typically parametrized by some vectors $\boldsymbol{\alpha} = (\alpha_i)$, $\boldsymbol{\beta} = (\beta_i)$, and $\boldsymbol{\gamma} = (\gamma_i)$, where $\alpha_i > 0$, $\beta_i \neq 0$, and $\gamma_i \in \mathbb{R}$.

When convenient, a system specified by

$$\Sigma \,:\, \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i + u_3 \sum_{i=1}^{3} d_i E_i$$

will be represented as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

The evaluation $\psi \cdot \Xi(\mathbf{1}, u)$ then becomes a matrix multiplication.

We start with single-input systems. (Only the inhomogeneous case need be considered as the homogeneous systems do not have full rank). The two-input homogeneous case follows as a corollary (by the lemma), although one still needs to verify that the systems obtained are not equivalent. (This verification shall be omitted as it is similar to the one made in the proof of the theorem.)

Theorem 3. *Every single-input (inhomogeneous) system is equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}} &:\ \alpha_2 E_1 + \gamma_1 E_3 + u\,\alpha_1 E_3 \\
\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}} &:\ (\gamma_1 + \beta_1) E_1 + \gamma_1 E_3 + u(E_1 + E_3) \\
\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}} &:\ (\beta_1 + \tfrac{1}{4}) E_1 + \gamma_1 E_2 + (\beta_1 - \tfrac{1}{4}) E_3 + u\,\alpha_1 E_2.
\end{aligned}$$

Here $\alpha_i > 0$, $\beta_1 \neq 0$, *and* $\gamma_1 \in \mathbb{R}$, *with different values of these parameters yielding distinct (non-equivalent) class representatives.*

Proof. Let $\Sigma : A + u\,B$ be a single-input system.

Suppose $B \odot B < 0$. Then (by proposition 3), there exists an automorphism ψ such that $\psi \cdot B = \alpha_1 E_3$ for some $\alpha_1 > 0$. Thus (by proposition 1) Σ is equivalent to $\Sigma' : A' + u\,\alpha_1 E_3$, where $A' = \psi \cdot A$. Now, as A and B are linearly independent, $A' \notin \langle E_3 \rangle$. Hence (by corollary 2) there exists an automorphism ψ' such that $\psi' \cdot \alpha_1 E_3 = \alpha_1 E_3$ and $\psi' \cdot A' = \alpha_2 E_1 + \gamma_1 E_3$ for some $\alpha_2 > 0$ and $\gamma_1 \in \mathbb{R}$. Therefore Σ' (and so also Σ) is equivalent to $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}} : \alpha_2 E_1 + \gamma_1 E_3 + u\,\alpha_1 E_3$.

Suppose $B \odot B = 0$. Then Σ is equivalent to $\Sigma' : A' + u(E_1 + E_3)$, where $A' \notin \langle E_1 + E_3 \rangle$. Hence, Σ is equivalent to either $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}} : (\gamma_1 + \beta_1)E_1 + \gamma_1 E_3 + u(E_1 + E_3)$ or $\Sigma'' : E_1 + \beta_1 E_2 + E_3 + u(E_1 + E_3)$ for some $\gamma_1 \in \mathbb{R}$ and $\beta_1 \neq 0$. However, Σ'' does not have full rank. As the full rank property is preserved by equivalence, it follows that Σ is equivalent to $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$.

Suppose $B \odot B > 0$. Then Σ is equivalent to $\Sigma' : A' + u\,\alpha_1 E_2$ for some $\alpha_1 > 0$, where $A' \notin \langle E_2 \rangle$. Hence, Σ is equivalent to either $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}} : (\beta_1 + \frac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \frac{1}{4})E_3 + u\,\alpha_1 E_2$ or $\Sigma'' : E_1 + \gamma_1 E_2 + E_3 + u\,\alpha_1 E_2$ for some $\gamma_1 \in \mathbb{R}$ and $\beta_1 \neq 0$. However, Σ'' does not have full rank and so Σ is equivalent to $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}}$.

