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A Classifier for Unimodular Isolated Complete Intersection Space Curve Singularities

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Abstract

C.T.C. Wall classified the unimodular complete intersection singularities. He indicated in the list only the μ-constant strata and not the complete classification in each case. In this article we give a complete list of space curve unimodular singularities and also the description of a classifier. Instead of computing the normal forms, the singularity is identified by certain invariants.

1 Introduction

Marc Giusti gave the complete list of simple isolated complete intersection singularities which are not hypersurfaces (cf. [GM83]). An implementation in SINGULAR for the classification of simple isolated complete intersection singularities over the complex numbers is given by Gerhard Pfister and Deeba Afzal in `classifyci.lib` as a SINGULAR library (cf. [ADPG1], ADPG2). Wall achieved the classification of contact unimodular singularities which are not hypersurfaces (cf. [Wal83]).

We report about a classifier for unimodular isolated complete intersection curve singularities in the computer algebra system SINGULAR (cf. [DGPS13],[GP07]). A basis for a classifier is a complete list of these singularities together with a list of invariants characterizing them. Since Wall gave only representatives of the μ-constant strata in his classification (cf. [Wal83]), we complete his list by computing the versal μ-constant deformation of the

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singularities. The new list obtained in this way contains all unimodular complete intersection curve singularities. In section 2 we characterize the 2-jet of the unimodular complete intersection singularities by using primary decomposition and Hilbert polynomials. In section 3 we give the complete list of unimodular complete intersection space curve singularities by fixing the 2-jet of the singularities and develop algorithms for each case. In section 4 we present examples.

Let us recall the basic definitions.

Let $\mathbb{C}[[x]] = \mathbb{C}[[x_1, ..., x_n]]$ be the local ring of formal power series and $\langle x\rangle = \langle x_1, ..., x_n\rangle$ its maximal ideal.

Definition 1.1. $f =< f_1, f_2, ..., f_p >$, is called *complete intersection* if $\dim \mathbb{C}[[x]]/\langle f_1, \ldots, f_i\rangle = n - i \ ,\forall \ i = 1, \ldots, p$.
Hypersurfaces are special cases of complete intersections for $p = 1$.

Definition 1.2. Let $f = \langle f_1, \ldots, f_p\rangle \subseteq \mathbb{C}[[x]]$ be a complete intersection. $f = \langle f_1, \ldots, f_p\rangle$ has an *isolated singularity* at 0, if

1. $\langle f_1, \ldots, f_p, M_1, \ldots M_k\rangle \subseteq \langle x\rangle$, $M_1, \ldots, M_k$ the $p \times p$–minors of $(\frac{\partial f_i}{\partial x_j})$.
2. $\langle x\rangle^c \subseteq \langle f_1, \ldots, f_p, M_1, \ldots M_k\rangle$ for some $c > 0$.

The *Milnor number* $\mu(f)$ is defined as follows

$$\mu(f) = \sum_{i=1}^{p} (-1)^{p-i} dim_{\mathbb{C}} \mathbb{C}[[x]]/C_i$$

with $C_i = \langle f_1, f_2, ..., f_{i-1}, \frac{\partial(f_1,\ldots,f_i)}{\partial x_{j_1} \ldots x_{j_i}}, 1 \leq j_1, \ldots < j_i \leq n\rangle$ (cf. [GM75]).

The *Tjurina number* of f is defined to be

$$dim_{\mathbb{C}} \mathbb{C}[[x]]^p / f\mathbb{C}[[x]]^p + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbb{C}[[x]].$$

Let $I_{n,p}$ be the set of all isolated complete intersection singularities. Then $G_c = Aut(\mathbb{C}[[x]]) \times GL_p(\mathbb{C}[[x]])$ acts on $I_{n,p}$ as follows:
let $(\phi, \psi) \in G_c$ and $f = (f_1, \ldots, f_p) \in I_{n,p}$ then $(\phi, \psi)(f) = \psi^{-1} \circ f \circ \phi$.

Definition 1.3. Let f and $g \in I_{n,p}$ are called *contact equivalent*, if there exists $(\phi, \psi) \in G_c$, such that $f = (\phi, \psi)(g)$.

$I_{n,p} \subseteq \mathbb{C}[[x]]^p$ caries a canonical topology. It is the topology such that the maps

$$\mathbb{C}[[x]]^p \to (\mathbb{C}[[x]]/\langle x\rangle^c)^p$$

are continuous $\forall$ c. Here we consider the classical topology of the affine space $(\mathbb{C}[[x]]/\langle x\rangle^c)^p$.

Definition 1.4. An element $f \in I_{n,p}$ is called *simple singularity*, if there exists a neighborhood of f in $I_{n,p}$ containing only finitely many orbits of G. In other words the modality of the singularity is zero.

Definition 1.5. f is called *unimodular singularity* if there exists a neighborhood of f in $I_{n,p}$ containing only one–dimensional families of orbits of G_c. In other words the modality of the singularity is 1.

Definition 1.6. $f \in I_{n,p}$ defines a *curve* if $\mathbb{C}[[x]]/f$ is of of dimension 1.

If f defines an irreducible curve , i:e $f \subseteq \mathbb{C}[[x]]$ is prime. Then the normalization of $\mathbb{C}[[x]]/f$ is $\mathbb{C}[[t]]$ and we have parametrization

$$\mathbb{C}[[x]]/f \cong K[[x_1(t), x_2(t), ..., x_n(t)]] \subseteq \mathbb{C}[[t]]$$

Definition 1.7.

$$\Gamma_f = \{ord(f) \mid f \in \mathbb{C}[[x_1(t), x_2(t), ..., x_n(t)]]\}$$

is the *semi group* of the curve.

Definition 1.8. $f = \langle f_1, \ldots, f_p \rangle \in I_{n,p}$ then $F = \langle F_1, \ldots, F_p \rangle, \quad F_i \in \mathbb{C}[[x, t]]$ where $t = \{t_1, \ldots, t_n\}$ is a *deformation of* f if

$$\mathbb{C}[[x]]/\langle f \rangle \cong \mathbb{C}[[x]]/\langle F(x, 0) \rangle.$$

Any deformation can be induced from the versal deformation by specifying parameters.

$F = f + \sum t_i m_i$ is a *versal deformation of* f where $m_1, \ldots, m_\tau$ is basis for

$$\mathbb{C}[[x]]^p / f\mathbb{C}[[x]]^p + \begin{pmatrix} \partial f_1/\partial x_1 \\ \vdots \\ \partial f_p/\partial x_1 \end{pmatrix} \mathbb{C}[[x]] + \ldots + \begin{pmatrix} \partial f_1/\partial x_n \\ \vdots \\ \partial f_p/\partial x_n \end{pmatrix} \mathbb{C}[[x]].$$

2 Characterization of normal form of 2-jet of singularities

Let $I = \langle f, g \rangle \subseteq \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$ defines a complete intersection singularity and I_2 be the 2-jet of I. According to C.T.C. Wall's classification the 2-jet of $\langle f, g \rangle$ is a homogenous ideal generated by 2 polynomials of degree 2. We want to give a description of the type of a singularity without producing the normal form. C.T.C. Wall's classification is based on the classification of the 2-jet I_2 of $\langle f, g \rangle$. Let $\bigcap_{i=1}^{s} Q_i$ be the irredundant primary decomposition of I_2 in $\mathbb{C}[x, y, z]$. Let $d_i = dim_{\mathbb{C}}(\mathbb{C}[x, y, z]/Q_i), i = 1, ..., s$ and h_i be the Hilbert polynomial of $\mathbb{C}[x, y, z]/Q_i$. According to C.T.C. Wall's classification we obtain unimodular singularities only in the following cases.

Table 1:

Type	Characterization	Normal form of I_2
P	$s=2,\ d_1=d_2=1 \quad h_1=h_2=2$	$\langle x^2, yz\rangle$
J	$s=2,\ d_1=d_2=1 \quad h_1=1, h_2=4$	$\langle xy+z^2, xz\rangle$
F	$s=2,\ d_1=1, d_2=2 \quad h_1=1, h_2=1+t$	$\langle xy, xz\rangle$
H	$s=2,\ d_1=1, d_2=2 \quad h_1=2, h_2=1+t$	$\langle xy, x^2\rangle$
G	$s=1$ and $\sqrt{I_2}^3 \subseteq I_2$	$\langle x^2, y^2\rangle$
K	$s=1$ and $\sqrt{I_2}^3 \not\subseteq I_2$	$\langle xy+z^2, x^2\rangle$

3 Unimodular complete intersection space curve singularities

We set

$$l_i(x,y) = \begin{cases} xy^q, & \text{if } i=2q \\ y^{q+2}, & \text{if } i=2q+1 \end{cases}$$

for brevity.

