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# ON 2-ABSORBING PRIMARY SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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**Abstract**

All rings are commutative with $1 \neq 0$, and all modules are unital. The purpose of this paper is to investigate the concept of 2-absorbing primary submodules generalizing 2-absorbing primary ideals of rings. Let $M$ be an $R$-module. A proper submodule $N$ of an $R$-module $M$ is called a *2-absorbing primary submodule* of $M$ if whenever $a, b \in R$ and $m \in M$ and $abm \in N$, then $am \in M$-rad$(N)$ or $bm \in M$-rad$(N)$ or $ab \in (N :_R M)$. It is shown that a proper submodule $N$ of $M$ is a 2-absorbing primary submodule if and only if whenever $I_1 I_2 K \subseteq N$ for some ideals $I_1, I_2$ of $R$ and some submodule $K$ of $M$, then $I_1 I_2 \subseteq (N :_R M)$ or $I_1 K \subseteq M\text{-}rad(N)$ or $I_2 K \subseteq M\text{-}rad(N)$. We prove that for a submodule $N$ of an $R$-module $M$ if $M\text{-}rad(N)$ is a prime submodule of $M$, then $N$ is a 2-absorbing primary submodule of $M$. If $N$ is a 2-absorbing primary submodule of a finitely generated multiplication $R$-module $M$, then $(N :_R M)$ is a 2-absorbing primary ideal of $R$ and $M\text{-}rad(N)$ is a 2-absorbing submodule of $M$.

## 1 Introduction and Preliminaries

Throughout this paper all rings are commutative with a nonzero identity and all modules are considered to be unitary. Prime submodules have an important

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role in the theory of modules over commutative rings. Let $M$ be a module over a commutative ring $R$. A *prime* (resp. *primary*) submodule is a proper submodule $N$ of $M$ with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a^k \in (N :_R M)$ for some positive integer $k$). In this case $p = (N :_R M)$ (resp. $p = \sqrt{(N :_R M)}$) is a prime ideal of $R$. There are several ways to generalize the concept of prime submodules. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [16]. A proper submodule $N$ of $M$ is *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Behboodi and Koohi in [13] defined another class of submodules and called it weakly prime. Their paper is on the basis of some recent papers devoted to this new class of submodules. Let $R$ be a ring and $M$ an $R$-module. A proper submodule $N$ of $M$ is said to be *weakly prime* when for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N$. To avoid the ambiguity, Behboodi renamed this concept and called submodules introduced in [13], *classical prime submodule*.

Badawi in [9] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal $I$ of $R$ to be a *2-absorbing ideal* of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This definition can obviously be made for any ideal of $R$. This concept has a generalization, called weakly 2-absorbing ideals, which has studied in [10]. A proper ideal $I$ of $R$ to be a *weakly 2-absorbing ideal of* $R$ if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [6] generalized the concept of 2-absorbing ideals to $n$-absorbing ideals. According to their definition, a proper ideal $I$ of $R$ is called an *$n$-absorbing* (resp. *strongly $n$-absorbing*) ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of $R$), then there are $n$ of the $x_i$'s (resp. $n$ of the $I_i$'s) whose product is in $I$. They proved that a proper ideal $I$ of $R$ is 2-absorbing if and only if $I$ is strongly 2-absorbing.

In [26], the concept of 2-absorbing and weakly 2-absorbing ideals generalized to submodules of a module over a commutative ring. Let $M$ be an $R$-module and $N$ a proper submodule of $M$. $N$ is said to be a *2-absorbing submodule* (resp. *weakly 2-absorbing submodule*) of $M$ if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Badawi et. al. in [11] introduced the concept of 2-absorbing primary ideals, where a proper ideal $I$ of $R$ is called *2-absorbing primary* if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Let $R$ be a ring, $M$ an $R$-module and $N$ a submodule of $M$. We will denote by $(N :_R M)$ *the residual of $N$ by $M$*, that is, the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of $M$ which is denoted by $ann_R(M)$ is $(0 :_R M)$. An $R$-module $M$ is called a *multiplication module* if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$ [17]. Finitely generated faithful multiplication modules are cancellation modules [25, Corollary to Theorem 9], where an $R$-module $M$ is defined to be a *cancellation module* if $IM = JM$ for ideals $I$ and $J$ of $R$ implies $I = J$. It is well-known that if $R$ is a commutative ring and $M$ a nonzero multiplication $R$-module, then every proper submodule of $M$ is contained in a maximal submodule of M and $K$ is a maximal submodule of $M$ if and only if there exists a maximal ideal $\mathfrak{m}$ of $R$ such that $K = \mathfrak{m}M$ [17, Theorem 2.5]. If $M$ is a finitely generated faithful multiplication $R$-module (hence cancellation), then it is easy to verify that $(IN :_R M) = I(N :_R M)$ for each submodule $N$ of $M$ and each ideal $I$ of $R$. For a submodule $N$ of $M$, if $N = IM$ for some ideal $I$ of $R$, then we say that $I$ is a presentation ideal of $N$. Clearly, every submodule of $M$ has a presentation ideal if and only if $M$ is a multiplication module. Let $N$ and $K$ be submodules of a multiplication $R$-module $M$ with $N = I_1M$ and $K = I_2M$ for some ideals $I_1$ and $I_2$ of $R$. The product of $N$ and $K$ denoted by $NK$ is defined by $NK = I_1I_2M$. Then by [3, Theorem 3.4], the product of $N$ and $K$ is independent of presentations of $N$ and $K$. Moreover, for $a, b \in M$, by $ab$, we mean the product of $Ra$ and $Rb$. Clearly, $NK$ is a submodule of $M$ and $NK \subseteq N \cap K$ (see [3]). Let $N$ be a proper submodule of a nonzero $R$-module $M$. Then the $M$-radical of $N$, denoted by $M$-rad$(N)$, is defined to be the intersection of all prime submodules of $M$ containing $N$. If $M$ has no prime submodule containing $N$, then we say $M$-rad$(N) = M$. It is shown in [17, Theorem 2.12] that if $N$ is a proper submodule of a multiplication $R$-module $M$, then $M$-rad$(N) = \sqrt{(N :_R M)}M$. In this paper we define the concept of 2-absorbing primary submodules. We give some basic results of this class of submodules and discuss on the relations among 2-absorbing ideals, 2-absorbing submodules, 2-absorbing primary ideals and 2-absorbing primary submodules.