It remains to be shown that no two of these equivalence representatives are equivalent. Let $\Sigma : A + uB$. If $\Sigma = \Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}}$, then $B \odot B < 0$. If $\Sigma = \Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$, then $B \odot B = 0$. If $\Sigma = \Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}}$, then $B \odot B > 0$. Thus, as $\odot$ is preserved by any automorphism, $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}}$ is not equivalent to either $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$ or $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}}$. Likewise $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$ is not equivalent to $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}}$.

Suppose $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}}$ is equivalent to $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha'\gamma'}}$. Then there exists an automorphism ψ such that

$$\psi \cdot \left[\begin{array}{c|c} \alpha_2 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha_1 \end{array}\right] = \left[\begin{array}{c|c} \alpha_2' & 0 \\ 0 & 0 \\ \gamma_1' & \alpha_1' \end{array}\right].$$

Thus $-\alpha_1^2 = \alpha_1 E_3 \odot \alpha_1 E_3 = \alpha_1' E_3 \odot \alpha_1' E_3 = -\alpha_1'^2$. Hence, as $\alpha_1, \alpha_1' > 0$, $\alpha = \alpha'$. Thus $\psi \cdot E_3 = E_3$. Therefore (by proposition 1) $\psi = \rho_3(t)$ for some $t > 0$. Then it follows that $\gamma_1 = \gamma_1'$ and $\alpha_2 = \alpha_2'$. That is to say $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}}$ and $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha'\gamma'}}$ are equivalent only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ and $\gamma_1 = \gamma_1'$.

Suppose $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$ is equivalent to $\Sigma^{(1,1)}_{2,\boldsymbol{\beta'\gamma'}}$. Then there exists an automorphism ψ such that

$$\psi \cdot \left[\begin{array}{c|c} \beta_1 + \gamma_1 & 1 \\ 0 & 0 \\ \gamma_1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \beta_1' + \gamma_1' & 1 \\ 0 & 0 \\ \gamma_1' & 1 \end{array}\right].$$

Hence, as $\psi \cdot (E_1 + E_3) = E_1 + E_3$, $\psi = \eta(t)$ for some $t \in \mathbb{R}$. We have

$$\eta(t) \cdot \left[\begin{array}{c|c} \beta_1 + \gamma_1 & 1 \\ 0 & 0 \\ \gamma_1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \beta_1 - \frac{t^2\beta_1}{2} + \gamma_1 & 1 \\ -t\beta_1 & 0 \\ -\frac{t^2\beta_1}{2} + \gamma_1 & 1 \end{array}\right].$$

Therefore $t = 0$ and so ψ is the identity automorphism. Consequently $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}}$ and $\Sigma^{(1,1)}_{2,\boldsymbol{\beta'\gamma'}}$ are equivalent only if $\beta_1 = \beta_1'$ and $\gamma_1 = \gamma_1'$.

Similar computations show that $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha'\beta'\gamma'}}$ only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$, $\boldsymbol{\beta} = \boldsymbol{\beta}'$ and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$. □

Corollary 3. *Every two-input homogeneous system is equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma^{(2,0)}_{1,\boldsymbol{\alpha\gamma}} &: \ \gamma_3 E_1 + \gamma_2 E_3 + u_1(\alpha_2 E_1 + \gamma_1 E_3) + u_2\,\alpha_1 E_3 \\
\Sigma^{(2,0)}_{2,\boldsymbol{\beta\gamma}} &: \ \gamma_3 E_1 + \gamma_2 E_3 + u_1((\gamma_1 + \beta_1)E_1 + \gamma_1 E_3) + u_2(E_1 + E_3) \\
\Sigma^{(2,0)}_{3,\boldsymbol{\alpha\beta\gamma}} &: \ \gamma_2(\beta_1 + \tfrac{1}{4})E_1 + \gamma_3 E_2 + \gamma_2(\beta_1 - \tfrac{1}{4})E_3 \\
&\qquad + u_1((\beta_1 + \tfrac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \tfrac{1}{4})E_3) + u_2\,\alpha_1 E_2.
\end{aligned}$$

Here $\alpha_i > 0$, $\beta_1 \neq 0$, and $\gamma_i \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Next we deal with the two-input inhomogeneous systems. The three-input case then follows as a corollary (as all three-input systems are clearly homogeneous).