Assume the 2-jet of $\langle f,g\rangle$ has normal form $\langle xy, z^2\rangle$. In this case according to C.T.C. Wall's classification the unimodular space curves are given in the table below

Table 2:

Type	Normal Form	μ	τ	Semigroup
$P_{k,l}$	$\langle xy, x^k+y^l+z^2\rangle$ $k \geq l \geq 3, k>3$	$l+k+1$	$l+k+1$	$\langle 1\rangle,\langle 1\rangle,\langle 1\rangle,\langle 1\rangle,\langle 1\rangle$ l, k even $\langle 1\rangle,\langle 1\rangle,\langle 2,k\rangle$ l is even, k is odd $\langle 2,k\rangle,\langle 2,l\rangle$ l, k odd

Proposition 3.1. *Let $I=< f,g >\subseteq \mathbb{C}[[x,y,z]]$ defines an isolated complete intersection singularity $(V(I),0)\subseteq(\mathbb{C}^3,0)$. Let μ be the Milnor number of I. Assume that the 2-jet of I has normal form $\langle x^2, yz\rangle$.*

If $(V(I),0)$ has 4 branches and all branches are smooth, let J_1, J_2, J_3 be the ideals of the strict transform of I blowing up $\mathbb{C}^3$ at 0 corresponding to the *three affine charts. Assume $(V(J_1),0)$ is an A_{l-3} singularity and $(V(J_2),0)$ is an A_{k-3} singularity. Then I is unimodular of type $P_{k,l}$, $k \geq 4$ and $l \geq 3$.*

If $(V(I),0)$ has 3 branches, two branches are smooth and the third branch has a semigroup generated by $(2,k)$ then I is unimodular of type $P_{k,\mu-k-1}$ if $(k,\mu) \neq (4,8)$ and $\mu - k > 3$.

If $(V(I),0)$ has 2 branches and the semigroup of the two branches are $(2,k)$ and $(2,l)$ then I is unimodular of type $P_{k,l}$ if $(k,l) \neq (3,3)$ and $(k,l) \neq (5,3)$.

Proof. Using lemma 3.2 (cf. [ADPG1]) we may assume $I = \langle x^2 + y^k + z^l + g, yz + h\rangle$, $3 \leq k \leq l \leq \infty, g \in \langle x,y,z\rangle^{l+1}, h \in \langle x,y,z\rangle^3$. According to Wall's classification we may assume that $I = \langle yz, x^2 + z^l + y^k\rangle$. Then $I = \langle z, x^2 + y^k\rangle \bigcap \langle y, x^2 + z^l\rangle$. If l and k are even then $(V(I),0)$ has 4 smooth branches. If $l+k$ is odd then $(V(I),0)$ has 3 branches, 2 of them smooth and the third defining an A_{k-1} respectively A_{l-1} singularity. If l and k are odd then $(V(I),0)$ has 2 branches, an A_{k-1} respectively A_{l-1} singularity. This proves the second and third part of the Proposition. For the first part we have to identify k and l. To do this we blow up of 0 of $\mathbb{C}^3$ and consider the strict transform in the 3 affine charts. We obtain $J_1 = \langle y, x^2 + z^{l-2}\rangle$, $J_2 = \langle z, x^2 + y^{k-2}\rangle$, $J_3 = \langle yz, 1 + z^l x^{l-2} + y^k x^{k-2}\rangle$. $(V(J_1),0)$ is an A_{l-3} singularity and $(V(J_2),0)$ is an A_{k-3} singularity. □

Algorithm 1 `Psingularity(I)`

Input: $I =< f, g >\in < x,y,z >^2 \mathbb{C}[[x,y,z]]$ and 2-jet of I having normal form (xy, z^2)

Output: the type of the singularity

compute μ =Milnor number of the I;
compute τ =Tjurina number of the I;
compute B =semigroups of I corresponding to the branches;
$T = findlk(I)$; *
if $\mu = \tau$ and $\mu = T[1] + T[2] + 1$ **then**
  if $T[1]$ and $T[2]$ even and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
    return $(P_{T[1],T[2]})$;
  if $T[1] + T[2]$ odd and $B = \langle 2, T[1]\rangle, \langle 1\rangle, \langle 1\rangle$ or $B = \langle 2, T[2]\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
    return $(P_{T[1],T[2]})$;
  if $T[1]$ and $T[2]$ odd and $B = (2, T[1]), (2, T[2])$ **then**
    return $(P_{T[1],T[2]})$;
return $(not \quad unimodular)$;

* `findlk(I)` is a procedure which computes k and l for the given $I = \langle xy, x^2 + y^l + z^k\rangle$.

Assume the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy, xz\rangle$. According to C.T.C. Wall's classification all unimodular curve singularities are in the μ-constant strata of the versal deformation of the curve singularities given in the Table 3.

Table 3:

Type	Normal Form	μ	τ	Semigroup
$FT_{4,4}$	$\langle xy+z^3, xz+y^3+\lambda yz^2\rangle$	10	10	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$
$FT_{k,l}$	$\langle xy+z^{l-1}, xz+y^{k-1}+yz^2\rangle$ $k \geq l \geq 4, k > 4$	$l+k+2$	$l+k+1$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ if k, l even $\langle 1\rangle, \langle 2, l-2\rangle, \langle 2, k-2\rangle$ if k, l odd $\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, k-2\rangle$ if k odd, l even
FW_{13}	$\langle xy+z^3, xz+y^4\rangle$	13	13	$\langle 1\rangle, \langle 4, 5, 11\rangle$
FW_{14}	$\langle xy+z^3, xz+zy^3\rangle$	14	14	$\langle 1\rangle, \langle 1\rangle, \langle 3, 4\rangle$
$FW_{1,0}$	$\langle xy+z^3, xz+z^2y^2+\lambda y^5\rangle$ $\lambda \neq 0, -1/4$	16	16	$\langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$
$FW_{1,i}$	$\langle xy+z^3, xz+z^2y^2+y^{5+i}\rangle$	$16+i$	$16+i-2$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle$ if i odd $\langle 1\rangle, \langle 2, 3\rangle, \langle 2, \mu-13\rangle$ if i even
$FW'_{1,i}$	$\langle xy+z^3, xz+2z^2y^2-y^5$ $+zy^2 l_i(z,y)\rangle$	$16+i$	$14+i-2$	$\langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ if i even $\langle 1\rangle, \langle 4, 6, \tau-2, \tau\rangle$ if i odd
FW_{18}	$\langle xy+z^3, xz+zy^4\rangle$	18	18	$\langle 1\rangle, \langle 1\rangle, \langle 3, 5\rangle$
FW_{19}	$\langle xy+z^3, xz+y^6\rangle$	19	19	$\langle 1\rangle, \langle 4, 7, 17\rangle$
FZ_{6m+6}	$\langle xy, xz+z^3+y^{3m+1}\rangle$	$6m+6$	$6m+6$	$\langle 1\rangle, \langle 1\rangle, \langle 3, 3m+1\rangle$
FZ_{6m+7}	$\langle xy, xz+z^3+zy^{2m+1}\rangle$	$6m+7$	$6m+7$	$\langle 1\rangle, \langle 1\rangle, \langle 2, 2m+1\rangle$
FZ_{6m+8}	$\langle xy, xz+z^3+y^{3m+2}\rangle$	$6m+8$	$6m+8$	$\langle 1\rangle, \langle 1\rangle, \langle 3, 3m+2\rangle$
$FZ_{m-1,0}$	$\langle xy, xz+z^3+z^2y^m+\lambda y^{3m}\rangle$ $\lambda \neq 0, -4/27$	$6m+4$	$6m+4$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 2m+i\rangle$
$FZ_{m-1,i}$	$\langle xy, xz+z^3+z^2y^m+y^{3m+i}\rangle$	$6m+4+i$	$5m+4+i$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 2m+i\rangle$ if i odd $\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ if i even

Proposition 3.2. *The unimodular complete intersection curve singularities with Milnor number* 13*,* 2 *branches and semigroup* $\langle 1\rangle, \langle 4, 5, 11\rangle$ *are* FW_{13} *with Tjurina number* 13 *defined by the ideal* $\langle xy + z^3, xz + y^4\rangle$ *and* $FW_{13,1}$ *with Tjurina number* 12 *defined by the ideal* $\langle xy + z^3, xz + y^4 + y^2z^2\rangle$*.*

Proof. In the list of C.T.C. Wall FW_{13} defined by the ideal $\langle xy+z^3, xz+y^4\rangle$ is the only singularity with $\mu = 13$, 2 branches and semigroup $\langle 1\rangle$, $\langle 4, 5, 11\rangle$.

