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Positive solutions for semilinear elliptic systems with sign-changing potentials

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Abstract

In this paper, we study the existence of positive solutions of the Dirichlet problem $-\Delta u = \lambda\, p(x) f(u,v)\,;\ -\Delta v = \lambda\, q(x) g(u,v), \quad \text{in } D,$ and $u = v = 0 \quad \text{on } \partial^\infty D$, where $D \subset \mathbb{R}^n$ $(n \geq 3)$ is an $C^{1,1}$-domain with compact boundary and $\lambda > 0$. The potential functions p, q are not necessarily bounded, may change sign and the functions $f,\, g:\ \mathbb{R}^2 \to \mathbb{R}$ are continuous with $f(0,0) > 0$, $g(0,0) > 0$. By applying the Leray-Schauder fixed point theorem, we establish the existence of positive solutions for λ sufficiently small.

1 Introduction

Let D be a $C^{1,1}$ domain of $\mathbb{R}^n$ $(n \geq 3)$ with compact boundary and let $\partial^\infty D = \partial D$ if D is bounded and $\partial^\infty D = \partial D \cup \{\infty\}$ whenever D is unbounded. This paper deals with the existence of positive continuous solutions (in the sense of distributions) for the following semilinear elliptic system

$$\begin{cases} -\Delta u = \lambda p(x) f(u,v), & \text{in } D, \\ -\Delta v = \lambda q(x) g(u,v), & \text{in } D, \\ u = v = 0 & \text{on } \partial^\infty D, \end{cases} \tag{1.1}$$

where the potential functions p, q are sign changing functions belonging to the Kato class $K(D)$ introduced and studied in [1] and [9] and f, g satisfy the following hypothesis.

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($\mathbf{H_1}$) The functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ are continuous with $f(0,0) > 0$ and $g(0,0) > 0$.

In recent years, a good amount of research is established for reaction-diffusion systems. reaction-diffusions systems model many phenomena in Biology, Ecology, combustion theory, chemical reactors, population dynamics etc. And the case $p(x) = q(x) = 1$ has been considered as a typical example when D is a bounded regular domain in $\mathbb{R}^n$ and many existence results where established by variational methods, topological methods and the method of sub and supersolution (see [5], [7], [4]).

Recently, Chen [2] studied the existence of positive solutions for the following system

$$\begin{cases} -\Delta u = \lambda p(x) f_1(v), & \text{in } D, \\ -\Delta v = \lambda q(x) g_1(v), & \text{in } D, \\ u = v = 0 \quad \text{on } \partial^\infty D, \end{cases} \tag{1.2}$$

where D is a bounded domain. He assumed that if p, q are continuous in $\overline{D}$ and

($\mathbf{H_2}$) There exists μ_1, $\mu_2 > 0$ such that

$$\begin{aligned} \int_D G(x,y)p_+(y)\,dy &> (1+\mu_1)\int_D G(x,y)p_-(y)\,dy \quad \forall x \in D, \\ \int_D G(x,y)q_+(y)\,dy &> (1+\mu_2)\int_D G(x,y)q_-(y)\,dy \quad \forall x \in D, \end{aligned}$$

where $G(x,y)$ is the Green's function of the Dirichlet Laplacian in D. Here p^+, q^+ are the positive parts of p and q, while p_-, q_- are the negative ones.

The main result of Chen [2] reads as follows.

Theorem A. Let p, q be nonzero continuous functions on $\overline{D}$ satisfying $\mathbf{H_2}$) and let $f_1, g_1 : [0,\infty) \to \mathbb{R}$ be continuous with $f_1(0) > 0$, $g_1(0) > 0$. Then there exists a positive number $\lambda^\star > 0$ such that (1.2) has a positive solution for $0 < \lambda < \lambda^\star$.

We note that in the case where f_1, g_1 are nonnegative nondecreasing continuous functions, $p(x) \leq 0$ in D and $q(x) \leq 0$ in D, system (1.2) was studied in [6] with nontrivial nonnegative boundary data and the existence of positive bounded solutions for (1.2) was established whenever λ is a small positive real number.