Theorem 4. *Every two-input inhomogeneous system is equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}} &: \ \gamma_3 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(\alpha_2 E_1 + \gamma_1 E_3) + u_2\,\alpha_1 E_3 \\
\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}} &: \ \gamma_3 E_1 + \beta_2 E_2 + \gamma_2 E_3 + u_1((\gamma_1 + \beta_1)E_1 + \gamma_1 E_3) + u_2(E_1 + E_3) \\
\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}} &: \ \gamma_1 E_1 + \gamma_2 E_2 + (\beta_2 + \gamma_1)E_3 + u_1(E_1 + \beta_1 E_2 + E_3) + u_2(E_1 + E_3) \\
\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}} &: \ \big(\beta_2(\beta_1 - \tfrac{1}{4}) + \gamma_2(\beta_1 + \tfrac{1}{4})\big)\,E_1 + \big(\beta_2(\beta_1 + \tfrac{1}{4}) + \gamma_2(\beta_1 - \tfrac{1}{4})\big)\,E_3 \\
&\qquad + \gamma_3 E_2 + u_1\left((\beta_1 + \tfrac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \tfrac{1}{4})E_3\right) + u_2\,\alpha_1 E_2 \\
\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}} &: \ \gamma_3 E_1 + \gamma_2 E_2 + (\beta_1 + \gamma_3)E_3 + u_1(E_1 + \gamma_1 E_2 + E_3) + u_2\,\alpha_1 E_2.
\end{aligned}$$

Here $\alpha_i > 0$, $\beta_i \neq 0$, and $\gamma_i \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Proof. Let $\Sigma : A + u_1 B_1 + u_2 B_2$ be a two-input system.

Suppose $B_2 \odot B_2 < 0$. Then Σ is equivalent to $\Sigma' : A' + u_1 B_1' + \alpha_1 E_3$ for some $\alpha_1 > 0$, where $B_1' \notin \langle E_3 \rangle$. Hence Σ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}}$: $\gamma_3 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(\alpha_2 E_1 + \gamma_1 E_3) + u_2\,\alpha_1 E_3$ for some $\alpha_2 > 0$ and $\gamma_1, \gamma_2, \gamma_3, \beta_1 \in \mathbb{R}$. As $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}}$ is inhomogeneous, it follows that $\beta_1 \neq 0$.

Suppose $B_2 \odot B_2 = 0$. Then Σ is equivalent to $\Sigma' : A' + u_1 B_1' + u_2(E_1 + E_3)$, where $B' \notin \langle E_1 + E_3 \rangle$. Hence, Σ is equivalent to either $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}} : \gamma_3 E_1 + \beta_2 E_2 + \gamma_2 E_3 + u_1((\gamma_1+\beta_1)E_1 + \gamma_1 E_3) + u_2(E_1+E_3)$ or $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}} : \gamma_1 E_1 + \gamma_2 E_2 + (\beta_2+\gamma_1)E_3 + u_1(E_1 + \beta_1 E_2 + E_3) + u_2(E_1+E_3)$ for some $\gamma_1, \gamma_2, \gamma_3, \beta_2 \in \mathbb{R}$ and $\beta_1 \neq 0$. As $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}}$ and $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$ are inhomogeneous, it follows that $\beta_2 \neq 0$.

Suppose $B_2 \odot B_2 > 0$. Then Σ is equivalent to $\widetilde{\Sigma} : \tilde{A} + u_1 \tilde{B}_1 + u_2\,\alpha_1 E_2$ for some $\alpha_1 > 0$, where $\tilde{B}_1 \notin \langle E_2 \rangle$. Hence, Σ is equivalent to either Σ' : $A' + u_1((\beta_1 + \frac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \frac{1}{4})E_3) + u_2\,\alpha_1 E_2$ or $\Sigma'' : A'' + u_1(E_1 + \gamma_1 E_2 + E_3) + u_2\,\alpha_1 E_2$ for some $\gamma_1 \in \mathbb{R}$ and $\beta_1 \neq 0$. We require that A', $(\beta_1 + \frac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \frac{1}{4})E_3$, and $\alpha_1 E_2$ are linearly independent. We have that $(\beta_1 - \frac{1}{4})E_1 + (\beta_1 + \frac{1}{4})E_3$, $(\beta_1 + \frac{1}{4})E_1 + (\beta_1 - \frac{1}{4})E_3$, and $\alpha_3 E_3$ are linearly independent. Thus $A' = \left(\beta_2(\beta_1 - \frac{1}{4}) + \gamma_2(\beta_1 + \frac{1}{4})\right) E_1 + \gamma_3 E_2 + \left(\beta_2(\beta_1 + \frac{1}{4}) + \gamma_2(\beta_1 - \frac{1}{4})\right) E_3$ for some $\gamma_2, \gamma_3 \in \mathbb{R}$ and $\beta_2 \neq 0$. Hence $\Sigma' = \Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$. We also require that A'', $E_1 + \gamma_1 E_2 + E_3$, and $\alpha_1 E_2$ are linearly independent. Thus $A'' = \gamma_3 E_1 + \gamma_2 E_2 + (\beta_1 + \gamma_3)E_3$ for some $\gamma_2, \gamma_3 \in \mathbb{R}$ and $\beta_1 \neq 0$. Therefore $\Sigma'' = \Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}}$.

