



## Flat local morphisms of rings with prescribed depth and dimension

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### Abstract

For any pairs of integers  $(n, m)$  and  $(d, e)$  such that  $0 \leq n \leq m$ ,  $0 \leq d \leq e$ ,  $d \leq n$ ,  $e \leq m$  and  $n - d \leq m - e$  we construct a local flat ring morphism of noetherian local rings  $u : A \rightarrow B$  such that  $\dim(A) = n$ ,  $\text{depth}(A) = d$ ,  $\dim(B) = m$  and  $\text{depth}(B) = e$ .

### 1 Introduction

While preparing [3], the present author was looking for an example of a flat local ring homomorphism of noetherian local rings  $u : (A, m) \rightarrow (B, n)$  such that  $A$  and  $B/mB$  are almost Cohen-Macaulay, while  $B$  is not almost Cohen-Macaulay. This means that, for example, one should construct such a morphism with  $\text{depth}(B) = \text{depth}(A) = 0$ ,  $\dim(B) = 2$  and  $\dim(A) = 1$ . Note that actually the flatness of the homomorphism  $u$  is the non-trivial point in the construction. After asking several people without obtaining a satisfactory answer, he decided to let it as an open question in [3]. The answer came soon, an example with the desired features being constructed by Tabaâ [6]. Using his idea, we construct a quite general example of this type, construction that can be useful in various situations.

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## 2 The construction

We start by pointing out the following easy and well-known fact.

**Lemma 2.1.** *Let  $k$  be a field,  $n, d \in \mathbb{N}$  such that  $0 \leq d \leq n$ . Then there exists  $m \in \mathbb{N}, m \geq n$  and a monomial ideal  $I \subset S := k[X_1, \dots, X_m]$  such that  $\dim(S/I)_{(X_1, \dots, X_m)} = n$  and  $\text{depth}(S/I)_{(X_1, \dots, X_m)} = d$ .*

**Proof:** Let  $r = n - d$ . If  $r = 0$  the assertion is clear. Assume that  $r > 0$ . If  $d = 0$  let  $S = k[X_0, X_1, \dots, X_r]$  and if  $d > 0$  let  $S = k[X_0, X_1, \dots, X_r, T_1, \dots, T_d]$ . Consider for example the monomial ideal  $I = (X_0) \cap (X_0, \dots, X_r)^{r+1}$ . Then we have  $\text{Ass}(S/I) = \{(X_0), (X_0, \dots, X_r)\}$ , hence  $\dim(S/I) = r + d = n$  and  $\text{depth}(S/I) = d$ . Now taking  $m = r + 1 + d$  and renumbering the indeterminates we get the desired relations.

**Remark 2.2.** *Clearly there are also many other choices for a monomial ideal with the properties of the above lemma. For more about this kind of construction one can see [5].*

**Theorem 2.3.** *Let  $0 \leq d_1 \leq n_1$  and  $0 \leq d_2 \leq n_2$  be natural numbers such that  $n_1 \leq n_2, d_1 \leq d_2$  and  $n_1 - d_1 \leq n_2 - d_2$ . Then there exists a local flat morphism of noetherian local rings  $u : (A, m) \rightarrow (B, n)$  such that  $\text{depth}(A) = d_1, \dim(A) = n_1, \text{depth}(B) = d_2$  and  $\dim(B) = n_2$ .*

**Proof:** Let  $k$  be a field and  $A = (k[X]/I)_{(X)}, X = (X_1, \dots, X_m)$  be a local ring obtained cf. 2.1 with  $\text{depth}(A) = d_1$  and  $\dim(A) = n_1$ . Let  $s = d_2 - d_1$  and  $t = n_2 - n_1$ . By assumption we have  $s \leq t$ . Hence let  $C = (k[Y]/J)_{(Y)}, Y = (Y_1, \dots, Y_p)$  be a local ring obtained cf. 2.1 with  $\text{depth}(C) = s$  and  $\dim(C) = t$ . Now let

$$D := A \otimes_k C = k[X]/I \otimes_k k[Y]/J = k[X, Y]/(Ik[X, Y] + Jk[X, Y])$$

and let  $B := D_{(X, Y)}$ . Then obviously the canonical morphism  $u : A \rightarrow B$  is flat and local, being a localisation of the base change of the flat morphism  $k \rightarrow C$ . We need the following probably well-known fact:

**Lemma 2.4.** *Let  $k$  be a field and  $m, p \in \mathbb{N}$ . Let also  $I$  and  $J$  be monomial ideals in  $k[X] = k[X_1, \dots, X_m]$  and  $k[Y] = k[Y_1, \dots, Y_p]$  respectively and set  $S := k[X, Y]$ . Then  $\text{Min}(IS + JS) = \{PS + QS \mid P \in \text{Min}(I), Q \in \text{Min}(J)\}$ . Consequently*

$$\dim(S/(IS + JS)) = \dim(k[X]/I) + \dim(k[Y]/J).$$

**Proof:** Using [2], 3.4 we obtain

$$\text{Min}(IS + JS) = \text{Min}(\sqrt{IS + JS}) = \text{Min}(\sqrt{\sqrt{IS} + \sqrt{JS}}) =$$

$$= \text{Min}(\sqrt{I}S + \sqrt{J}S) = \{PS + QS \mid P \in \text{Min}(I), Q \in \text{Min}(J)\}.$$

Returning at the proof of the Theorem, by 2.4 we get  $\dim(B) = \dim(A) + \dim(C) = n_1 + t = n_2$  and by [1], Lemma 2 we have that  $\text{depth}(B) = \text{depth}(A) + \text{depth}(C) = s + d_1 = d_2$ . This concludes the proof of 2.3.

**Example 2.5.** Let  $k$  be a field, let  $A = C = (k[X, Y]/(X^2, XY))_{(X, Y)}$  and let  $B = A \otimes_k C = (k[X, Y, U, V]/(X^2, XY, U^2, UV))_{(X, Y, U, V)}$ . The canonical morphism  $u : A \rightarrow B$  is the morphism obtained performing the above construction. This is the example from [6], namely we have  $\dim(A) = 1, \dim(B) = 2, \text{depth}(A) = \text{depth}(B) = 0$ .

**Remark 2.6.** Let  $(A, m)$  be a noetherian local ring. Then the natural number  $\text{cmd}(A) = \dim(A) - \text{depth}(A)$  is called the Cohen-Macaulay defect of  $A$ . Thus  $A$  is Cohen-Macaulay if and only if  $\text{cmd}(A) = 0$  and  $A$  is almost Cohen-Macaulay if and only if  $\text{cmd}(A) \leq 1$  (see [3]).

**Example 2.7.** Using the above construction, one can also get examples of flat local morphisms of noetherian local rings, whose closed fiber has prescribed Cohen-Macaulay defect, or even more general, has prescribed dimension  $n$  and depth  $d \leq n$ . Indeed, by ([4], 15.1, 23.3), the flatness of  $u$  implies that  $n = \dim(B/mB) = n_2 - n_1$  and  $d = \text{depth}(B/mB) = d_2 - d_1$ , so that it is enough to choose appropriate values for  $n_1 \leq n_2$  and  $d_1 \leq d_2$  and perform the above construction.

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