## 2 Properties of 2-absorbing primary submodules

**Definition 2.1.** A proper submodule $N$ of an $R$-module $M$ is called a *2-absorbing primary submodule* (resp. *weakly 2-absorbing primary submodule*) of $M$ if whenever $a, b \in R$ and $m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $am \in M$-rad$(N)$ or $bm \in M$-rad$(N)$ or $ab \in (N :_R M)$.

**Example 2.2.** Let $p$ be a fixed prime integer and $N_0 = \mathbb{N} \cup \{0\}$. Each proper $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$ is of the form $G_t = \langle 1/p^t + \mathbb{Z}\rangle$ for some $t \in N_0$. In [15, Example 1] it was shown that every submodule $G_t$ is not primary. For each $t \in N_0$, $(G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty)) = 0$. Note that $p^2\left(\frac{1}{p^{t+2}} + \mathbb{Z}\right) = \frac{1}{p^t} + \mathbb{Z} \in G_t$, but neither $p^2 \in (G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty)) = 0$ nor $p\left(\frac{1}{p^{t+2}} + \mathbb{Z}\right) \in G_t$. Hence $\mathbb{Z}(p^\infty)$ has no 2-absorbing submodule. Since every prime submodule is 2-absorbing, then $\mathbb{Z}(p^\infty)$ has no prime submodule. Therefore $\mathbb{Z}(p^\infty)$-rad$(G_t) = \mathbb{Z}(p^\infty)$, and so $G_t$ is a 2-absorbing primary submodule of $\mathbb{Z}(p^\infty)$.

**Theorem 2.3.** *Let $N$ be a proper submodule of an $R$-module $M$. Then the following conditions are equivalent:*

1. *$N$ is a 2-absorbing primary submodule of $M$;*
2. *For every elements $a, b \in R$ such that $ab \notin (N :_R M)$, $(N :_M ab) \subseteq (M\text{-}rad(N) :_M a) \cup (M\text{-}rad(N) :_M b)$;*
3. *For every elements $a, b \in R$ such that $ab \notin (N :_R M)$, $(N :_M ab) \subseteq (M\text{-}rad(N) :_M a)$ or $(N :_M ab) \subseteq (M\text{-}rad(N) :_M b)$.*

*Proof.* (1)⇒(2) Suppose that $a, b \in R$ such that $ab \notin (N :_R M)$. Let $m \in (N :_M ab)$. Then $abm \in N$, and so either $ma \in M\text{-}rad(N)$ or $bm \in M\text{-}rad(N)$. Therefore either $m \in (M\text{-}rad(N) :_M a)$ or $m \in (M\text{-}rad(N) :_M b)$. Hence $(N :_M ab) \subseteq (M\text{-}rad(N) :_M a) \cup (M\text{-}rad(N) :_M b)$.
(2)⇒(3) Notice to the fact that if a submodule (a subgroup) is a subset of the union of two submodules (two subgroups), then it is a subset of one of them. Thus we have $(N :_M ab) \subseteq (M\text{-}rad(N) :_M a)$ or $(N :_M ab) \subseteq (M\text{-}rad(N) :_M b)$.
(3)⇒(1) is straightforward. □

**Lemma 2.4.** *Let $M$ be a finitely generated multiplication $R$-module. Then for any submodule $N$ of $M$, $\sqrt{(N :_R M)} = (M\text{-}rad(N) :_R M)$.*

*Proof.* By [21, Theorem 4], $(M\text{-rad}(N) :_R M) \subseteq \sqrt{(N :_R M)}$. Now we prove the other containment without any assumption on $M$. Let $K$ be a prime submodule of $M$ containing $N$. Then clearly $(K : M)$ is a prime ideal that contains $(N : M)$. Therefore $\sqrt{(N :_R M)} \subseteq (K : M)$, so $\sqrt{(N :_R M)} \subseteq (M\text{-rad}(N) :_R M)$. □

**Proposition 2.5.** *Let $M$ be a finitely generated multiplication $R$-module and $N$ be a submodule of $M$. Then $M$-rad$(N)$ is a primary submodule of $M$ if and only if $M$-rad$(N)$ is a prime submodule of $M$.*

*Proof.* Suppose that $M$-rad$(N)$ is a primary submodule of $M$. Let $a \in R$ and $m \in M$ be such that $am \in M$-rad$(N)$ and $m \notin M$-rad$(N)$. Since $M$-rad$(N)$ is primary, it follows $a \in \sqrt{(M\text{-rad}(N) :_R M)} = \sqrt{\sqrt{(N :_R M)}} = \sqrt{(N :_R M)} = (M\text{-rad}(N) :_R M)$, by Lemma 2.4. Thus $M$-rad$(N)$ is a prime submodule of $M$. The converse part is clear. □

**Theorem 2.6.** *Let $M$ be a finitely generated multiplication $R$-module. If $N$ is a 2-absorbing primary submodule of $M$, then*

1. *$(N :_R M)$ is a 2-absorbing primary ideal of $R$.*
2. *$M$-rad$(N)$ is a 2-absorbing submodule of $M$.*

*Proof.* (1) Let $a, b, c \in R$ be such that $abc \in (N :_R M)$, $ac \notin \sqrt{(N :_R M)}$ and $bc \notin \sqrt{(N :_R M)}$. Since, by Lemma 2.4, $\sqrt{(N :_R M)} = (M\text{-rad}(N) :_R M)$, there exist $m_1, m_2 \in M$ such that $acm_1 \notin M$-rad$(N)$ and $bcm_2 \notin M$-rad$(N)$. But $ab(cm_1 + cm_2) \in N$, because $abc \in (N :_R M)$. So $a(cm_1 + cm_2) \in M$-rad$(N)$ or $b(cm_1 + cm_2) \in M$-rad$(N)$ or $ab \in (N :_R M)$, since $N$ is 2-absorbing primary. If $ab \in (N :_R M)$, then we are done. Thus assume that $a(cm_1 + cm_2) \in M$-rad$(N)$. So $acm_2 \notin M$-rad$(N)$, because $acm_1 \notin M$-rad$(N)$. Therefore $ab \in (N :_R M)$, since $N$ is 2-absorbing primary and $abcm_2 \in N$. Similarly if $b(cm_1 + cm_2) \in M$-rad$(N)$, then $ab \in (N :_R M)$. Consequently $(N :_R M)$ is a 2-absorbing primary ideal.
(2) By [11, Theorem 2.3] we have two cases.