It remains to be shown that no two of these equivalence representatives are equivalent. As $\odot$ is preserved by any automorphism, it follows that $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}}$ is not equivalent to $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}}$, $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$, $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$, or $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}}$. Likewise, $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}}$ is not equivalent to $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$ or $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}}$; $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$ is not equivalent to $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$ or $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}}$.

Suppose $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{2,\boldsymbol{\beta'\gamma'}}$. Then there exists an automorphism $\eta(t)$ fixing $E_1 + E_3$ such that

$$\eta(t) \cdot \left[\begin{array}{c|cc} \gamma_1 & 1 & 1 \\ \gamma_2 & \beta_1 & 0 \\ \beta_2 + \gamma_1 & 1 & 1 \end{array}\right] = \left[\begin{array}{c|cc} \gamma_3 & \beta_1 + \gamma_1 & 1 \\ \beta_2 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right]$$

i.e.,

$$\left[\begin{array}{c|cc} \frac{t^2\beta_2}{2} + \gamma_1 + t\gamma_2 & 1 + t\beta_1 & 1 \\ t\beta_2 + \gamma_2 & \beta_1 & 0 \\ \frac{1}{2}\left(2 + t^2\right)\beta_2 + \gamma_1 + t\gamma_2 & 1 + t\beta_1 & 1 \end{array}\right] = \left[\begin{array}{c|cc} \gamma_3 & \beta_1 + \gamma_1 & 1 \\ \beta_2 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right].$$

Thus $\beta_1 = 0$, a contradiction. Hence $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$ is not equivalent to $\Sigma^{(2,1)}_{2,\boldsymbol{\beta'\gamma'}}$ (for any admissible parameters). Similarly, $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$ is not equivalent to $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha'\beta'\gamma'}}$.

Suppose $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha'\beta'\gamma'}}$. Then there exists an automorphism ψ such that

$$\psi \cdot \left[\begin{array}{c|cc} \gamma_3 & \alpha_2 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right] = \left[\begin{array}{c|cc} \gamma_3' & \alpha_2' & 0 \\ \beta_1' & 0 & 0 \\ \gamma_2' & \gamma_1' & \alpha_1' \end{array}\right].$$

Thus $-\alpha_1^2 = -\alpha_1'^2$ and so $\alpha_1 = \alpha_1'$. Therefore ψ fixes E_3. Hence $\psi = \rho_3(t)$ for some $t \in \mathbb{R}$. Now

$$\rho_3(t) \cdot \left[\begin{array}{c|cc} \gamma_3 & \alpha_2 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right] = \left[\begin{array}{c|cc} \beta_1 \sin t + \gamma_3 \cos t & \alpha_2 \cos t & 0 \\ \beta_1 \cos t - \gamma_3 \sin t & -\alpha_2 \sin t & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right].$$

Thus $\gamma_1 = \gamma_1'$. Therefore $\alpha_2^2 = \alpha_2'^2$ and so $\alpha_2 = \alpha_2'$. Hence $\psi \cdot \alpha_2 E_1 = \alpha_2 E_1$, i.e., ψ fixes E_1. Hence, (by corollary 1) ψ is the identity automorphism. Accordingly $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha'\beta'\gamma'}}$ only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$, $\beta_1 = \beta_1'$, and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$.