The versal deformation of FW_{13} is given by $\langle xy+z^3+\nu_1z^2+\nu_2z+\nu_3, xz+y^4+\lambda_1y^2z^2+\lambda_2yz^2+\lambda_3z^2+\lambda_4y^2z+\lambda_5yz+\lambda_6z+\lambda_7y^3+\lambda_8y^2+\lambda_9\rangle$. FW_{13}

defines a weighted homogenous isolated complete intersection singularity with weights $(w_1, w_2, w_3) = (11, 4, 5)$ and degrees $(d_1, d_2) = (15, 16)$. The versal μ-constant deformation of FW_{13} is given by $\langle xy + z^3, xz + y^4 + \lambda_1 y^2 z^2\rangle$.

Using the coordinate change $x \to \xi^{11}x, y \to \xi^4 y, z \to \xi^5 z$ we have $I_\lambda = \langle xy + z^3, xz + y^4 + \xi^2\lambda_1 y^2 z^2\rangle$. Choose ξ such that $\xi^2\lambda_1 = 1$. So we obtain $\langle xy + z^3, xz + y^4 + y^2 z^2\rangle$. It has 2 branches and same semigroup as FW_{13} and $\tau = 12$.

It can be distinguished from FW_{13} by the Tjurina number. □

Proposition 3.3. *The unimodular complete intersection curve singularities with Milnor number* 14*,* 3 *branches two of them are smooth and the third branch has semigroup* $\langle 3, 4\rangle$ *are* FW_{14} *with Tjurina number* 14 *defined by the ideal* $\langle xy + z^3, xz + zy^3\rangle$ *and* $FW_{14,1}$ *with the Tjurina number* 13 *defined by the ideal* $\langle xy + z^3, xz + zy^3 + y^5\rangle$*.*

Proposition 3.4. *The unimodular complete intersection curve singularities with Milnor number* 16*,* 3 *branches and semigroup* $\langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ *are* $FW_{1,0}$ *with Tjurina number* $\tau = 16$ *defined by the ideal* $\langle xy + z^3, xz + z^2y^2 + \lambda y^5\rangle$ *and* $FW_{1,0,1}$ *Tjurina number* $\tau = 15$ *defined by the ideal* $\langle xy + z^3, xz + z^2y^2 + \lambda y^5 + y^6\rangle$*.*

Proof. The proofs of propositions 3.3 and 3.4 are similar to the proof of proposition 3.2. □

Proposition 3.5. *The unimodular complete intersection curve singularities with Milnor number* 18*,* 3 *branches and semigroup* $\langle 1\rangle, \langle 1\rangle, \langle 3, 5\rangle$ *are* FW_{18} *with Tjurina number* $\tau = 18$ *defined by the ideal* $\langle xy+z^3, xz+zy^4\rangle$ $\tau = 18$*,* $FW_{18,1}$ *with Tjurina number* $\tau = 17$ *defined by the ideal* $\langle xy + z^3, xz + zy^4 + y^7\rangle$ *and* $FW_{18,2}$ *with Tjurina number* $\tau = 16$ *defined by the ideal* $\langle xy+z^3, xz+zy^4+y^6\rangle$*.*

Proof. In the list of C.T.C. Wall FW_{18} defined by the ideal $\langle xy+z^3, xz+zy^4\rangle$ is the only singularity with $\mu = 18$, 3 branches and semigroup $\langle 1\rangle, \langle 1\rangle, \langle 3, 5\rangle$.

The versal deformation of FW_{18} is given by $\langle xy+z^3+\nu_1 z^2+\nu_2 z+\nu_3, xz+y^4z+\lambda_1 y^2z^2+\lambda_2 yz^2+\lambda_3 z^2+\lambda_4 y^3z+\lambda_5 y^2z+\lambda_6 yz+\lambda_7 z+\lambda_8 y^7+\lambda_9 y^6+\lambda_{10} y^5+\lambda_{11} y^4 + \lambda_{12} y^3 + \lambda_{13} y^2 + \lambda_{14} y + \lambda_{15}\rangle$. FW_{18} defines a weighted homogenous isolated complete intersection singularity with weights $(w_1, w_2, w_3) = (12, 3, 5)$ and degrees $(d_1, d_2) = (15, 17)$. The versal μ-constant deformation of FW_{18} is given by $\langle xy + z^3, xz + y^4z + \lambda_8 y^7 + \lambda_9 y^6\rangle$.

If $\lambda_9 \neq 0$ then we have $I = \langle xy + z^3, xz + y^4z + uy^6\rangle$ where $u = \lambda_8 y + \lambda_7$.

Using the coordinate change $x \to \xi^{12}x, y \to \xi^3 y, z \to \xi^5 z$ we may assume $I = \langle xy + z^3, xz + y^4z + \xi\bar{u}y^6\rangle$. Choose ξ such that $\xi\bar{u} = 1$. So $I = \langle xy + z^3, xz + y^4z + y^6\rangle$. It has Tjurina number $\tau = 17$, two branches and the same semigroup as FW_{18}.

If $\lambda_9 = 0$ we again apply the same transformation and obtain $I = \langle xy + z^3, xz + y^4z + y^7\rangle$ by choosing $\lambda_8^4 = 1$. It has Tjurina number $\tau = 16$, also 3 branches and the same semigroup as FW_{18}. $FW_{18,1}$ and $FW_{18,2}$ can be differentiated from FW_{18} by the Tjurina numbers. □

Proposition 3.6. *The unimodular complete intersection curve singularities with Milnor number* 19*, 2 branches and semigroup* $\langle 1\rangle, \langle 4, 7, 17\rangle$ *are* FW_{19} *with Tjurina number* $\tau = 19$ *defined by the ideal* $\langle xy + z^3, xz + y^6\rangle$*,* $FW_{19,1}$ *with Tjurina number* $\tau = 18$ *defined by the ideal* $\langle xy+z^3, xz+y^6+y^4z^2\rangle$ *and* $FW_{19,2}$ *with Tjurina number* $\tau = 17$ *defined by the ideal* $\langle xy + z^3, xz + y^6 + y^3z^2\rangle$ *.*

Proof. The proof can be done similarly to the proof of proposition 3.5. □

Proposition 3.7. *The unimodular complete intersection singularities having Milnor number of the form* $\mu = 6m + 6$ *where* m *is a positive integer with* 3 *branches , 2 branches are smooth and third branch has semigroup* $\langle 3, 3m + 1\rangle$ *are* FZ_{6m+6} *defined by the ideal* $\langle xy, xz + z^3 + y^{3m+1}\rangle$ *with Tjurina number* $\tau = 6m + 6$ *and* $FZ_{6m+6,i+1}$*, i=0,1,...,m-1, defined by the ideal* $\langle xy, xz + z^3 + y^{3m+1} + y^{3m-i}z\rangle$ *with Tjurina number* $\tau = \mu - i - 1$*.*

Proof. In C.T.C. Wall's list FZ_{6m+6}, $m \geq 1$ defined by the ideal $\langle xy, xz + z^3 + y^{3m+1}\rangle$ are the singularities with $\mu = 6m + 6$, 3 branches and semigroup $\langle 1\rangle, \langle 1\rangle, \langle 3, 3m + 1\rangle$.