Our aim in this paper is to extend and improve, by a modified proof, the result of Chen [2] in a number of ways. First, the domain D will be bounded or an exterior domain. Second, the functions p, q are not necessarily continuous in $\overline{D}$.

Indeed p, q may be singular on the boundary of D. Third, the nonlinear terms $f_1(v)$ and $g_1(u)$ considered in [2] are more restrictive than the class $f(u,v)$ and $g(u,v)$ considered in our case. More precisely, we will establish the existence of a positive solution for (1.1) in the case where $f(0,0)>0$, $g(0,0)>0$ and the potentials of p , q satisfy hypothesis $(\mathbf{H_2})$ and belong to the Kato class introduced and studied in [1] and [9]. A nonexistence of positive bounded solution will be also given in the case where f and g are sublinear functions with $f(0,0)=0$ and $g(0,0)=0$. To this aim, we give in the sequel some notations and we recall some properties of the Kato class.

Definition 1.1. *(See [1] and [9].) A Borel measurable function k in D belongs to the Kato class $K(D)$ if*

$$\lim_{\alpha\to 0}\sup_{x\in D}\int_{D\cap B(x,\alpha)}\frac{\rho(y)}{\rho(x)}G(x,y)|k(y)|dy=0$$

and satisfies further

$$\lim_{M\to\infty}\sup_{x\in D}\int_{D\cap\{|y|\geq M\}}\frac{\rho(y)}{\rho(x)}G(x,y)|k(y)|dy=0\ \ (\textit{ whenever } D \textit{ is unbounded}),$$

where $\rho(x)=\min(1,\delta(x))$ and $\delta(x)$ denotes the euclidian distance from x to the boundary of D.

We remark that in the case where D is bounded an if d denotes its diameter, then

$$\frac{1}{1+d}\delta(x)\leq\rho(x)\leq\delta(x).$$

So in this case, we can replace $\rho(x)$ by $\delta(x)$ in the Definition 1.1.
Next, we give some examples of functions belonging to $K(D)$.

Example 1.1. *(see [1] and [9])*

(1) *Let D be a bounded domain of $\mathbb{R}^n$.*

   (a) *Let $q(y)=\frac{1}{(\delta(x))^\lambda}$. Then $q\in K(D)$ if and only if $\lambda<2$.*

   (b) *Let $p>\frac{n}{2}$, then for $\lambda<2-\frac{n}{p}$, we have $\frac{1}{\delta(.)^\lambda}L^p(D)\subset K(D)$. In particular $L^p(D)\subset K(D)$.*

   (c) *Let $D=B(0,1)$ and let q be a Borel radial function in D, then $q\in K(D)$ if and only if $\int_0^1 r(1-r)|q(r)|\,dr<\infty$.*

(2) *Let D be a $C^{1,1}$-exterior domain in $\mathbb{R}^n$ $(n\geq 3)$. The function $x\to\frac{1}{(|x|+1)^{\mu-\lambda}\,\delta(x)^\lambda}\in K(D)$ if and only if $\lambda<2<\mu$.*

(3) *Let $D = \overline{B(0,1)^c}$ be the exterior of the unit closed ball in $\mathbb{R}^n$ $(n \geq 3)$ and let q be a Borel radial function in D, then $q \in K(D)$ if and only if $\int_1^\infty (r-1)\,|q(r)|\,dr < \infty$.*

For any nonnegative Borel measurable function φ in D, we denote by $V\varphi$ the Green potential of φ defined on D by

$$V\varphi(x) = \int_D G(x,y)\varphi(y)dy.$$

Recall that if $\varphi \in L^1_{loc}(D)$ and $V\varphi \in L^1_{loc}(D)$, then we have in the distributional sense (see [3] p. 52)

$$\Delta(V\varphi) = -\varphi \ \text{ in } D. \tag{1.3}$$

Our main results are as follows.

