



Fuzzy soft set approach to ideal theory of regular AG-groupoids

Feng Feng, Madad Khan, Violeta Leoreanu-Fotea,
Saima Anis and Nadeem Ajaib

Abstract

This paper aims to apply fuzzy sets and soft sets in combination to investigate algebraic properties of regular AG-groupoids. We initiate $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideals (right ideals, bi-ideals and quasi-ideals) over AG-groupoids and explore some related properties. Moreover, we give a number of characterizations for regular AG-groupoids by virtue of various types of fuzzy soft ideals over them.

1 Introduction

Uncertainty plays an essential role in the real world. In various fields such as data analysis, pattern recognition, expert systems and intelligent decision making, it has been found that mathematical models based on classical sets might not be suitable for dealing with problems containing vagueness and uncertainty. Also it is worth noting that uncertainties emerging from miscellaneous domains are very distinct and cannot be captured within a unique framework. In response to this, a number of useful mathematical theories such as probability theory, fuzzy sets [1], rough sets [2] and interval mathematics [3] have been developed.

Molodtsov's soft set theory [4] is a relatively new mathematical model for coping with uncertainty from a parametrization point of view. In mathematical modelling, sometimes it is impossible to find an exact solution of

Key Words: Groupoid, AG-groupoid, Soft set, Fuzzy set, Ideal.
2010 Mathematics Subject Classification: Primary 20N02, 03E72; Secondary 06D72.
Received: December 20, 2014.
Accepted: January 7, 2015.

the model of extremely high complexity. Alternatively, we are interested in finding numerical or approximate solutions. But this process can be done in a reverse way, that is, we can model and manipulate many real-life problems in an approximate way. By means of parameterization, a soft set can produce a series of approximate descriptions for a complicated entity being perceived from different aspects. This is in accordance with the way that humans use to treat uncertain concepts in their daily life. In recent years, soft set theory has been applied to various kinds of algebraic structures [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 15, 16, 18, 19, 20, 21]. But so far as we know, the algebras which have been investigated in virtue of soft sets are all associative, and very little attention has been paid to non-associative algebraic structures in the literature.

An Abel-Grassmann's groupoid (or briefly, AG-groupoid) is a non-associative algebraic structure lies in between a groupoid and a commutative semigroup. If an AG-groupoid contains left identity then this left identity is unique. An AG-groupoid with right identity becomes a commutative semigroup with identity. Moreover, every commutative AG-groupoid becomes a commutative semigroup. In most cases, an AG-groupoid behaves like a commutative semigroup. For instance, $a^2b^2 = b^2a^2$ holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity.

The aim of this study is to investigate ideal theory of regular AG-groupoids in virtue of fuzzy soft sets. In particular, we will introduce various types of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideals over AG-groupoids and try to obtain some characterizations for regular AG-groupoids based on algebraic ideal theory of them.

2 AG-Groupoids

Let (S, \cdot) be a groupoid and $a, b, c, d \in S$. It is said that S is an AG-groupoid, if it satisfies the left invertive law $(ab)c = (cb)a$. It can be verified that the medial law $(ab)(cd) = (ac)(bd)$ holds in every AG-groupoid. If S is an AG-groupoid with left identity e , then $S = S^2$ and the paramedial law $(ab)(cd) = (db)(ca)$ holds. Also we have

$$a(bc) = b(ac). \quad (1)$$

Moreover, it is easy to see that every one sided ideal of S is a quasi-ideal of S . In [22], it shown that $L[a] = a \cup Sa$ is a principal left ideal of S generated by a . Note also that $B[a] = a \cup a^2 \cup (aS)a$ is a principal bi-ideal and $Q[a] = a \cup (Sa \cap aS)$ is a principle quasi-ideal of S generated by a . A subset B of S is called generalized bi-ideal of S if $(BS)B \subseteq B$.

Lemma 1. *Let S be an AG-groupoid. If $a = a(ax)$ for some $x \in S$, then there exists $y \in S$ such that $a = a^2y$.*

Proof. Assume that $a = a(ax)$ for some x in S . By the medial law, we get

$$a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y, \text{ where } y = (ax)x.$$

□

Lemma 2. *Let S be an AG-groupoid with left identity. If $a = a^2x$ for some $x \in S$, then $a = (ay)a$ for some $y \in S$.*

Proof. Suppose that $a = a^2x$ for some $x \in S$. Note first that

$$a = a^2x = (aa)x = ((a^2x)(a^2x))x.$$

Then by the left invertive law, and the medial law, it follows that

$$((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2).$$

According to Eq. (1), we get

$$(xx^2)(a^2a^2) = a^2((xx^2)a^2) = (aa)((xx^2)a^2),$$

and hence

$$(aa)((xx^2)a^2) = (((xx^2)a^2)a)a = ((aa^2)(xx^2))a,$$

by the left invertive law. But we also have

$$(aa^2)(xx^2) = (x^2a^2)(xa) = (x^2x)(a^2a),$$

by both the paramedial law and the medial law. This implies that

$$((aa^2)(xx^2))a = ((x^2x)(a^2a))a.$$

Using Eq. (1), the paramedial law and the medial law, we further deduce that

$$\begin{aligned} ((x^2x)(a^2a))a &= (a^2((x^2x)a))a = ((aa)((x^2x)a))a \\ &= ((a(x^2x))(aa))a = (a((a(x^2x))a))a = (ay)a, \end{aligned}$$

where $y = (a(x^2x))a \in S$.