The versal deformation of FZ_{6m+6} is given by $\langle xy+\nu_1z^2+\nu_2z+\nu_3, xz+z^3+ y^{3m+1} + \sum_{i=0}^{3m} \alpha_i y^{3m-i}z + \sum_{i=0}^{3m} \beta_i y^{3m-i}\rangle$. FZ_{6m+6} defines a weighted homogenous isolated complete intersection singularity with weights $(w_1, w_2, w_3) = (6m + 2, 3, 3m + 1)$ and degrees $(d_1, d_2) = (6m + 5, 9m + 3)$. The versal μ-constant deformation of FZ_{6m+6} is given by $\langle xy, xz + z^3 + y^{3m+1} + \sum_{i=0}^{m-1} \alpha_i y^{3m-i}z\rangle$.

Consider $\phi \in Aut_{\mathbb{C}}(\mathbb{C}[[x, y, z]])$ defined by $\phi(x) = \xi^{6m+2}x$, $\phi(y) = \xi^3y$ and $\phi(z) = \xi^{3m+1}z$. If $\alpha_{m-1} \neq 0$, then $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1}z (\sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1})\rangle$. Let $u_{m-1} = \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1}$ then $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1} zu_{m-1}\rangle$. By applying the transformation ϕ we get $I = \langle xy, xz + z^3 + y^{3m+1} + \xi y^{2m+1}z\bar{u}_{m-1}\rangle$. Choose ξ such that $\xi\bar{u}_{m-1} = 1$ This implies $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1}z\rangle$.

Now we assume $\alpha_{m-1} = 0$. This implies $I = \langle xy, xz+z^3+y^{3m+1}+y^{2m+2}z (\sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}\rangle$. Let $u_{m-2} = \sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}$ then $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+2}zu_{m-2}\rangle$. After applying ϕ we may assume that $I = \langle xy, xz + z^3 + y^{3m+1} + \xi^4y^{2m+2}z\bar{u}_{m-2}\rangle$. Choose ξ such that $\xi^4\bar{u}_{m-2} = 1$.

This implies $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+2}z\rangle$. If $\alpha_{m-2} = 0$ then we assume $\alpha_{m-3} \neq 0$.

We may iterate this process and we get m different unimodular singularities $I = \langle xy, xz+z^3+y^{3m+1}+y^{3m-i}z\rangle, \quad i = 0, 1, ..., m-1$ having Tjurina number $\tau = \mu - i - 1$ and the same semigroup as FZ_{6m+6}. These singularities can be distinguished from FZ_{6m+6} by the Tjurina numbers. □

Proposition 3.8. *The unimodular complete intersection singularities having Milnor number of the form $\mu = 6m + 7$ where m is a positive integer with 4 branches , 3 branches are smooth and the fourth branch has semigroup generated by $\langle 2, 2m + 1\rangle$ are FZ_{6m+7} with Tjurina number $\tau = 6m + 7$ defined by the ideal $\langle xy, xz + z^3 + zy^{2m+1}\rangle$ and $FZ_{6m+7,i}$, $i = 1, ..., m$ with Tjurina number $\tau = \mu - i$ defined by the ideal $\langle xy, xz + z^3 + zy^{2m+1} + y^{4m+2-i}\rangle$.*

Proposition 3.9. *The unimodular complete intersection singularities having Milnor number of the form $\mu = 6m + 8$ where m is a positive integer with 3 branches and the semigroup $\langle 1\rangle, \langle 1\rangle, \langle 3, 3m + 2\rangle$ are FZ_{6m+8} defined by the ideal $\langle xy, xz + z^3 + y^{3m+2}\rangle$ with Tjurina number $\tau = 6m + 8$ and $FZ_{6m+8,i}$, $i = 1, ..., m$ defined by the ideal $\langle xy, xz + z^3 + y^{3m+2} + y^{3m+2-i}z\rangle$ with Tjurina number $\tau = \mu - i$.*

Proof. The proofs of propositions 3.8 and 3.9 are similar to proof of proposition 3.7. □

Summarizing the results of the above propositions we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal form $\langle xy, xz\rangle$.

Proposition 3.10. *Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^3, 0)$ be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy, xz\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Table 3 and 4.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.2 - 3.9. □

Algorithm 2 `Fsingularity(I)`

Input: $I =< f, g >\in< x, y, z >^2 \mathbb{C}[[x, y, z]]$ and 2-jet of I having normal form (xy, xz).
Output: the type of the singularity

compute μ =Milnor number of the I;
compute τ =Tjurina number of the I;
compute B =semigroups of I corresponding to the branches;
if $\mu = 10$ and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
  if $\mu = \tau$ **then**
    return $(FT_{4,4})$;
if $\mu = 13$ and $B = \langle 1\rangle, \langle 4, 5, 11\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FW_{13});
  if $\mu - \tau = 1$ **then**
    return $(FW_{13,1})$;
if $\mu = 14$ and $A = \langle 1\rangle, \langle 1\rangle, \langle 3, 4\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FW_{14});
  if $\mu - \tau = 1$ **then**
    return $(FW_{14,1})$;
if $\mu = 18$ and $A = \langle 1\rangle, \langle 1\rangle, \langle 3, 5\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FW_{18});
  else
    return $(FW_{18,\mu-\tau})$;
if $\mu = 19$ and $B = \langle 1\rangle, \langle 4, 7, 17\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FW_{19});
  else
    return $(FW_{19,\mu-\tau})$;
if $\mu = 16$ and $B = \langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
  if $\mu = \tau$ **then**
    return $(FW_{1,0})$;
  if $\mu - \tau = 1$ **then**
    return $(FW_{1,0,1})$;
if $\mu \equiv 0 \bmod 6$, $\mu > 11$ and $B = \langle 1\rangle, \langle 1\rangle, \langle 3, 3(\mu - 6)/6 + 1\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FZ_μ);
  else
    return $(FZ_{\mu,\mu-\tau})$;

†

if $\mu \equiv 1 \bmod 6$, $\mu > 12$ and $B = \langle 1\rangle, \langle 1\rangle, \langle 2, 2(\mu-7)/6+1\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FZ_{μ});
  else
    return $(FZ_{\mu,\mu-\tau})$;
if $\mu \equiv 2 \bmod 6$, $\mu > 13$ and $B = \langle 1\rangle, \langle 1\rangle, \langle 3, 3(\mu-8)/6+2\rangle$ **then**
  if $\mu = \tau$ **then**
    return (FZ_{μ});
  else
    return $(FZ_{\mu,\mu-\tau})$;
if $\mu \equiv 4 \bmod 6$ and $\mu > 9$ **then**
  $m = (\mu-4)/6$;
  if $\mu = \tau$ **then**
    if m is even and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
      return $(FZ_{m-1,0})$;
    if m is odd and $B = \langle 1\rangle, \langle 1\rangle, \langle 2, 3(\mu-4)/6\rangle$ **then**
      return $(FZ_{m-1,0})$;
if $\mu \neq \tau$ **then**
  if $\mu - \tau = 1$ **then**
    $T = findlk(I)$;
    if $\mu = T[1] + T[2] + 2$ **then**
      if $T[1]$ and $T[2]$ even and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
        return $(FT_{T[1],T[2]})$;
      if $T[1] + T[2]$ odd and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, k-2\rangle$ or $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, l-2\rangle$ **then**
        return $(FT_{T[1],T[2]})$;
      if $T[1]$ and $T[2]$ odd and $B = \langle 1\rangle, \langle 2, l-2\rangle, \langle 2, k-2\rangle$ **then**
        return $(FT_{T[1],T[2]})$;
  if μ is odd and $\mu > 16$ **then**
    if $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle\langle 2, 3\rangle$ **then**
      return $(FW_{1,\mu-16})$;
    if $B = \langle 1\rangle, \langle 4, 6, \mu-4, \mu-2\rangle$ **then**
      return $(FW'_{1,\mu-16})$;
  if μ is even and $\mu > 16$ **then**
    if $B = \langle 1\rangle, \langle 2, 3, \langle 2, \mu-13\rangle$ **then**
      return $(FW_{1,\mu-16})$;
    if $B = \langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
      return $(FW'_{1,\mu-16})$;
  if μ is even, $\mu \geq 11$ and $B = \langle 1, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
    return $(FZ_{\mu-\tau-1,6\tau-5\mu-4})$;
  if μ is odd, $\mu \geq 11$ and $B = \langle 1, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 4\tau-3\mu-4, 2\rangle$ **then**
    return $(FZ_{\mu-\tau-1,6\tau-5\mu-4})$;