Theorem 1.2. *Let p, q be in the Kato class $K(D)$ and assume that hypotheses $(\mathbf{H}_1) - (\mathbf{H}_2)$ are satisfied. Then there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, problem (1.1) has a positive continuous solution in D.*

For the nonexistence of positive bounded solutions, we establish

Theorem 1.3. *Let p, q be two nontrivial functions in the Kato class $K(D)$. Assume that the functions f, g : $\mathbb{R}^2 \to \mathbb{R}$ are measurable and there exists a positive constant M such that for all u, v we have,*

$$|f(u,v)| \leq M(|u| + |v|)$$
$$|g(u,v)| \leq M(|u| + |v|).$$

Then there exists $\lambda_0 > 0$ such that the problem (1.1) has no bounded positive continuous solution in D for each $\lambda \in (0, \lambda_0)$.

Throughout this paper, we denote by $B(D)$ the set of Borel measurable functions in D and by $C_0(D)$ the set of continuous ones satisfying $\lim_{x \to \xi \in \partial^\infty D} u(x) = 0$.
Finally, for a bounded real function ω defined on a set S we denote by $\|\omega\|_\infty = \sup\limits_{x \in S} |\omega(x)|$.

2 Proof of Theorems 1.2 and 1.3

We begin this section by giving a continuity result.

Proposition 2.1. *(see [1] and [9]) Let φ be a nonnegative function in $K(D)$. Then we have* *i) The function* $y \to \frac{\delta(y)}{(1+|y|)^{n-1}}\varphi(y)$ *is in* $L^1(D)$*. In particular* $\varphi \in L^1_{loc}(D)$*.*

ii) $V\varphi \in C_0(D)$*.*

iii) Let h_0 *be a positive harmonic function in* D *which is continuous and bounded in* $\overline{D}$*. Then the family of functions*

$$\left\{ \int_D G.,y)h_0(y)p(y)dy : |p| \leq \varphi \right\}$$

is relatively compact in $C_0(D)$*.*

Next, we recall first the Leray-Schauder fixed point theorem.

Lemma 2.2. *(Leray-Schauder fixed point theorem) Let* X *be a Banach space with norm* $\|.\|$ *and* x_0 *be a point of* X*. Suppose that* $T : X \times [0,1] \to X$ *is continuous and compact with* $T(x,0) = x_0$*, for each* $x \in X$*, and there exists a fixed constant* $M > 0$ *such that each solution* $(x,\sigma) \in X \times [0,1]$ *of the* $T(x,\sigma) = x$ *satisfies* $\|x\| \leq M$*. Then* $T(.,1)$ *has a fixed point.*

Using this Lemma, we obtain the following general existence result.

Lemma 2.3. *Suppose that* p *and* q *are in the Kato class* $K(D)$ *and* f*,* g *are continuous and bounded from* $\mathbb{R}^2$ *to* $\mathbb{R}$*. Then for every* $\lambda \in (0,\infty)$*, problem (1.1) has a solution* $(u_\lambda, v_\lambda) \in C_0(D) \times C_0(D)$*.*

Proof. For $\lambda \in \mathbb{R}$, we consider the operator $T_\lambda :\ C_0(D) \times C_0(D) \times [0,1] \to C_0(D) \times C_0(D)$ defined by

$$T_\lambda((u,v),\sigma) = (\sigma\,\lambda V(p\,f(u,v)), \sigma\,\lambda V(q\,g(u,v))).$$

By Proposition 2.1, the operator T_λ is well defined, continuous, compact and $T_\lambda((u,v),0) = (0,0) := x_0 \in C_0(D) \times C_0(D)$. Let $(u,v) \in C_0(D) \times C_0(D)$ and $\sigma \in [0,1]$ such that $T_\lambda((u,v),\sigma) = (u,v)$. Then, since f, g are bounded and p, q are in $K(D)$ we deduce by using Proposition 2.1 that

$$\begin{array}{rcl} \max(\|u\|_\infty, \|v\|_\infty) & = & \sigma\,\lambda\,\max(\|V(pf(u,v))\|_\infty, \|V(qg(u,v))\|_\infty) \\ & \leq & \lambda\,\max(\|Vp\|_\infty \|f\|_\infty, \|Vq\|_\infty \|g\|_\infty) \\ & = & M. \end{array}$$

Hence by Leray-Schauder fixed point theorem, the operator $T_\lambda(.,1)$ has a fixed point. Namely, there exists $(u,v) \in C_0(D) \times C_0(D)$ such that $(u,v) = (\lambda\,V(p\,f(u,v)), \lambda\,V(q\,g(u,v)))$. So, using (1.3) and Proposition 2.1, we deduce that (u,v) is a solution of system (1.1).