**Case 1**. $\sqrt{(N :_R M)} = p$ is a prime ideal of $R$. Since $M$ is a multiplication module, $M$-rad$(N) = \sqrt{(N :_R M)}M = pM$, where $pM$ is a prime submodule of $M$ by [17, Corollary 2.11]. Hence in this case $M$-rad$(N)$ is a 2-absorbing submodule of $M$.

**Case 2**. $\sqrt{(N :_R M)} = p_1 \cap p_2$, where $p_1$, $p_2$ are distinct prime ideals of $R$ that are minimal over $(N :_R M)$. In this case, we have $M$-rad$(N) = \sqrt{(N :_R M)}M = (p_1 \cap p_2)M = ([p_1 + \text{ann}M] \cap [p_2 + \text{ann}M])M = p_1M \cap p_2M$, where $p_1M$, $p_2M$ are prime submodules of $M$ by [17, Corollary 2.11, 1.7]. Consequently, $M$-rad$(N)$ is a 2-absorbing submodule of $M$ by [26, Theorem 2.3]. □

**Theorem 2.7.** *Let $M$ be a (resp. finitely generated multiplication) $R$-module and $N$ be a submodule of $M$. If $M$-rad$(N)$ is a (resp. primary) prime submodule of $M$, then $N$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Suppose that $M$-rad$(N)$ is a prime submodule of $M$. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$, $am \notin M$-rad$(N)$. Since $M$-rad$(N)$ is a prime submodule and $abm \in M$-rad$(N)$, then $b \in (M\text{-rad}(N) :_R M)$. So $bm \in M$-rad$(N)$. Consequently $N$ is a 2-absorbing primary submodule of $M$. Now assume that $M$ is a finitely generated multiplication module and $M$-rad$(N)$ is a primary submodule of $M$, then $M$-rad$(N)$ is a prime submodule of $M$, by Proposition 2.5. Therefore $N$ is 2-absorbing primary. □

In [2, Theorem 1(3)], it was shown that for any faithful multiplication module $M$ not necessary finitely generated, $M$-rad$(IM) = \sqrt{I}M$ for any ideal $I$ of $R$.

**Theorem 2.8.** *Let $M$ be a (resp. finitely generated faithful multiplication) faithful multiplication $R$-module. If $M$-rad$(N)$ is a (resp. primary) prime submodule of $M$, then $N^n$ is a 2-absorbing primary submodule of $M$ for every positive integer $n \geq 1$.*

*Proof.* Assume that $M$ is a (resp. finitely generated faithful multiplication) faithful multiplication module and $M$-rad$(N)$ is a (resp. primary) prime submodule of $M$. There exists an ideal $I$ of $R$ such that $N = IM$. Thus

$$M - \mathrm{rad}(N^n) = \sqrt{I^n}M = M - \mathrm{rad}(N),$$

which is a (resp. primary) prime submodule of $M$. Hence for every positive integer $n \geq 1$, $N^n$ is a 2-absorbing primary submodule of $M$, by Theorem 2.7. □

Recall that a commutative ring $R$ with $1 \neq 0$ is called a divided ring if for every prime ideal $p$ of $R$, we have $p \subseteq xR$ for every $x \in R \backslash p$. Generalizing this idea to modules we say that an $R$-module $M$ is divided if for every prime submodule $N$ of $M$, $N \subseteq Rm$ for all $m \in M \backslash N$.

**Theorem 2.9.** *If $M$ is a divided $R$-module, then every proper submodule of $M$ is a 2-absorbing primary submodule of $M$. In particular, every proper submodule of a chained module is a 2-absorbing primary submodule.*

*Proof.* Let $N$ be a proper submodule of $M$. Since the prime submodules of a divided module are linearly ordered, we conclude that $M$-rad$(N)$ is a prime submodule of $M$. Hence $N$ is a 2-absorbing primary submodule of $M$ by Theorem 2.7. □

**Remark 2.10.** Let $I = (0 :_R M)$ and $R' = R/I$. It is easy to see that $N$ is a 2-absorbing primary $R$-submodule of $M$ if and only if $N$ is a 2-absorbing primary $R'$-submodule of $M$. Also, $(N :_R M)$ is a 2-absorbing primary ideal of $R$ if and only if $(N :_{R'} M)$ is a 2-absorbing primary ideal of $R'$.

**Theorem 2.11.** *Let $S$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. If $N$ is a 2-absorbing primary submodule of $M$ and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing primary submodule of $S^{-1}M$.*

*Proof.* If $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s} \in S^{-1}N$, then $ua_1a_2m \in N$ for some $u \in S$. It follows that $ua_1m \in M$-rad$(N)$ or $ua_2m \in M$-rad$(N)$ or $a_1a_2 \in (N :_R M)$, so we conclude that $\frac{a_1}{s_1}\frac{m}{s} = \frac{ua_1m}{us_1s} \in S^{-1}(M\text{-rad}(N)) \subseteq S^{-1}M\text{-rad}(S^{-1}N)$ or $\frac{a_2}{s_2}\frac{m}{s} = \frac{ua_2m}{us_2s} \in S^{-1}M\text{-rad}(S^{-1}N)$ or $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. □

**Theorem 2.12.** *Let $I$ be a 2-absorbing primary ideal of a ring $R$ and $M$ a faithful multiplication $R$-module such that $Ass_R(M/\sqrt{I}M)$ is a totally ordered set. Then $abm \in IM$ implies that $am \in \sqrt{I}M$ or $bm \in \sqrt{I}M$ or $ab \in I$ whenever $a, b \in R$ and $m \in M$.*