Table 4:

Type	Normal Form	μ	τ	Semigroup
$FW_{13,1}$	$\langle xy+z^3, xz+y^4+y^2z^2\rangle$	13	12	$\langle 1\rangle, \langle 4,5,11\rangle$
$FW_{14,1}$	$\langle xy+z^3, xz+zy^3+y^5\rangle$	14	13	$\langle 1\rangle, \langle 1\rangle, \langle 3,4\rangle$
$FW_{1,0,1}$	$\langle xy+z^3, xz+z^2y^2+\lambda y^5+y^6\rangle$ $\lambda \neq 0, -1/4$	16	15	$\langle 1\rangle, \langle 2,3\rangle, \langle 2,3\rangle$
$FW_{18,1}$	$\langle xy+z^3, xz+zy^4+y^7\rangle$	18	17	$\langle 1\rangle, \langle 1\rangle, \langle 3,5\rangle$
$FW_{18,2}$	$\langle xy+z^3, xz+zy^4+y^6\rangle$	18	16	$\langle 1\rangle, \langle 1\rangle, \langle 3,5\rangle$
$FW_{19,1}$	$\langle xy+z^3, xz+y^6+y^4z^2\rangle$	19	18	$\langle 1\rangle, \langle 4,7,17\rangle$
$FW_{19,2}$	$\langle xy+z^3, xz+y^6+y^3z^2\rangle$	19	17	$\langle 1\rangle, \langle 4,7,17\rangle$
$FZ_{6m+6,i}$	$\langle xy, xz+z^3+y^{3m+1}+y^{3m-i}z\rangle$ $i=0,1,....,m-1$	$6m+6$	$6m+5-i$	$\langle 1\rangle, \langle 1\rangle, \langle 3,3m+1\rangle$
$FZ_{6m+7,i}$	$\langle xy, xz+z^3+zy^{2m+1}+y^{4m+2-i}\rangle$ $i=1,....,m$	$6m+7$	$6m+7-i$	$\langle 1\rangle, \langle 1\rangle, \langle 2,2m+1\rangle$
$FZ_{6m+8,i}$	$\langle xy, xz+z^3+y^{3m+2}+y^{3m+2-i}z\rangle$ $i=1,....,m$	$6m+8$	$6m+8-i$	$\langle 1\rangle, \langle 1\rangle, \langle 3,3m+2\rangle$

Proposition 3.11. *Assume the 2-jet of $\langle f,g\rangle$ has normal form $\langle x^2\ ,y^2\rangle$. According to C.T.C. Wall's classification the unimodular space curve singularities are given in the table below*

Table 5:

Type	Normal Form	μ	τ	Semigroup
G_{2n+3} $n\geq 3$	(x^2+z^3, y^2+z^n)	$2n+3$	$2n+3$	$(2,3),(2,3)$
G_{2n+6} $n\geq 2$	(x^2+z^3, y^2+xz^n)	$2n+6$	$2n+6$	$(4,6,2n+3)$

Algorithm 3 `Gsingularity(I)`

Input: $I =< f, g >\in< x, y, z >^2 \mathbb{C}[[x, y, z]]$ having 2-jet of the form (x^2, y^2).
Output: the type of the singularity

1: compute μ =Milnor number of I;
2: compute τ =Tjurina number of the I;
3: compute B =semigroups of I corresponding to the branches;
4: **if** $\mu = \tau$ **then**
5:   **if** μ is even and $B = (4, 6, \mu - 3)$ **then**
6:     **return** (G_μ);
7:   **if** μ is odd and $B = (2, 3), (2, 3)$ **then**
8:     **return** (G_μ);
9: **return** $(not \quad unimodular)$;

Assume the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy, x^2\rangle$. According to C.T.C. Wall's classification all unimodular curve singularities are in the μ-constant strata of the versal deformation of the curve singularities given in the table below

Table 6:

Type	Normal Form	μ	τ	Semigroup
HA_{11}	$\langle xy + z^3, x^2 + z^3 + yz^2 + y^3\rangle$	11	11	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle$
HA_{r+11}	$\langle xy + z^3, x^2 + z^3 + yz^2 + y^{3+r}\rangle$ $r \geq 1$	$r + 11$	$r + 10$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle, \quad \mu \quad odd$ $\langle 1\rangle, \langle 2, 3\rangle, \langle 2, 2 + r\rangle, \mu \quad even$
HB_{r+13}	$\langle xy + z^3, x^2 + yz^2 + y^{4+r}\rangle$ $r \geq 0$	$r + 13$	$r + 12$	$\langle 3, 4, 5\rangle, \langle 2, 3 + r\rangle$
HC_{13}	$\langle xy + z^3, x^2 + z^3 + y^4\rangle$	13	13	$\langle 2, 3\rangle, \langle 3, 4\rangle$
HC_{14}	$\langle xy + z^3, x^2 + z^3 + zy^3\rangle$	14	14	$\langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$
HC_{15}	$\langle xy + z^3, x^2 + z^3 + y^5\rangle$	15	15	$\langle 2, 3\rangle, \langle 3, 5\rangle$
HD_{13}	$\langle xy + z^3, x^2 + zy^2\rangle$	13	13	$\langle 1\rangle, \langle 4, 5, 7\rangle$
HD_{14}	$\langle xy + z^3, x^2 + y^3\rangle$	14	14	$\langle 5, 6, 9\rangle$

Proposition 3.12. *The unimodular complete intersection curve singularities with Milnor number* 13*,* 2 *branches and with semigroups* $\langle 2, 3\rangle, \langle 3, 4\rangle$ *are* HC_{13} *with Tjurina number* 13 *defined by the ideal* $\langle xy+z^3, x^2+z^3+y^4\rangle$ *and* $HC_{13,1}$ *with Tjurina number* $\tau = 12$ *defined by the ideal* $\langle xy + z^3, x^2 + z^3 + y^4 + y^3z\rangle$*.*

The singularities having the same Milnor number but being irreducible with semigroup $\langle 5,6,9\rangle$ are HD_{13} defined by the ideal $\langle xy+z^3, x^2+zy^2\rangle$ with Tjurina number 13 and $HD_{13,1}$ with Tjurina number $\tau=12$ defined by the ideal $\langle xy+z^3+yz^2, x^2+zy^2+zy^{13}\rangle$.

Proof. In the list of C.T.C. Wall HC_{13} defined by the ideal $I=\langle xy+z^3, x^2+z^3+y^4\rangle$ is the only singularity with $\mu=13$, 2 branches and semigroup $\langle 2,3\rangle,\langle 3,4\rangle$.

We may choose an automorphism $\psi\in Aut_{\mathbb{C}}(\mathbb{C}[[x,y,z]])$ such that $\psi(I)=\langle xy, x^2+z^3+y^4\rangle$. The versal deformation is given by $\langle xy+\nu_1z^2+\nu_2yz+\nu_3z+\nu_4y+\nu_5, x^2+z^3+y^4+\lambda_1y^3z+\lambda_2y^2z+\lambda_3yz+\lambda_4z+\lambda_5y^3+\lambda_6y^2+\lambda_7y+\lambda_8\rangle$. HC_{13} defines a weighted homogenous isolated complete intersection singularity with weights $\langle w_1,w_2,w_3\rangle=\langle 6,3,4\rangle$ and degrees $\langle d_1,d_2\rangle=\langle 9,12\rangle$. The versal μ-constant deformation of HC_{13} is given by $\langle xy, x^2+z^3+y^4+\lambda_1y^3z\rangle$. Using the coordinate change $x\to\xi^6x, y\to\xi^3y, z\to\xi^4z$, we may assume $I_{\lambda_1}=\langle xy+z^3, x^2+z^3+y^4+\xi\lambda_1y^3z\rangle$. Choose ξ such that $\xi\lambda_1=1$. So $\langle xy, x^2+z^3+y^4+y^3z\rangle$ has two branches with same semigroup as HC_{13} and Tjurina number $\tau=12$. It can be differentiated from HC_{13} by the Tjurina number.

In C.T.C. Wall'list HD_{13} defined by the ideal $\langle xy+z^3, x^2+zy^2\rangle$ is the only singularity with $\mu=13$, 2 branches and semigroup $\langle 1\rangle,\langle 4,5,7\rangle$.