Suppose $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha'\beta'\gamma'}}$. Then there exists an automorphism ψ such that

$$\begin{aligned}
&\psi \cdot \left[\begin{array}{c|cc} \left(\beta_1 - \frac{1}{4}\right)\beta_2 + \left(\beta_1 + \frac{1}{4}\right)\gamma_2 & \beta_1 + \frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ \left(\beta_1 + \frac{1}{4}\right)\beta_2 + \left(\beta_1 - \frac{1}{4}\right)\gamma_2 & \beta_1 - \frac{1}{4} & 0 \end{array}\right] \\
&= \left[\begin{array}{c|cc} \left(\beta_1' - \frac{1}{4}\right)\beta_2' + \left(\beta_1' + \frac{1}{4}\right)\gamma_2' & \beta_1' + \frac{1}{4} & 0 \\ \gamma_3' & \gamma_1' & \alpha_1' \\ \left(\beta_1' + \frac{1}{4}\right)\beta_2' + \left(\beta_1' - \frac{1}{4}\right)\gamma_2' & \beta_1' - \frac{1}{4} & 0 \end{array}\right].
\end{aligned}$$

Thus $-\alpha_1^2 = -\alpha_1'^2$ and so $\alpha_1 = \alpha_1'$. Therefore ψ fixes E_2. Hence $\psi = \rho_2(t)$ or $\psi = \varsigma \circ \rho_2(t)$ for some $t \in \mathbb{R}$. Now

$$\rho_2(t) \cdot \begin{bmatrix} \frac{1}{4} + \beta_1 \\ \gamma_1 \\ -\frac{1}{4} + \beta_1 \end{bmatrix} = \begin{bmatrix} \frac{e^{-\theta}}{4} + e^{\theta}\beta_1 \\ \gamma_1 \\ -\frac{e^{-\theta}}{4} + e^{\theta}\beta_1 \end{bmatrix}$$

$$(\varsigma \circ \rho_2(t)) \cdot \begin{bmatrix} \frac{1}{4} + \beta_1 \\ \gamma_1 \\ -\frac{1}{4} + \beta_1 \end{bmatrix} = \begin{bmatrix} -\frac{e^{-\theta}}{4} - e^{\theta}\beta_1 \\ \gamma_1 \\ \frac{e^{-\theta}}{4} - e^{\theta}\beta_1 \end{bmatrix}$$

Thus $\gamma_1 = \gamma_1'$. Hence $\beta_1 = (\beta_1 + \frac{1}{4})^2 - (\beta_1 - \frac{1}{4})^2 = (\beta_1' + \frac{1}{4})^2 - (\beta_1' - \frac{1}{4})^2 = \beta_1'$. For $\psi = \varsigma \circ \rho_2(t)$ we then get $\left(-\frac{e^{-\theta}}{4} - e^{\theta}\beta_1\right) + \left(\frac{e^{-\theta}}{4} - e^{\theta}\beta_1\right) = -2e^{\theta}\beta_1 = 2\beta_1$, a contradiction. Therefore $\psi = \rho_2(t)$. Then $\left(\frac{e^{-\theta}}{4} + e^{\theta}\beta_1\right) + \left(-\frac{e^{-\theta}}{4} + e^{\theta}\beta_1\right) = 2e^{\theta}\beta_1 = 2\beta_1$. Thus $\theta = 0$ and so ψ is the identity automorphism. Therefore $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{4,\boldsymbol{\alpha'\beta'\gamma'}}$ only if $\alpha_1 = \alpha_1'$, $\boldsymbol{\beta} = \boldsymbol{\beta}'$, and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$.