*Proof.* Let $a, b \in R$, $m \in M$ and $abm \in IM$. If $(\sqrt{I}M :_R am) = R$ or $(\sqrt{I}M :_R bm) = R$, we are done. Suppose that $(\sqrt{I}M :_R am)$ and $(\sqrt{I}M :_R bm)$ are proper ideals of $R$. Since $\mathrm{Ass}_R(M/\sqrt{I}M)$ is a totally ordered set, $(\sqrt{I}M :_R am) \cup (\sqrt{I}M :_R bm)$ is an ideal of $R$, and so there is a maximal ideal $\mathfrak{m}$ such that $(\sqrt{I}M :_R am) \cup (\sqrt{I}M :_R bm) \subseteq \mathfrak{m}$. We have $am \notin \mathrm{T}_{\mathfrak{m}}(M) := \{m' \in M : (1-x)m' = 0, \text{ for some } x \in \mathfrak{m}\}$, since $am \in \mathrm{T}_{\mathfrak{m}}(M)$ implies that $(1-x)am = 0$ for some $x \in \mathfrak{m}$, thus $(1-x)am \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$, a contradiction. So by [17, Theorem 1.2], there are $x \in \mathfrak{m}$ and $m' \in M$ such that $(1-x)M \subseteq Rm'$. Thus, $(1-x)m = rm'$ some $r \in R$. Moreover, $(1-x)abm = sm'$ for some $s \in I$, because $abm \in IM$. Hence $(abr - s)m' = 0$ and so $(1-x)(abr - s)M \subseteq (abr - s)Rm' = 0$. Thus $(1-x)(abr-s) = 0$, because $M$ is faithful. Therefore, $(1-x)abr = (1-x)s \in I$. Then $(1-x)ar \in \sqrt{I}$ or $(1-x)b \in \sqrt{I}$ or $abr \in I$, since $I$ is 2-absorbing primary. If $(1-x)ar \in \sqrt{I}$, then $(1-x)a \in \sqrt{I}$ or $(1-x)r \in \sqrt{I}$ or $ar \in \sqrt{I}$, because by [11, Theorem 2.2] $\sqrt{I}$ is a 2-absorbing ideal of $R$. If $(1-x)a \in \sqrt{I}$, then $(1-x)am \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$ that is a contradiction. If $(1-x)r \in \sqrt{I}$, then $(1-x)^2 m = (1-x)rm' \in \sqrt{I}M$ which implies that $(1-x)^2 \in (\sqrt{I}M :_R m) \subseteq (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$, a contradiction. Similarly we can see that $(1-x)b \notin \sqrt{I}$. Now, $ar \in \sqrt{I}$ implies that $(1-x)am = arm' \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$ which is a contradiction.
If $arb \in I$, then $ar \in \sqrt{I}$ or $br \in \sqrt{I}$ or $ab \in I$ which the first two cases are impossible, thus $ab \in I$. □

Let $R$ be a ring with the total quotient ring $K$. A nonzero ideal $I$ of $R$ is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The concept of an invertible submodule was introduced in [23] as a generalization of the concept of an invertible ideal. Let $M$ be an $R$-module and let $S = R \backslash \{0\}$. Then $T = \{t \in S \mid tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$ is a multiplicatively closed subset of $R$. Let $N$ be a submodule of $M$ and $N' = \{x \in R_T \mid xN \subseteq M\}$. A submodule $N$ is said to be *invertible in* $M$, if $N'N = M$, [23]. A nonzero $R$-module $M$ is called *Dedekind* provided that each nonzero submodule of $M$ is invertible.

We recall from [20] that, a finitely generated torsion-free multiplication module $M$ over a domain $R$ is a Dedekind module if and only if $R$ is a Dedekind domain.

**Theorem 2.13.** *Let $R$ be a Noetherian domain, $M$ a torsion-free multiplication $R$-module. Then the following statements are equivalent:*

1. *$M$ is a Dedekind module;*

2. *If $N$ is a nonzero 2-absorbing primary submodule of $M$, then either $N = \mathfrak{M}^n$ for some maximal submodule $\mathfrak{M}$ of $M$ and some positive integer $n \geq 1$ or $N = \mathfrak{M}_1^n \mathfrak{M}_2^m$ for some maximal submodules $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $M$ and some positive integers $n, m \geq 1$;*

3. *If $N$ is a nonzero 2-absorbing primary submodule of $M$, then either $N = P^n$ for some prime submodule $P$ of $M$ and some positive integer $n \geq 1$ or $N = P_1^n P_2^m$ for some prime submodules $P_1$ and $P_2$ of $M$ and some positive integers $n, m \geq 1$.*

*Proof.* By the fact that every multiplication module over a Noetherian ring is a Noetherian module, $M$ is Noetherian and so finitely generated.
(1) $\Rightarrow$ (2) Let $N$ be a 2-absorbing primary submodule of $M$. There exists a proper ideal $I$ of $R$ such that $N = IM$. So $(N :_R M) = I$ is a 2-absorbing primary ideal of $R$, by Theorem 2.6. Since $R$ is a Dedekind domain, then we have either $I = \mathfrak{m}^n$ for some maximal ideal $\mathfrak{m}$ of $R$ and some positive integer $n \geq 1$ or $I = \mathfrak{m}_1^n \mathfrak{m}_2^m$ for some maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $R$ and some positive integers $n, m \geq 1$, by [9, Theorem 2.11]. Thus, either $N = \mathfrak{m}^n M = (\mathfrak{m}M)^n$ or $N = (\mathfrak{m}_1 M)^n (\mathfrak{m}_2 M)^m$ as desired.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) It is sufficient to show that $R$ is a Dedekind domain, for this let $\mathfrak{m}$ be a maximal ideal of $R$. Let $I$ be an ideal of $R$ such that $\mathfrak{m}^2 \subset I \subset \mathfrak{m}$. So $\sqrt{I} = \mathfrak{m}$ and then $M$-rad$(IM) = \mathfrak{m}M$, since $M$ is a faithful multiplication $R$-module. Then $IM$ is a 2-absorbing primary submodule of $M$, Theorem 2.7. By assumption, either $IM = P^n$ for some prime submodule $P$ of $M$ and some positive integer $n \geq 1$ or $IM = P_1^n P_2^m$ for some prime submodules $P_1$ and $P_2$ of $M$ and some positive integers $n, m \geq 1$. Now, since $M$ is cancellation, either $I = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ of $R$ or $I = \mathfrak{p}_1^n \mathfrak{p}_2^m$ for some prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of $R$, which any two cases have a contradiction. Hence there are no ideals properly between $\mathfrak{m}^2$ and $\mathfrak{m}$. Consequently $R$ is a Dedekind domain by [19, Theorem 39.2, p. 470]. □