The versal deformation of HD_{13} is given by $\langle xy+z^3+\nu_1yz^2+\nu_2z^2+\nu_3yz+\nu_4z+\nu_5y+\nu_6, x^2+zy^2+\lambda_1z^3+\lambda_2yz^2+\lambda_3z^2+\lambda_4yz+\lambda_5z+\lambda_6y+\lambda_7\rangle$. HD_{13} defines a weighted homogenous isolated complete intersection singularity with weights $\langle w_1,w_2,w_3\rangle=\langle 7,5,4\rangle$ and degrees $\langle d_1,d_2\rangle=\langle 12,14\rangle$. The versal μ-constant deformation of HD_{13} is given by $\langle\ xy+z^3+\nu_1yz^2,\ x^2+y^2z\ \rangle$.

Using the coordinate change $x\to\xi^7x, y\to\xi^5y, z\to\xi^4z$, we may assume $I_\lambda=\langle xy+z^3+\xi\nu_1yz^2, x^2+y^2z\rangle$. Choose ξ such that $\xi\nu_1=1$. So $\langle xy+z^3+yz^2, x^2+y^2z\rangle$ has 2 branches with same semigroup as HD_{13} and Tjurina number $\tau=12$. It can be differentiated from HD_{13} by the Tjurina number. □

Proposition 3.13. *The unimodular complete intersection curve singularities with Milnor number 14, 3 branches and with semigroup $\langle 1\rangle,\langle 2,3\rangle$, $\langle 2,\ 3\rangle$ are HC_{14} with Tjurina number 14 defined by the ideal $\langle xy+z^3, x^2+z^3+zy^3\rangle$ and $HC_{14,1}$ with Tjurina number $\tau=13$ defined by the ideal $\langle xy+z^3, x^2+z^3+zy^3+y^5\rangle$. The singularities having same Milnor number but irreducible with semigroup $\langle 5,6,9\rangle$ are HD_{14} defined by the ideal $\langle xy+z^3, x^2+y^3\rangle$ with Tjurina number 13 and $HD_{14,1}$ with Tjurina number $\tau=13$ defined by the ideal $\langle xy+z^3, x^2+y^3+z^4\rangle$.*

Proposition 3.14. *The unimodular complete intersection curve singularities with Milnor number 15, 2 branches and semigroup $\langle 2,3\rangle,\langle 3,5\rangle$ are HC_{15} with*

Tjurina number 15 defined by the ideal $\langle xy+z^3, x^2+z^3+y^5\rangle$ *and* $HC_{15,1}$ *with Tjurina number* $\tau = 14$ *defined by the ideal* $\langle xy+z^3, x^2+z^3+y^5+y^4z\rangle$.

Proof. The proofs of Propositions 3.13 and 3.14 are similar to proof of Proposition 3.12. □

Summarizing the results of the propositions above we complete the list of unimodular complete intersection singularities in case of $\langle f, g\rangle$ having 2-jet with normal form $\langle xy, x^2\rangle$.

Table 7:

Type	Normal Form	μ	τ	Semigroup
$HC_{13,1}$	$\langle xy+z^3, x^2+z^3+y^4+y^3z\rangle$	13	12	$\langle 2,3\rangle, \langle 3,4\rangle$
$HC_{14,1}$	$\langle xy+z^3, x^2+z^3+zy^3+y^5\rangle$	14	13	$\langle 1\rangle, \langle 2,3\rangle, \langle 2,3\rangle$
$HC_{15,1}$	$\langle xy+z^3, x^2+z^3+y^5+y^4z\rangle$	15	14	$\langle 2,3\rangle, \langle 3,5\rangle$
$HD_{13,1}$	$\langle xy+z^3+yz^2, x^2+zy^2\rangle$	13	12	$\langle 1\rangle, \langle 4,5,7\rangle$
$HD_{14,1}$	$\langle xy+z^3, x^2+y^3+z^4\rangle$	14	13	$\langle 5,6,9\rangle$

Proposition 3.15. *Let* $(V(\langle f,g\rangle),0) \subseteq (\mathbb{C}^3,0)$ *be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of* $\langle f,g\rangle$ *has normal form* $\langle xy, x^2\rangle$. $(V(\langle f,g\rangle),0)$ *is unimodular if and only if it is isomorphic to a complete intersection in the Table 6 and 7.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.12 - 3.14. □

Algorithm 4 `Hsingularity(I)`

Input: $I =< f, g >\in< x, y, z >^2 \mathbb{C}[[x, y, z]]$ 2-jet of I having normal form (xy, x^2)

Output: the type of the singularity

compute μ =Milnor number of the I and τ =Tjurina number of the I;
compute B =semigroups of I corresponding to the branches;
if $\mu = 13$ **then**
  if $\mu = \tau$ **then**
    if $B = \langle 2, 3\rangle, \langle 3, 4\rangle$ **then**
      return (HC_{13});
    if $B = \langle 1\rangle, \langle 4, 5, 7\rangle$ **then**
      return (HD_{13});
  else
    if $B = \langle 2, 3\rangle, \langle 3, 4\rangle$ **then**
      return $(HC_{13,\mu-\tau})$;
    if $B = \langle 1\rangle, \langle 4, 5, 7\rangle$ **then**
      return $(HD_{13,\mu-\tau})$;
if $\mu = 14$ **then**
  if $\mu = \tau$ **then**
    if $B = \langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
      return (HC_{14});
    if $B = \langle 5, 6, 9\rangle$ **then**
      return (HD_{14});
  else
    if $B = \langle 1\rangle, \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
      return $(HC_{14,\mu-\tau})$;
    if $B = \langle 5, 6, 9\rangle$ **then**
      return $(HD_{14,\mu-\tau})$;
if $\mu = 15$ and $B = \langle 2, 3\rangle, \langle 3, 5\rangle$ **then**
  if $\mu = \tau$ **then**
    return (HC_{15});
  else
    return $(HC_{15,\mu-\tau})$;
if $\mu = 11$ and $A = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle$ **then**
  if $\mu = \tau$ **then**
    return (HA_{11});
if $\mu \neq \tau$ **then**
  if $\mu - \tau = 1$ and $\mu > 11$ **then**
    if μ is even and $B = \langle 1\rangle, \langle 2, 3\rangle, \langle 2, \mu - 9\rangle$ **then**
      return (HA_μ);
    if μ is odd and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle$ **then**
      return (HA_μ);
    if μ is even and $B = \langle 1\rangle, \langle 1\rangle, \langle 3, 4, 5\rangle$ **then**
      return (HB_μ);
    if μ is odd and $B = \langle 3, 4, 5\rangle, \langle 2, \mu - 10\rangle$ **then**
      return (HB_μ);
return *not unimodular*;

Assume the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy + z^2, xz\rangle$. According to C.T.C. Wall's classification all unimodular curve singularities are in the μ-constant strata of the versal deformation of the curve singularities given in the table below

Table 8:

Type	Normal Form	μ	τ	Semigroup
J_{6m+7}	$\langle xy + z^2, xz + y^{3m+3}\rangle$ $\lambda \neq 0, -4/27$	$6m + 7$	$6m + 7$	$\langle 1\rangle, \langle 3, 3m + 4, 6m + 5\rangle$
J_{6m+8}	$\langle xy + z^2, xz + zy^{2m+2}\rangle$	$6m + 8$	$6m + 8$	$\langle 1\rangle, \langle 1\rangle, \langle 2, 2m + 3\rangle$
J_{6m+9}	$\langle xy + z^2, xz + y^{3m+4}\rangle$	$6m + 9$	$6m + 9$	$\langle 1\rangle, \langle 3m + 5, 6m + 7\rangle$
$J_{m+1,0}$	$\langle xy + z^2, xz + z2y^m + \lambda y^{3m+2}\rangle$	$6m + 5$	$6m + 5$	$\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$
$J_{m+1,i}$	$\langle xy + z^2, xz + z2y^m + y^{3m+2+i}\rangle$	$6m + i + 5$	$5m + i + 5$	$\langle 1\rangle, \langle 1\rangle, \langle 2, 2m + i + 2\rangle$, if i is odd $\langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ if i is even

Proposition 3.16. *The unimodular complete intersection singularities having Milnor number of the form $\mu = 6m+7$ where m is a positive integer with 2 branches and semigroup generated by $\langle 1\rangle, \langle 3, 3m+4, 6m+5\rangle$ are J_{6m+7} defined by the ideal $\langle xy+z^2, xz+y^{3m+3}\rangle$ with Tjurina number $\tau = 6m+7$ and $J_{6m+7,i}$, $i = 0, ..., m-1$ defined by the ideal $\langle xy + z^2, xz + y^{3m+3} + y^{(3m+1)-i}z\rangle$ with Tjurina number $\tau = \mu - i$.*

Proof. In C.T.C. Wall's list J_{6m+7}, $m \geq 1$ defined by $\langle xy + z^2, xz + y^{3m+3}\rangle$ are the singularities with $\mu = 6m + 7$, 2 branches and semigroup $\langle 1\rangle, \langle 3, 3m + 4, 6m + 5\rangle$.