Proof of Theorem 1.2. Fix a large number $M > 0$ and an infinitely continuously differentiable function ψ with compact support on $\mathbb{R}^2$ such that $\psi = 1$ in the open ball with center 0 and radius M and $\psi = 0$ on the exterior of the ball with center 0 and radius $2M$. Define the bounded functions $\widetilde{f}$, $\widetilde{g}$ on $\mathbb{R}^2$ by $\widetilde{f}(u, v) = \psi(u, v)f(u, v)$ and $\widetilde{g}(u, v) = \psi(u, v)g(u, v)$. By Lemma 2.3, the Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda p(x)\widetilde{f}(u, v), & \text{in } D, \\ -\Delta v = \lambda q(x)\widetilde{g}(u, v), & \text{in } D, \\ u = v = 0 \quad \text{on } \partial^\infty D, \end{cases} \tag{2.1}$$

has a solution $(u_\lambda, v_\lambda) \in C_0(D) \times C_0(D)$ satisfying

$$(u_\lambda, v_\lambda) = (\lambda V(p\widetilde{f}(u_\lambda, v_\lambda))\lambda V(q\widetilde{g}(u_\lambda, v_\lambda))).$$

Moreover

$$\max(\|u_\lambda\|_\infty, \|v_\lambda\|_\infty) \leq \lambda \max(\|Vp\|_\infty\|\widetilde{f}\|_\infty, \|Vq\|_\infty\|\widetilde{g}\|_\infty) \tag{2.2}$$

Put $\mu = \min(\mu_1, \mu_2)$ and consider $\gamma \in (0, \frac{\mu}{2+\mu})$. Since $\widetilde{f}$ and $\widetilde{g}$ are continuous, then there exists $\delta \in (0, M)$ such that if $\max(|\zeta|, |\xi|) < \delta$, we have $\widetilde{f}(0, 0)(1 - \gamma) < \widetilde{f}(\zeta, \xi) < \widetilde{f}(0, 0)(1 + \gamma)$ and $\widetilde{g}(0, 0)(1 - \gamma) < \widetilde{g}(\zeta, \xi) < \widetilde{g}(0, 0)(1 + \gamma)$.
Using (2.2) , we deduce that there exists $\lambda_0 > 0$ such that $\|u_\lambda\|_\infty < \delta$ and $\|v_\lambda\|_\infty < \delta$ for any $\lambda \in (0, \lambda_0)$. This together with the fact that $0 < \delta < M$ implies that for $\lambda \in (0, \lambda_0)$, we have $\widetilde{f}(u_\lambda, v_\lambda) = f(u_\lambda, v_\lambda)$ and $\widetilde{g}(u_\lambda, v_\lambda) = g(u_\lambda, v_\lambda)$.
Now, for each $x \in D$ we have

$$\begin{aligned} u_\lambda &= \lambda V(p_+\widetilde{f}(u_\lambda, v_\lambda)) - \lambda V(p_-\widetilde{f}(u_\lambda, v_\lambda)) \\ &> \lambda f(0, 0)(1 - \gamma)V(p_+) - \lambda f(0, 0)(1 + \gamma)V(p_-) \\ &> \lambda f(0, 0)[(1 - \gamma)(1 + \mu_1) - (1 + \gamma)]V(p_-) \\ &> \lambda f(0, 0)(1 - \gamma)\left[1 + \mu_1 - \frac{1 + \gamma}{1 - \gamma}\right]V(p_-) \\ &> \lambda f(0, 0)(1 - \gamma)\left[1 + \mu - \frac{1 + \gamma}{1 - \gamma}\right]V(p_-). \end{aligned}$$