**Proposition 2.14.** *Let $M$ be a multiplication $R$-module and $K$, $N$ be submodules of $M$. Then*

1. *$\sqrt{(KN :_R M)} = \sqrt{(K :_R M)} \cap \sqrt{(N :_R M)}$.*
2. *$M$-rad$(KN) = M$-rad$(K) \cap M$-rad$(N)$.*
3. *$M$-rad$(K \cap N) = M$-rad$(K) \cap M$-rad$(N)$.*

*Proof.* (1) By hypothesis there exist ideals $I$, $J$ of $R$ such that $K = IM$ and $N = JM$. Now assume $r \in \sqrt{(K :_R M)} \cap \sqrt{(N :_R M)}$. Therefore there exist positive integers $m$, $n$ such that $r^m M \subseteq IM$ and $r^n M \subseteq JM$. Hence $r^{m+n} M \subseteq r^m JM \subseteq IJM = KN$. So $r \in \sqrt{(KN :_R M)}$. Consequently $\sqrt{(K :_R M)} \cap \sqrt{(N :_R M)} \subseteq \sqrt{(KN :_R M)}$. The other inclusion trivially holds.
(2) By part (1) and [17, Corollary 1.7],

$$\begin{aligned} M - \mathrm{rad}(KN) &= \sqrt{(KN :_R M)}M = (\sqrt{(K :_R M)} \cap \sqrt{(N :_R M)})M \\ &= ([\sqrt{(K :_R M)} + \mathrm{ann}M] \cap [\sqrt{(N :_R M)} + \mathrm{ann}M])M \\ &= \sqrt{(K :_R M)}M \cap \sqrt{(N :_R M)}M \\ &= M - \mathrm{rad}(K) \cap M - \mathrm{rad}(N). \end{aligned}$$

(3) See [1, Theorem 15(3)]. □

**Theorem 2.15.** *Let $M$ be a multiplication $R$-module and $N_1, N_2, \ldots, N_n$ be 2-absorbing primary submodules of $M$ with the same $M$-radical. Then $N = \cap_{i=1}^n N_i$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Notice that $M$-rad$(N) = \cap_{i=1}^n M$-rad$(N_i)$, by Proposition 2.14. Suppose that $abm \in N$ for some $a, b \in R$ and $m \in M$ and $ab \notin (N :_R M)$. Then $ab \notin (N_i :_R M)$ for some $1 \leq i \leq n$. Hence $am \in M$-rad$(N_i)$ or $bm \in M$-rad$(N_i)$. □

**Lemma 2.16.** *Let $M$ be an $R$-module and $N$ a 2-absorbing primary submodule of $M$. Suppose that $abK \subseteq N$ for some elements $a, b \in R$ and some submodule $K$ of $M$. If $ab \notin (N :_R M)$, then $aK \subseteq M$-rad$(N)$ or $bK \subseteq M$-rad$(N)$.*

*Proof.* Suppose that $aK \nsubseteq M$-rad$(N)$ and $bK \nsubseteq M$-rad$(N)$. Then $ak_1 \notin M$-rad$(N)$ and $bk_2 \notin M$-rad$(N)$ for some $k_1, k_2 \in K$. Since $abk_1 \in N$ and $ab \notin (N :_R M)$ and $ak_1 \notin M$-rad$(N)$, we have $bk_1 \in M$-rad$(N)$. Since $abk_2 \in N$ and $ab \notin (N :_R M)$ and $bk_2 \notin M$-rad$(N)$, we have $ak_2 \in M$-rad$(N)$. Now, since $ab(k_1 + k_2) \in N$ and $ab \notin (N :_R M)$, we have $a(k_1 + k_2) \in M$-rad$(N)$ or $b(k_1 + k_2) \in M$-rad$(N)$. Suppose that $a(k_1 + k_2) = ak_1 + ak_2 \in M$-rad$(N)$. Since $ak_2 \in M$-rad$(N)$, we have $ak_1 \in M$-rad$(N)$, a contradiction. Suppose that $b(k_1 + k_2) = bk_1 + bk_2 \in M$-rad$(N)$. Since $bk_1 \in M$-rad$(N)$, we have $bk_2 \in M$-rad$(N)$, a contradiction again. Thus $aK \subseteq M$-rad$(N)$ or $bK \subseteq M$-rad$(N)$. □

The following theorem offers a characterization of 2-absorbing primary submodules.

**Theorem 2.17.** *Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. The following conditions are equivalent:*

1. *$N$ is a 2-absorbing primary submodule of $M$;*
2. *If $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of $R$ and some submodule $K$ of $M$, then either $I_1I_2 \subseteq (N :_R M)$ or $I_1K \subseteq M$-rad$(N)$ or $I_2K \subseteq M$-rad$(N)$;*
3. *If $N_1N_2N_3 \subseteq N$ for some submodules $N_1,\ N_2$ and $N_3$ of $M$, then either $N_1N_2 \subseteq N$ or $N_1N_3 \subseteq M$-rad$(N)$ or $N_2N_3 \subseteq M$-rad$(N)$.*

*Proof.* (1)$\Rightarrow$(2) Suppose that $N$ is a 2-absorbing primary submodule of $M$ and $I_1I_2K \subseteq N$ for some ideals $I_1, I_2$ of $R$ and some submodule $K$ of $M$ and $I_1I_2 \nsubseteq (N :_R M)$. We show that $I_1K \subseteq M$-rad$(N)$ or $I_2K \subseteq M$-rad$(N)$. Suppose that $I_1K \nsubseteq M$-rad$(N)$ and $I_2K \nsubseteq M$-rad$(N)$. Then there are $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1K \nsubseteq M$-rad$(N)$ and $a_2K \nsubseteq M$-rad$(N)$. Since $a_1a_2K \subseteq N$ and neither $a_1K \subseteq M$-rad$(N)$ nor $a_2K \subseteq M$-rad$(N)$, we have $a_1a_2 \in (N :_R M)$ by Lemma 2.16.
Since $I_1I_2 \nsubseteq (N :_R M)$, we have $b_1b_2 \notin (N :_R M)$ for some $b_1 \in I_1$ and $b_2 \in I_2$. Since $b_1b_2K \subseteq N$ and $b_1b_2 \notin (N :_R M)$, we have $b_1K \subseteq M$-rad$(N)$ or $b_2K \subseteq M$-rad$(N)$ by Lemma 2.16. We consider three cases.