The versal deformation of J_{6m+7} is given by $\langle xy+z^2+\nu_1 z+\nu_2, xz+y^{3m+3}+ \sum_{i=0}^{3m+1} \alpha_i y^{(3m+1)-i} z+ \sum_{i=0}^{3m+2} \beta_i y^{(3m+2)-i}\rangle$. J_{6m+7} defines a weighted homogenous isolated complete intersection singularity with weights $(w_1, w_2, w_3) = (6m + 5, 3, 3m + 4)$ and degrees $(d_1, d_2) = (6m + 8, 9m + 9)$. The versal μ-constant deformation of J_{6m+7} is given by $\langle xy + z^2, xz + y^{3m+3} + \sum_{i=0}^{m-1} \alpha_i y^{(3m+1)-i} z\rangle$.

Consider $\phi \in Aut_{\mathbb{C}}(\mathbb{C}[[x, y, z]]$ defined by $\phi(x) = \xi^{6m+5}x$, $\phi(y) = \xi^3 y$ and $\phi(z) = \xi^{3m+4}z$. If $\alpha_{m-1} \neq 0$ then $I = \langle xy + z^2,\ xz + y^{3m+3} + y^{2m+2}z$ $(\sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1})\rangle$.

Let $u_{m-1} = \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1}$. Then $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+2}zu\rangle$. By applying the transformation ϕ we get $I = \langle xy+z^2, xz+y^{3m+3}+\xi y^{2m+2}z\bar{u}_{m-1}\rangle$. Choose ξ such that $\xi\bar{u}_{m-1} = 1$. This implies $I = \langle xy+z^2, xz+y^{3m+3} + y^{2m+2}z\rangle$.

If $\alpha_{m-1} = 0$ then $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+3}z(\sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}\rangle$. Let $u_{m-2} = \sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i}z + \alpha_{m-2}$ then $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+3}zu_{m-2}\rangle$. After applying ϕ we may assume that $I = \langle xy + z^2, xz + y^{3m+3} + \xi^4 y^{2m+3}z\bar{u}_{m-2}\rangle$. Choose ξ such that $\xi^4\bar{u}_{m-2} = 1$. This implies $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+3}z\rangle$.

If $\alpha_{m-2} = 0$. We may iterate this process and we get m different unimodular space curve singularities $I = \langle xy + z^2, xz + y^{3m+3} + y^{(3m+1)-i}z\rangle \quad, i = 0, 1, ..., m-1$ having Tjurina number $\tau = \mu - i$. These singularities can be distinguished from J_{6m+7} by the Tjurina numbers. □

Proposition 3.17. *The unimodular complete intersection singularities having Milnor number of the form $\mu = 6m + 8$ where m is a positive integer with 3 branches and semigroup generated by $\langle 1\rangle, \langle 1\rangle, \langle 2, 2m+3\rangle$ are J_{6m+8} with Tjurina number $\tau = 6m + 8$ defined by the ideal $\langle xy + z^2, xz + zy^{2m+2}\rangle$ and $J_{6m+8,i}$, $i = 1, ..., m$ with Tjurina number $\tau = \mu - i$ defined by the ideal $\langle xy + z^2, xz + zy^{2m+2} + y^{4m+4-i}\rangle$.*

Proposition 3.18. *The unimodular complete intersection singularities having Milnor number of the form $\mu = 6m + 9$ where m is a positive integer with 2 branches and semigroup generated by $\langle 1\rangle, \langle 3m + 5, 6m + 7\rangle$ are J_{6m+9} with Tjurina number $\tau = 6m + 9$ defined by the ideal $\langle xy + z^2, xz + y^{3m+4}\rangle$ and $J_{6m+9,i}$, $i = 1, ..., m$ with Tjurina number $\tau = \mu - i$ defined by the ideal $\langle xy + z^2, xz + y^{3m+4} + y^{(3m+3)-i}\rangle$.*

Proof. The proofs of Propositions 3.17 and 3.18 can be done similarly to the proof of Proposition 3.16. □

We complete the list of unimodular singularities in this case as

Proposition 3.19. *Let $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^3, 0)$ be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy + z^2, xz\rangle$. $(V(\langle f, g\rangle), 0)$ is unimodular if and only if it is isomorphic to a complete intersection in Table 8 and 9.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.16 - 3.18. □

Table 9:

Type	Normal Form	μ	τ	Semigroup
$J_{6m+7,i}$	$\langle xy+z^2, xz+y^{3m+3}+y^{(3m+1)-i}z\rangle$	$6m+7$	$6m+7-i$	$\langle 1\rangle, \langle 3, 3m+4, 6m+5\rangle$
$J_{6m+8,i}$	$\langle xy+z^2, xz+zy^{2m+2}+y^{(4m+4)-i}\rangle$	$6m+8$	$6m+8-i$	$\langle 1\rangle, \langle 1\rangle, \langle 2, 2m+3\rangle$
$J_{6m+9.i}$	$\langle xy+z^2, xz+y^{3m+4}+y^{(3m+3)-i}\rangle$	$6m+9$	$6m+9-i$	$\langle 1\rangle, \langle 3m+5, 6m+7\rangle$

Assume the 2-jet of $\langle f, g\rangle$ has normal form $\langle xy+z^2, x^2\rangle$. According to C.T.C. Wall's classification all unimodular curve singularities are in the μ-constant strata of the versal deformation of the curve singularities given in the table below

Table 10:

Type	Normal Form	μ	τ	Semigroup
$K_{1,0}$	$\langle xy+z^2, x^2+z^2y+\lambda y^4\rangle$ $\lambda \neq 0, 1/4$	11	11	$\langle 2,3\rangle, \langle 2,3\rangle$
$K_{1,i}$	$\langle xy+z^2, x^2+z^2y+y^4+y^{4+i}\rangle$	$11+i$	$11+i-1$	$\langle 1\rangle, \langle 1\rangle, \langle 2,3\rangle$, if i is odd $\langle 2,3\rangle, \langle 2,3+i\rangle$, if i is even
$K'_{1,i}$	$\langle xy+z^2, x^2+2z^2y+y^4+zyl_i\langle z,y\rangle\rangle$	$11+i$	$11+i-1$	$\langle 2,3\rangle, \langle 2,3\rangle$, if i is odd $\langle 2,3\rangle, \langle 2,8+i\rangle$, if i is even
K_{13}	$\langle xy+z^2, x^2+zy^3\rangle$	13	13	$\langle 1\rangle, \langle 3,5,7\rangle$
K_{14}	$\langle xy+z^2, x^2+y^5\rangle$	14	14	$\langle 4,7,10\rangle$

Proposition 3.20. *The unimodular complete intersection curve singularities with Milnor number* 13, 2 *branches and semigroup* $\langle 1\rangle, \langle 3,5,7\rangle$ *are* K_{13} *with Tjurina number* 13 *defined by the ideal* $\langle xy+z^2, x^2+zy^3\rangle$ *and* $K_{13,1}$ *with Tjurina number* $\tau = 12$ *defined by the ideal* $\langle xy+z^2, x^2+zy^3+y^5\rangle$.

Proof. In the list of C.T.C. Wall K_{13} defined by the ideal $\langle xy+z^2, x^2+zy^3\rangle$ is the only singularity with $\mu = 13$, 2 branches and semigroup $\langle 1\rangle, \langle 3,5,7\rangle$.