Likewise, $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{2,\boldsymbol{\beta'\gamma'}}$, $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{3,\boldsymbol{\beta'\gamma'}}$, and $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}}$ is equivalent to $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha'\beta'\gamma'}}$, respectively, only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$, $\boldsymbol{\beta} = \boldsymbol{\beta}'$, and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$. □

Corollary 4. *Every three-input (homogeneous) system is equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma^{(3,0)}_{1,\boldsymbol{\alpha\beta\gamma}} :\ & \gamma_6 E_1 + \gamma_5 E_2 + \gamma_4 E_3 + u_1(\gamma_3 E_1 + \beta_1 E_2 + \gamma_2 E_3) \\
& + u_2(\alpha_2 E_1 + \gamma_1 E_3) + u_3\,\alpha_1 E_3 \\
\Sigma^{(3,0)}_{2,\boldsymbol{\beta\gamma}} :\ & \gamma_6 E_1 + \gamma_5 E_2 + \gamma_4 E_3 + u_1(\gamma_3 E_1 + \beta_2 E_2 + \gamma_2 E_3) \\
& + u_2((\gamma_1 + \beta_1)E_1 + \gamma_1 E_3) + u_3(E_1 + E_3) \\
\Sigma^{(3,0)}_{3,\boldsymbol{\beta\gamma}} :\ & \gamma_6 E_1 + \gamma_5 E_2 + \gamma_4 E_3 + u_1(\gamma_1 E_1 + \gamma_2 E_2 + (\beta_2 + \gamma_1)E_3) \\
& + u_2(E_1 + \beta_1 E_2 + E_3) + u_3(E_1 + E_3) \\
\Sigma^{(3,0)}_{4,\boldsymbol{\alpha\beta\gamma}} :\ & \gamma_4 E_1 + \gamma_4 E_2 + \gamma_3 E_3 + u_2((\beta_1 + \tfrac{1}{4})E_1 + \gamma_1 E_2 + (\beta_1 - \tfrac{1}{4})E_3) \\
& + u_3\,\alpha_1 E_2 + u_1\Big(\left(\beta_2(\beta_1 - \tfrac{1}{4}) + \gamma_2(\beta_1 + \tfrac{1}{4})\right) E_1 \\
& \qquad + \left(\beta_2(\beta_1 + \tfrac{1}{4}) + \gamma_2(\beta_1 - \tfrac{1}{4})\right) E_3 + \gamma_3 E_2 \Big) \\
\Sigma^{(3,0)}_{5,\boldsymbol{\alpha\beta\gamma}} :\ & \gamma_6 E_1 + \gamma_5 E_2 + \gamma_4 E_3 + u_1(\gamma_3 E_1 + \gamma_2 E_2 + (\beta_1 + \gamma_3)E_3) \\
& + u_2(E_1 + \gamma_1 E_2 + E_3) + u_3\,\alpha_1 E_2.
\end{aligned}$$

Here $\alpha_i > 0$, $\beta_i \neq 0$, and $\gamma_i \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

5 Conclusion

Two systems (on a connected Lie group G)

$$\Sigma \,:\, \Xi\,(\mathbf{1}, u) \quad \text{and} \quad \Sigma' \,:\, \Xi'\,(\mathbf{1}, u)$$

are detached feedback equivalent (shortly DF-equivalent) if there exists a diffeomorphism $\Phi : \mathsf{G} \times \mathbb{R}^{\ell} \to \mathsf{G} \times \mathbb{R}^{\ell}$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$$

for $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. It turns out that Σ and Σ' are detached feedback equivalent if and only if there exists a Lie algebra automorphism $\psi \in d\,\mathsf{Aut}(\mathsf{G})$ such that $\psi \cdot \Gamma = \Gamma'$ (cf. [10]). Detached feedback equivalence is a weaker equivalence relation than state space equivalence.

A classification, under detached feedback equivalence, of systems evolving on $\mathsf{SO}(2,1)_0$ was obtained in [7]. Furthermore a full list of (detached feedback) equivalence representatives was identified. We now compare this classification (under DF-equivalence) to the classification obtained in this paper. Specifically, we match (families of) state space equivalence class representatives to detached feedback equivalence class representatives.