**Case 1**. Suppose that $b_1K \subseteq M$-rad$(N)$ but $b_2K \nsubseteq M$-rad$(N)$. Since $a_1b_2K \subseteq N$ and neither $b_2K \subseteq M$-rad$(N)$ nor $a_1K \subseteq M$-rad$(N)$, we conclude that $a_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $b_1K \subseteq M$-rad$(N)$ but $a_1K \nsubseteq M$-rad$(N)$, we conclude that $(a_1 + b_1)K \nsubseteq M$-rad$(N)$. Since $(a_1 + b_1)b_2K \subseteq N$ and neither $b_2K \subseteq M$-rad$(N)$ nor $(a_1 + b_1)K \subseteq M$-rad$(N)$, we conclude that $(a_1 + b_1)b_2 \in (N :_R M)$ by Lemma 2.16. Since $(a_1 + b_1)b_2 = a_1b_2 + b_1b_2 \in (N :_R M)$ and $a_1b_2 \in (N :_R M)$, we conclude that $b_1b_2 \in (N :_R M)$, a contradiction.

**Case 2**. Suppose that $b_2K \subseteq M$-rad$(N)$ but $b_1K \nsubseteq M$-rad$(N)$. Similar to the previous case we reach to a contradiction.

**Case 3**. Suppose that $b_1K \subseteq M$-rad$(N)$ and $b_2K \subseteq M$-rad$(N)$. Since $b_2K \subseteq M$-rad$(N)$ and $a_2K \nsubseteq M$-rad$(N)$, we conclude that $(a_2 + b_2)K \nsubseteq M$-rad$(N)$. Since $a_1(a_2 + b_2)K \subseteq N$ and neither $a_1K \subseteq M$-rad$(N)$ nor $(a_2 + b_2)K \subseteq M$-rad$(N)$, we conclude that $a_1(a_2 + b_2) = a_1a_2 + a_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2 \in (N :_R M)$ and $a_1a_2 + a_1b_2 \in (N :_R M)$, we conclude that $a_1b_2 \in (N :_R M)$. Since $b_1K \subseteq M$-rad$(N)$ and $a_1K \nsubseteq M$-rad$(N)$, we conclude that $(a_1 + b_1)K \nsubseteq M$-rad$(N)$. Since $(a_1 + b_1)a_2K \subseteq N$ and neither $a_2K \subseteq M$-rad$(N)$ nor $(a_1 + b_1)K \subseteq M$-rad$(N)$, we conclude that $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2 \in (N :_R M)$ and $a_1a_2 + b_1a_2 \in (N :_R M)$, we conclude that $b_1a_2 \in (N :_R M)$. Now, since $(a_1 + b_1)(a_2 + b_2)K \subseteq N$ and neither $(a_1 + b_1)K \subseteq M$-rad$(N)$ nor $(a_2 + b_2)K \subseteq M$-rad$(N)$, we conclude that

$(a_1 + b_1)(a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2, a_1b_2, b_1a_2 \in (N :_R M)$, we have $b_1b_2 \in (N :_R M)$, a contradiction. Consequently $I_1K \subseteq M\text{-rad}(N)$ or $I_2K \subseteq M\text{-rad}(N)$.
(2)⇒(1) is trivial.
(2)⇒(3) Let $N_1N_2N_3 \subseteq N$ for some submodules $N_1,\ N_2$ and $N_3$ of $M$ such that $N_1N_2 \nsubseteq N$. Since $M$ is multiplication, there are ideals $I_1,\ I_2$ of $R$ such that $N_1 = I_1M,\ N_2 = I_2M$. Clearly $I_1I_2N_3 \subseteq N$ and $I_1I_2 \nsubseteq (N :_R M)$. Therefore $I_1N_3 \subseteq M\text{-rad}(N)$ or $I_2N_3 \subseteq M\text{-rad}(N)$, which implies that $N_1N_3 \subseteq M\text{-rad}(N)$ or $N_2N_3 \subseteq M\text{-rad}(N)$.
(3)⇒(2) Suppose that $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of $R$ and some submodule $K$ of $M$. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 = K$ in part (3). □

**Theorem 2.18.** *Let $M$ be a multiplication $R$-module and $N$ a submodule of $M$. If $(N :_R M)$ is a 2-absorbing primary ideal of $R$, then $N$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Let $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of $R$ and some submodule $K$ of $M$. Since $M$ is multiplication, then there is an ideal $I_3$ of $R$ such that $K = I_3M$. Hence $I_1I_2I_3 \subseteq (N :_R M)$ which implies that either $I_1I_2 \subseteq (N :_R M)$ or $I_1I_3 \subseteq \sqrt{(N :_R M)}$ or $I_2I_3 \subseteq \sqrt{(N :_R M)}$, by [11, Theorem 2.19]. If $I_1I_2 \subseteq (N :_R M)$, then we are done. So, suppose that $I_1I_3 \subseteq \sqrt{(N :_R M)}$. Thus $I_1I_3M = I_1K \subseteq \sqrt{(N :_R M)}M = M\text{-rad}(N)$. Similary if $I_2I_3 \subseteq \sqrt{(N :_R M)}$, then we have $I_2K \subseteq M\text{-rad}(N)$. It completes the proof, by Theorem 2.17. □

The following example shows that Theorem 2.18 is not satisfied in general.