The versal deformation of K_{13} is given by $\langle xy+z^2+\nu_1 z+\nu_2, x^2+zy^3+\lambda_1 yz^2+\lambda_2 z^2+\lambda_3 y^2 z+\lambda_4 yz+\lambda_5 z+\lambda_6 y^5+\lambda_7 y^4+\lambda_8 y^3+\lambda_9 y^2+\lambda_{10} y+\lambda_{11}\rangle$. K_{13} defines a weighted homogenous isolated complete intersection singularity with weights $(w_1, w_2, w_3) = (7,3,5)$ and degrees $(d_1, d_2) = (10, 14)$. The versal μ-constant deformation of K_{13} is given by $\langle xy+z^2, x^2+zy^3+\lambda_6 y^5\rangle$.

Algorithm 5 `Jsingularity(I)`

Input: $I =< f, g > \in < x, y, z >^2 \mathbb{C}[[x, y, z]]$ and 2-jet of I having normal form $(xy + z^2, xz)$

Output: the type of the singularity

compute μ =Milnor number of the I;
compute τ =Tjurina number of the I;
compute B =Semigroup of I corresponding to each branch.;
if $\mu \equiv 1 \bmod 6$ and $B = \langle 1\rangle, \langle 3, 3(\mu-7)/6+4, 6(\mu-7)/6+5\rangle$ **then**
  if $\mu = \tau$ **then**
    return (J_μ);
  else
    return $(J_{\mu,\mu-\tau})$;
if $\mu \equiv 2 \bmod 6$ and $B = \langle 1\rangle, \langle 1\rangle\langle 2, 2(\mu-8)/6+3\rangle$ **then**
  if $\mu = \tau$ **then**
    return (J_μ);
  else
    return $(J_{\mu,\mu-\tau})$;
if $\mu \equiv 3 \bmod 6$ and $B = \langle 1\rangle, \langle 3, 3(m-9)/6+5, 6(m-9)/6+7\rangle$ **then**
  if $\mu = \tau$ **then**
    return (J_μ);
  else
    return $(J_{\mu,\mu-\tau})$;
if $\mu \equiv 5 \bmod 6$ and $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
  if $\mu = \tau$ **then**
    return $(J_{(\mu-5)/6+1,0})$;
if $\mu \neq \tau$ **then**
  if μ is even and $B = \langle 1\rangle, \langle 1\rangle, \langle 2, 4\tau-3\mu-3\rangle$ **then**
    return $(J_{\mu-\tau+1,6\tau-5\mu-5})$;
  if μ is odd **then**
    if $B = \langle 1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$ **then**
      return $(J_{\mu-\tau+1,6\tau-5\mu-5})$;
return *not unimodular*;

Using the coordinate change $x \to \xi^7 x, y \to \xi^3 y, z \to \xi^5 z$, we may assume $I_{\lambda_6} = \langle xy + z^2, x^2 + zy^3 + \xi\lambda_6 y^5\rangle$. Choose ξ such that $\xi\lambda_6 = 1$. So we obtain $\langle xy + z^2, x^2 + zy^3 + y^5\rangle$. It has 2 branches and same semigroup as K_{13} but $\tau = 12$. It can be differentiated from K_{13} by the Tjurina number. □

Proposition 3.21. *The unimodular complete intersection curve singularities with Milnor number* 14*, irreducible having semigroup* $\langle 4, 7, 10\rangle$ *are* K_{14} *with Tjurina number* 14 *defined by the ideal* $\langle xy+z^2, x^2+y^5\rangle$ *and* $K_{14,1}$ *with Tjurina number* $\tau = 13$ *defined by the ideal* $\langle xy + z^2, x^2 + y^5 + y^2z^2\rangle$.

We complete the list of unimodular singularities in this case as

Table 11:

$K_{13,1}$	$\langle xy + z^2, x^2 + zy^3 + y^5\rangle$	13	12	$\langle 1\rangle, \langle 3, 5, 7\rangle$
$K_{14,1}$	$\langle xy + z^2, x^2 + y^5 + y^2z^2\rangle$	14	13	$\langle 4, 7, 10\rangle$

Proposition 3.22. *Let* $(V(\langle f, g\rangle), 0) \subseteq (\mathbb{C}^3, 0)$ *be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the* 2*-jet of* $\langle f, g\rangle$ *has normal form* $\langle xy + z^2, x^2\rangle$*. Then* $(V(\langle f, g\rangle), 0)$ *is unimodular if and only if it is isomorphic to a complete intersection in Table 10 and 11.*

Proof. The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.20 and 3.21. □

The following Algorithm is the basis for classifying the unimodular complete intersection curve singularities when $char(K) = 0$.

4 Singular examples

```
> ring R=0,(x,y,z),ds;
> ideal I=xy+11y2+9yz+z3,x2+22xy+121y2+18xz+198yz+81z2+z3+y4;
> classifyicis1(I);
HC_13:(xy+z3,x2+z3+y4)

> ideal J=x2+xy+2y2+2xz+z2,x2+2xy+xz+2yz+xy12+y12z;
> classifyicis1(J);
J_6*5+8:(xy+z2,xz+zy12)
```

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Algorithm 6 `Ksingulariy(I)`

Input: $I =< f, g >\in< x, y, z >^2\, \mathbb{C}[[x, y, z]]$ having 2-jet of the form $(xy + z^2, x^2)$
Output: the type of the singularity

compute μ =Milnor number of the I;
compute τ =Tjurina number of the I;
compute B =semigroups of I corresponding to the branches;
if $\mu = 13$ and $B = \langle 1\rangle, \langle 3, 5, 7\rangle$ **then**
  if $\mu = \tau$ **then**
    return (K_{13});
  if $\mu - \tau = 1$ **then**
    return $(K_{13,1})$;
if $\mu = 14$ and $B = \langle 4, 7, 10\rangle$ **then**
  if $\mu = \tau$ **then**
    return (K_{14});
  if $\mu - \tau = 1$ **then**
    return $(K_{14,1})$;
if $\mu = 11$ and $B = \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
  if $\mu = \tau$ **then**
    return $(K_{1,0})$;
if $\mu \neq \tau$ **then**
  if $\mu - \tau = 1$ and $\mu > 11$ **then**
    if μ is even **then**
      if $B = \langle 1\rangle, \langle 1\rangle, \langle 2, 3\rangle$ **then**
        return $(K_{1,\mu-11})$;
      if $B = \langle 2, 3\rangle, \langle 2, 3\rangle$ **then**
        return $(K^{'}_{1,\mu-11})$;
    if μ is odd **then**
      if $B = \langle 2, 3\rangle, \langle 2, \mu - 8\rangle$ **then**
        return $(K_{1,\mu-11})$;
      if $B = \langle 4, 6, \mu - 3\rangle$ **then**
        return $(K^{'}_{1,\mu-11})$;
return *not unimodular*;

Algorithm 7 `classifyicis1(I)`[Unimodular curve singularities]

Input: $I = \langle f, g\rangle \subseteq \langle x, y, z\rangle^2\mathbb{C}[[x, y, z]]$ isolated complete intersection curve singularity.

Output: The type of the singularity $(V(I), 0)$.

compute I_2 the 2-jet of I;
compute $I_2 = \bigcap_{i=1}^{s} Q_i$ the irredundant primary decomposition over $\mathbb{C}$;
compute d_i =Krull dimension of $\mathbb{C}[x, y, z]/Q_i$;
compute $h_i \in \mathbb{Q}[t]$ the Hilbert polynomial corresponding to each Q_i;
if $s = 2$ **then**
  if $d_1 = d_2 = 1$ **then**
    if $h_1 = h_2 = 2$ **then**
      return `Psingularity(I)`; via Algorithm 1
    if $h_1 = 1$ and $h_2 = 4$ **then**
      return (`Jsingularity`); via Algorithm 5
  if $d_1 = 1, d_2 = 2$ **then**
    if $h_1 = 1, h_2 = 1 + t$ **then**
      return (`Fsingularity(I)`); via Algorithm 2
    if $h_1 = 2$ and $h_2 = 1 + t$ **then**
      return (`Hsingulrity(I)`); via Algorithm 4
if $s = 1$ **then**
  compute R the radical of I_2
  if $R^3 \nsubseteq I_2$ **then**
    return (`Gsingularity(I)`); via Algorithm 3
  else
    return (`Ksingualrtiy(I)`); via Algorithm 6
return (*not unimodular*);

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