For the single-input systems we have

- $\Sigma^{(1,1)}_{1,\boldsymbol{\alpha\gamma}} : \left[\begin{array}{c|c} \alpha_2 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha_1 \end{array}\right]$ is DF-equivalent to $\Sigma : \alpha_2 E_2 + uE_3$;
- $\Sigma^{(1,1)}_{2,\boldsymbol{\beta\gamma}} : \left[\begin{array}{c|c} \beta_1+\gamma_1 & 1 \\ 0 & 0 \\ \gamma_1 & 1 \end{array}\right]$ is DF-equivalent to $\Sigma : E_3 + u(E_2 + E_3)$;
- $\Sigma^{(1,1)}_{3,\boldsymbol{\alpha\beta\gamma}} : \left[\begin{array}{c|c} \beta_1+\frac{1}{4} & 0 \\ \gamma_1 & \alpha_1 \\ \beta_1-\frac{1}{4} & 0 \end{array}\right]$ is
  - DF-equivalent to $\Sigma : \sqrt{\beta_1}E_1 + uE_2$ if $\beta_1 > 0$
  - DF-equivalent to $\Sigma : \sqrt{-\beta_1}E_3 + uE_2$ if $\beta_1 < 0$.

For the two-input homogeneous systems we have

- $\Sigma^{(2,0)}_{1,\boldsymbol{\alpha\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & \alpha_2 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right]$ and $\Sigma^{(2,0)}_{1,\boldsymbol{\alpha\gamma}} : \left[\begin{array}{c|c} \beta_1+\gamma_1 & 1 \\ 0 & 0 \\ \gamma_1 & 1 \end{array}\right]$ are DF-equivalent to $\Sigma : u_1E_2 + u_2E_3$;
- $\Sigma^{(2,0)}_{3,\boldsymbol{\alpha\beta\gamma}} : \left[\begin{array}{c|cc} (\beta_1+\frac{1}{4})\gamma_2 & \beta_1+\frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ (\beta_1-\frac{1}{4})\gamma_2 & \beta_1-\frac{1}{4} & 0 \end{array}\right]$ is
  - DF-equivalent to $\Sigma : u_1E_2 + u_2E_3$ if $\beta_1 < 0$
  - DF-equivalent to $\Sigma : u_1E_1 + u_2E_2$ if $\beta_1 > 0$.

For the two-input inhomogeneous systems we have

- $\Sigma^{(2,1)}_{1,\boldsymbol{\alpha\beta\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & \alpha_2 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right]$ is DF-equivalent to $\Sigma : |\beta_1|E_1 + u_1E_2 + u_2E_3$;
- $\Sigma^{(2,1)}_{2,\boldsymbol{\beta\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & \beta_1+\gamma_1 & 1 \\ \beta_2 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right]$ is DF-equivalent to $\Sigma : |\beta_2|E_1 + u_1E_2 + u_2E_3$;
- $\Sigma^{(2,1)}_{3,\boldsymbol{\beta\gamma}} : \left[\begin{array}{c|cc} \gamma_1 & 1 & 1 \\ \gamma_2 & \beta_1 & 0 \\ \beta_2+\gamma_1 & 1 & 1 \end{array}\right]$ and $\Sigma^{(2,1)}_{4,\boldsymbol{\beta\gamma}} : \left[\begin{array}{c|cc} \gamma_3 & 1 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \\ \beta_1+\gamma_3 & 1 & 0 \end{array}\right]$ are DF-equivalent to $\Sigma : E_3 + u_1E_1 + u_2(E_2+E_3)$;
- $\Sigma^{(2,1)}_{5,\boldsymbol{\alpha\beta\gamma}} : \left[\begin{array}{c|cc} (\beta_1-\frac{1}{4})\beta_2 + (\beta_1+\frac{1}{4})\gamma_2 & \beta_1+\frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ (\beta_1+\frac{1}{4})\beta_2 + (\beta_1-\frac{1}{4})\gamma_2 & \beta_1-\frac{1}{4} & 0 \end{array}\right]$ is
  - DF-equivalent to $\Sigma : \sqrt{-\beta_1\beta_2^2}E_1 + u_1E_2 + u_2E_3$ if $\beta_1 < 0$
  - DF-equivalent to $\Sigma : \sqrt{\beta_1\beta_2^2}E_3 + u_1E_1 + u_2E_2$ if $\beta_1 > 0$.

The three-input case is trivial; any three-input system is DF-equivalent to $\Sigma : u_1E_1 + u_2E_2 + u_3E_3$.

A summary of the classification results (in matrix form) is appended as a table.