Now, since $\gamma \in (0, \frac{\mu}{2+\mu})$, then $1 + \mu - \frac{1+\gamma}{1-\gamma} > 0$ and it follows that $\lambda f(0, 0)(1 - \gamma)\left[1 + \mu - \frac{1+\gamma}{1-\gamma}\right]V(p_-) \geq 0$. Consequently, for each $\lambda \in (0, \lambda_0)$ and for each $x \in D$ we have $u_\lambda(x) > 0$. Similarly, we obtain $v_\lambda(x) > 0$ for each $x \in D$.

Proof of Theorem 1.3 Suppose that (1.1) has a bounded positive solution (u,v) for $\lambda>0$. Then $f(u,v)$ and $g(u,v)$ are bounded. Put $\widetilde{u}=\lambda V(p\,f(u,v))$ and $\widetilde{v}=\lambda V(q\,g(u,v))$. Since $f(u,v)$ and $g(u,v)$ are bounded, then the functions $\widetilde{u}\,,\,\widetilde{v}\in C_0(D)$. The functions $z=u-\widetilde{u}$ and $\omega=v-\widetilde{v}$ are harmonic in the distributional sense and continuous in D, so they are harmonic in the classical sense. Moreover, since $u=\widetilde{u}=v=\widetilde{v}=0$ on $\partial^\infty D$ then $u=\widetilde{u}$ and $v=\widetilde{v}$ in D. Which implies

$$\|u\|_\infty \le \lambda\, V(|p|f(u,v)) \le \lambda\, M\, \|V(|p|)\|_\infty\, (\|u\|_\infty+\|v\|_\infty)\,,$$
$$\|v\|_\infty \le \lambda\, V(|q|g(u,v)) \le \lambda\, M\, \|V(|q|)\|_\infty\, (\|u\|_\infty+\|v\|_\infty).$$

By adding these inequalities, we obtain

$$(\|u\|_\infty+\|v\|_\infty) \le \lambda\, M\, [\|V(|p|)\|_\infty+\|V(|q|)\|_\infty]\, (\|u\|_\infty+\|v\|_\infty).$$

This gives a contradiction if $\lambda\, M\, [\|V(|p|)\|_\infty+\|V(|q|)\|_\infty]<1$.

Example 2.1. *Let $p\,,\,q$ be two measurable radial functions on the exterior of the unit ball $D=\overline{B(0,1)}^c$, $n\ge 3$. Assume that there exists $\varepsilon>0$ such that each $t>1$ and $x\in D$, we have*

$$\int_1^t \frac{r^{n-1}}{(|x|\vee r)^{n-2}}\,(1-(|x|\wedge r)^{2-n})\,p^+(r)\,dr \ge (1+\varepsilon)\int_1^t \frac{r^{n-1}}{(|x|\vee r)^{n-2}}\,(1-(|x|\wedge r)^{2-n})\,p^-(r)\,dr\,, \quad \text{and}$$

$$\int_1^t \frac{r^{n-1}}{(|x|\vee r)^{n-2}}\,(1-(|x|\wedge r)^{2-n})\,q^+(r)\,dr \ge (1+\varepsilon)\int_1^t \frac{r^{n-1}}{(|x|\vee r)^{n-2}}\,(1-(|x|\wedge r)^{2-n})\,q^-(r)\,dr\,,$$

then hypothesis ($\mathbf{H_2}$) is satisfied. Indeed (see [1]), for a nonnegative radial function k, the function $x\to\int_D G_D(x,y)k(|y|)\,dy$ is radial and

$$\int_D G_D(x,y)k(|y|)\,dy = a_n\int_1^\infty \frac{r^{n-1}}{(|x|\vee r)^{n-2}}\,(1-(|x|\wedge r)^{2-n})\,k(r)\,dr\,,$$

where $|x|\wedge t=\min(|x|,t)$, $|x|\vee t=\max(|x|,t)$ and $a_n>0$.

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