**Example 2.19.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and $N = 6\mathbb{Z} \times 0$ a submodule of $M$. Observe that $\mathbb{Z} \times 0$, $2\mathbb{Z} \times \mathbb{Z}$ and $3\mathbb{Z} \times \mathbb{Z}$ are some of the prime submodules of $M$ containing $N$. Also $(N :_\mathbb{Z} M) = 0$ is a 2-absorbing primary ideal of $\mathbb{Z}$. On the other hand, since $2.3.(1, 0) = (6, 0) \in N$, $2.3 \notin (N :_\mathbb{Z} M)$, $2.(1, 0) = (2, 0) \notin M\text{-rad}(N) \subseteq (\mathbb{Z} \times 0) \cap (2\mathbb{Z} \times \mathbb{Z}) \cap (3\mathbb{Z} \times \mathbb{Z}) = 6\mathbb{Z} \times 0 = N$ and $3.(1, 0) = (3, 0) \notin M\text{-rad}(N) = N$, so $N$ is not a 2-absorbing primary submodule of $M$.

**Theorem 2.20.** *Let $M$ be a multiplication $R$-module and $N_1$ and $N_2$ be primary submodules of $M$. Then $N_1 \cap N_2$ is a 2-absorbing primary submodule of $M$. If in addition $M$ is finitely generated faithful, then $N_1N_2$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Since $N_1$ and $N_2$ are primary submodules of $M$, then $(N_1 :_R M)$ and $(N_2 :_R M)$ are primary ideals of $R$. Hence $(N_1 :_R M)(N_2 :_R M)$ and $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$ are 2-absorbing primary ideals of $R$, by [11, Theorem 2.4]. Therefore, Theorem 2.18 implies that $N_1 \cap N_2$ is a 2-absorbing primary submodule of $M$. If $M$ is a finitely generated faithful multiplication $R$-module, then $(N_1N_2 :_R M) = (N_1 :_R M)(N_2 :_R M)$. So, again by Theorem 2.18 we deduce that $N_1N_2$ is a 2-absorbing primary submodule of $M$. $\square$

Let $M$ be a multiplication $R$-module and $N$ a primary submodule of $M$. We know that $\sqrt{(N :_R M)}$ is a prime ideal of $R$ and so $P = M\text{-rad}(N) = \sqrt{(N :_R M)}M$ is a prime submodule of $M$. In this case we say that $N$ is a *$P$-primary submodule of $M$*.

**Corollary 2.21.** *Let $M$ be a multiplication $R$-module and $P_1$ and $P_2$ be prime submodules of $M$. Suppose that $P_1^n$ is a $P_1$-primary submodule of $M$ for some positive integer $n \geq 1$ and $P_2^m$ is a $P_2$-primary submodule of $M$ for some positive integer $m \geq 1$.*

1. *$P_1^n \cap P_2^m$ is a 2-absorbing primary submodule of $M$.*
2. *If in addition $M$ is finitely generated faithful, then $P_1^nP_2^m$ is a 2-absorbing primary submodule of $M$.*

**Theorem 2.22.** *Let $M$ be a multiplication $R$-module and $N$ be a submodule of $M$ that has a primary decomposition. If $M$-rad$(N) = \mathfrak{M}_1 \cap \mathfrak{M}_2$ where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two maximal submodules of $M$, then $N$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Assume that $N = N_1 \cap \cdots \cap N_n$ is a primary decomposition. By Proposition 2.14(3), $M\text{-rad}(N) = M\text{-rad}(N_1) \cap \cdots \cap M\text{-rad}(N_n) = \mathfrak{M}_1 \cap \mathfrak{M}_2$. Since $M\text{-rad}(N_i)$'s are prime submodules of $M$, then $\{M\text{-rad}(N_1), \ldots, M\text{-rad}(N_n)\} = \{\mathfrak{M}_1, \mathfrak{M}_2\}$, by [3, Theorem 3.16]. Without loss of generality we may assume that for some $1 \leq t < n$, $\{M\text{-rad}(N_1), \ldots, M\text{-rad}(N_t)\} = \{\mathfrak{M}_1\}$ and $\{M\text{-rad}(N_{t+1}), \ldots, M\text{-rad}(N_n)\} = \{\mathfrak{M}_2\}$. Set $K_1 := N_1 \cap \cdots \cap N_t$ and $K_2 := N_{t+1} \cap \cdots \cap N_n$. By [8, Lemma 1.2.2], $K_1$ is an $\mathfrak{M}_1$-primary submodule and $K_2$ is an $\mathfrak{M}_2$-primary submodule of $M$. Therefore, by Theorem 2.20, $N = K_1 \cap K_2$ is 2-absorbing primary. $\square$

**Lemma 2.23.** *([22, Corollary 1.3]) Let $M$ and $M'$ be $R$-modules with $f : M \to M'$ an $R$-module epimorphism. If $N$ is a submodule of $M$ containing $Ker(f)$, then $f(M\text{-}rad(N)) = M'\text{-}rad(f(N))$.*

**Theorem 2.24.** *Let $f : M \to M'$ be a homomorphism of $R$-modules.*

1. *If $N'$ is a 2-absorbing primary submodule of $M'$, then $f^{-1}(N')$ is a 2-absorbing primary submodule of $M$.*

2. *If $f$ is epimorphism and $N$ is a 2-absorbing primary submodule of $M$ containing $Ker(f)$, then $f(N)$ is a 2-absorbing primary submodule of $M'$.*

*Proof.* (1) Let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(N')$. Then $abf(m) \in N'$. Hence $ab \in (N' :_R M')$ or $af(m) \in M'$-rad$(N')$ or $bf(m) \in M'$-rad$(N')$, and thus $ab \in (f^{-1}(N') :_R M)$ or $am \in f^{-1}(M'\text{-rad}(N'))$ or $bm \in f^{-1}(M'\text{-rad}(N'))$. By using the inclusion $f^{-1}(M'\text{-rad}(N')) \subseteq M\text{-rad}(f^{-1}(N'))$, we conclude that $f^{-1}(N')$ is a 2-absorbing primary submodule of $M$.
(2) Let $a, b \in R$, $m' \in M'$ and $abm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abm) \in f(N)$. Since $Ker(f) \subseteq N$, we have $abm \in N$. It implies that $ab \in (N :_R M)$ or $am \in M$-rad$(N)$ or $bm \in M$-rad$(N)$. Hence $ab \in (f(N) :_R M')$ or $am' \in f(M\text{-rad}(N)) = M'\text{-rad}(f(N))$ or $bm' \in f(M\text{-rad}(N)) = M'\text{-rad}(f(N))$. Consequently $f(N)$ is a 2-absorbing primary submodule of $M'$. □

As an immediate consequence of Theorem 2.24(2) we have the following Corollary.