Type	Equivalence representatives ($\alpha_i > 0,\ \beta_i \neq 0,\ \gamma_i \in \mathbb{R}$)
$(1,1)$	$\left[\begin{array}{c\|c} \alpha_2 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha_1 \end{array}\right], \quad \left[\begin{array}{c\|c} \beta_1+\gamma_1 & 1 \\ 0 & 0 \\ \gamma_1 & 1 \end{array}\right], \quad \left[\begin{array}{c\|c} \beta_1+\frac{1}{4} & 0 \\ \gamma_1 & \alpha_1 \\ \beta_1-\frac{1}{4} & 0 \end{array}\right]$
$(2,0)$	$\left[\begin{array}{c\|cc} \gamma_3 & \alpha_2 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right], \quad \left[\begin{array}{c\|cc} \gamma_3 & \beta_1+\gamma_1 & 1 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right]$ $\left[\begin{array}{c\|cc} (\beta_1+\frac{1}{4})\gamma_2 & \beta_1+\frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ (\beta_1-\frac{1}{4})\gamma_2 & \beta_1-\frac{1}{4} & 0 \end{array}\right]$
$(2,1)$	$\left[\begin{array}{c\|cc} \gamma_3 & \alpha_2 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right], \quad \left[\begin{array}{c\|cc} \gamma_3 & \beta_1+\gamma_1 & 1 \\ \beta_2 & 0 & 0 \\ \gamma_2 & \gamma_1 & 1 \end{array}\right]$ $\left[\begin{array}{c\|cc} \gamma_1 & 1 & 1 \\ \gamma_2 & \beta_1 & 0 \\ \beta_2+\gamma_1 & 1 & 1 \end{array}\right], \quad \left[\begin{array}{c\|cc} \gamma_3 & 1 & 0 \\ \gamma_2 & \gamma_1 & \alpha_1 \\ \beta_1+\gamma_3 & 1 & 0 \end{array}\right]$ $\left[\begin{array}{c\|cc} (\beta_1-\frac{1}{4})\beta_2+(\beta_1+\frac{1}{4})\gamma_2 & \beta_1+\frac{1}{4} & 0 \\ \gamma_3 & \gamma_1 & \alpha_1 \\ (\beta_1+\frac{1}{4})\beta_2+(\beta_1-\frac{1}{4})\gamma_2 & \beta_1-\frac{1}{4} & 0 \end{array}\right]$
$(3,0)$	$\left[\begin{array}{c\|ccc} \gamma_6 & \gamma_3 & \alpha_2 & 0 \\ \gamma_5 & \beta_1 & 0 & 0 \\ \gamma_4 & \gamma_2 & \gamma_1 & \alpha_1 \end{array}\right], \quad \left[\begin{array}{c\|ccc} \gamma_6 & \gamma_3 & \beta_1+\gamma_1 & 1 \\ \gamma_5 & \beta_2 & 0 & 0 \\ \gamma_4 & \gamma_2 & \gamma_1 & 1 \end{array}\right]$ $\left[\begin{array}{c\|ccc} \gamma_5 & \gamma_1 & 1 & 1 \\ \gamma_4 & \gamma_2 & \beta_1 & 0 \\ \gamma_3 & \beta_2+\gamma_1 & 1 & 1 \end{array}\right], \quad \left[\begin{array}{c\|ccc} \gamma_6 & \gamma_3 & 1 & 0 \\ \gamma_5 & \gamma_2 & \gamma_1 & \alpha_1 \\ \gamma_4 & \beta_1+\gamma_3 & 1 & 0 \end{array}\right]$ $\left[\begin{array}{c\|ccc} \gamma_6 & (\beta_1-\frac{1}{4})\beta_2+(\beta_1+\frac{1}{4})\gamma_2 & \beta_1+\frac{1}{4} & 0 \\ \gamma_5 & \gamma_3 & \gamma_1 & \alpha_1 \\ \gamma_4 & (\beta_1+\frac{1}{4})\beta_2+(\beta_1-\frac{1}{4})\gamma_2 & \beta_1-\frac{1}{4} & 0 \end{array}\right]$
$\left[\begin{array}{c\|ccc} A & B_1 & \cdots & B_\ell \end{array}\right] \longleftrightarrow A + u_1 B_1 + \cdots + u_\ell B_\ell$	

Classification of systems on $\mathsf{SO}(2,1)_0$ (matrix form)

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