**Corollary 2.25.** *Let $M$ be an $R$-module and $L \subseteq N$ be submodules of $M$. If $N$ is a 2-absorbing primary submodule of $M$, then $N/L$ is a 2-absorbing primary submodule of $M/L$.*

**Theorem 2.26.** *Let $K$ and $N$ be submodules of $M$ with $K \subset N \subset M$. If $K$ is a 2-absorbing primary submodule of $M$ and $N/K$ is a weakly 2-absorbing primary submodule of $M/K$, then $N$ is a 2-absorbing primary submodule of $M$.*

*Proof.* Let $a, b \in R$, $m \in M$ and $abm \in N$. If $abm \in K$, then $am \in M$-rad$(K) \subseteq M$-rad$(N)$ or $bm \in M$-rad$(K) \subseteq M$-rad$(N)$ or $ab \in (K :_R M) \subseteq (N :_R M)$ as it is needed.
So suppose that $abm \notin K$. Then $0 \neq ab(m + K) \in N/K$ that implies, $a(m + K) \in M/K\text{-rad}(N/K) = \frac{M-\text{rad}(N)}{K}$ or $b(m + K) \in M/K$-rad$(N/K)$ or $ab \in (N/K :_R M/K)$. It means that $am \in M$-rad$(N)$ or $bm \in M$-rad$(N)$ or $ab \in (N :_R M)$, which completes the proof. □

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is of the form $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$. In addition, if $M_i$ is a multiplication $R_i$-module, for $i = 1, 2$, then $M$ is a multiplication $R$-module. In this case, for each submodule $N = N_1 \times N_2$ of $M$ we have $M$-rad$(N) = M_1$-rad$(N_1) \times M_2$-rad$(N_2)$.

**Theorem 2.27.** *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where $M_1$ is a multiplication $R_1$-module and $M_2$ is a multiplication $R_2$-module.*

1. *A proper submodule $K_1$ of $M_1$ is a 2-absorbing primary submodule if and only if $N = K_1 \times M_2$ is a 2-absorbing primary submodule of $M$.*
2. *A proper submodule $K_2$ of $M_2$ is a 2-absorbing primary submodule if and only if $N = M_1 \times K_2$ is a 2-absorbing primary submodule of $M$.*
3. *If $K_1$ is a primary submodule of $M_1$ and $K_2$ is a primary submodule of $M_2$, then $N = K_1 \times K_2$ is a 2-absorbing primary submodule of $M$.*

*Proof.* (1) Suppose that $N = K_1 \times M_2$ is a 2-absorbing primary submodule of $M$. From our hypothesis, $N$ is proper, so $K_1 \neq M_1$. Set $M' = \frac{M}{\{0\}\times M_2}$. Hence $N' = \frac{N}{\{0\}\times M_2}$ is a 2-absorbing primary submodule of $M'$ by Corollary 2.25. Also observe that $M' \cong M_1$ and $N' \cong K_1$. Thus $K_1$ is a 2-absorbing primary submodule of $M_1$. Conversely, if $K_1$ is a 2-absorbing primary submodule of $M_1$, then it is clear that $N = K_1 \times M_2$ is a 2-absorbing primary submodule of $M$.
(2) It can be easily verified similar to (1).
(3) Assume that $N = K_1 \times K_2$ where $K_1$ and $K_2$ are primary submodules of $M_1$ and $M_2$, respectively. Hence $(K_1 \times M_2) \cap (M_1 \times K_2) = K_1 \times K_2 = N$ is a 2-absorbing primary submodule of $M$, by parts (1) and (2) and Theorem 2.20. □

**Theorem 2.28.** *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be a finitely generated multiplication $R$-module where $M_1$ is a multiplication $R_1$-module and $M_2$ is a multiplication $R_2$-module. If $N = N_1 \times N_2$ is a proper submodule of $M$, then the followings are equivalent.*

1. *$N$ is a 2-absorbing primary submodule of $M$.*
2. *$N_1 = M_1$ and $N_2$ is a 2-absorbing primary submodule of $M_2$ or $N_2 = M_2$ and $N_1$ is a 2-absorbing primary submodule of $M_1$ or $N_1$, $N_2$ are primary submodules of $M_1$, $M_2$, respectively.*

*Proof.* (1)⇒(2) Suppose that $N = N_1 \times N_2$ is a 2-absorbing primary submodule of $M$. Then $(N : M) = (N_1 : M_1) \times (N_2 : M_2)$ is a 2-absorbing primary ideal of $R = R_1 \times R_2$ by Theorem 2.6. From Theorem 2.3 in [11], we have $(N_1 : M_1) = R_1$ and $(N_2 : M_2)$ is a 2-absorbing primary ideal of $R_2$ or $(N_2 : M_2) = R_2$ and $(N_1 : M_1)$ is a 2-absorbing primary ideal of $R_1$ or $(N_1 : M_1)$ and $(N_2 : M_2)$ are primary ideals of $R_1$, $R_2$, respectively. Assume that $(N_1 : M_1) = R_1$ and $(N_2 : M_2)$ is a 2-absorbing primary ideal of $R_2$.

Thus $N_1 = M_1$ and $N_2$ is a 2-absorbing primary submodule of $M_2$ by Theorem 2.18. Similarly if $(N_2 : M_2) = R_2$ and $(N_1 : M_1)$ is a 2-absorbing primary ideal of $R_1$, then $N_2 = M_2$ and $N_1$ is a 2-absorbing primary submodule of $M$. And if the last case hold, then clearly we conclude that $N_1$, $N_2$ are primary submodules of $M_1$, $M_2$, respectively.
(2)⇒(1) It is clear from Theorem 2.27. □

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