□

Lemma 3. *Let S be an AG-groupoid with left identity. Then we have*

- (1) $(aS)a^2 = (aS)a.$
- (2) $(aS)((aS)a) = (aS)a.$
- (3) $S((aS)a) = (aS)a.$
- (4) $(Sa)(aS) = a(aS).$
- (5) $(aS)(Sa) = (aS)a.$
- (6) $(a(aS))S = (aS)a.$
- (7) $((Sa)S)(Sa) = (aS)(Sa).$
- (8) $(Sa)S = (aS).$
- (9) $S(Sa) = Sa.$
- (10) $Sa^2 = a^2S.$

Proof. The proof is straightforward and thus omitted. \square

3 Fuzzy sets and fuzzy soft sets

The theory of fuzzy sets [1] provides an efficient framework for mathematical description of uncertain concepts. Recall that a fuzzy subset f of a given set S is defined by (and usually identified with) its membership function $f : S \rightarrow [0, 1]$. For $x \in S$, the membership value $f(x)$ essentially specifies the degree to which $x \in S$ belongs to the fuzzy subset f . By $f \subseteq g$, we mean that $f(x) \leq g(x)$ for all $x \in S$. Clearly $f = g$ if $f \subseteq g$ and $g \subseteq f$. That is, $f(x) = g(x)$ for all $x \in S$.

Let f and g be any fuzzy subsets of an AG-groupoid S . Then, the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in S \text{ with } a = bc, \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of S defined as

$$f(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t , where $t \in (0, 1]$.

Let $\gamma, \delta \in [0, 1]$ with $\gamma < \delta$, X be a nonempty set and $Y \subseteq X$. In the sequel, $\chi_Y^{(\gamma, \delta)}$ will be used to represent any fuzzy subset of X such that

$$\chi_Y^{(\gamma, \delta)}(x) = \begin{cases} s, & \text{if } x \in Y, \\ t, & \text{otherwise,} \end{cases}$$

where s, t are chosen (but fixed) from $[0, 1]$ with $s \geq \delta$ and $t \leq \gamma$. It is obvious that $\chi_Y^{(0,1)}$ coincides with the characteristic function χ_Y of Y . In what follows, $\mathcal{F}(X)$ denotes the set of all fuzzy subsets of X .

Let $\gamma, \delta \in [0, 1]$ with $\gamma < \delta$, x_r be a fuzzy point and $f \in \mathcal{F}(X)$. Then

- (1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
- (2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
- (3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

For $f, g \in \mathcal{F}(X)$, we write $f \subseteq \vee q_{(\gamma, \delta)} g$ if $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. In addition, f and g are said to be (γ, δ) -equal, denoted by $f =_{(\gamma, \delta)} g$, if $f \subseteq \vee q_{(\gamma, \delta)} g$ and $g \subseteq \vee q_{(\gamma, \delta)} f$. It is shown in [23] that the “ (γ, δ) -equal relation” is an equivalence relation on $\mathcal{F}(X)$.

Lemma 4. (see [23]) *Let f and g be fuzzy subsets of X . Then $f \subseteq \vee q_{(\gamma, \delta)} g$ if and only if $\max\{g(x), \gamma\} \geq \min\{f(x), \delta\}$ for all $x \in X$.*

Lemma 5. (see [23]) *Let $f, g, h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma, \delta)} g$ and $g \subseteq \vee q_{(\gamma, \delta)} h$, then we have $f \subseteq \vee q_{(\gamma, \delta)} h$.*

Note that $f =_{(\gamma, \delta)} g$ if and only if

$$\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$$

for all $x \in X$.

Lemma 6. *Let $\gamma, \delta \in [0, 1]$ with $\gamma < \delta$, A and B be nonempty subsets of an AG-groupoid S with left identity. Then we have*

- (1) $A \subseteq B$ if and only if $\chi_A^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)} \chi_B^{(\gamma, \delta)}$.
- (2) $\chi_A^{(\gamma, \delta)} \cap \chi_B^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(A \cap B)}^{(\gamma, \delta)}$.
- (3) $\chi_A^{(\gamma, \delta)} \circ \chi_B^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(AB)}^{(\gamma, \delta)}$.

Proof. The proof is straightforward and thus omitted. \square

Soft set theory was proposed by Molodtsov [4] in 1999, which provides a general mechanism for uncertainty modelling in a wide variety of applications. Let U be an initial universe of objects and E_U (or simply E) be the set of all parameters associated with objects in U , called a *parameter space*. In most cases parameters are considered to be attributes, characteristics or properties of objects in U . A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U , where $A \subseteq E$ and $F: A \rightarrow \mathcal{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set \mathfrak{S} . By means of parametrization, a soft set produces a series of approximate descriptions of a complicated object being perceived from various points of view. It is apparent that a soft set $\mathfrak{S} = (F, A)$ over a universe U can be viewed as a parameterized family of subsets of U .

By combining fuzzy sets with soft set, Maji et al. [24] initiated the following hybrid model called fuzzy soft set.

Definition 1. (see [24]) A pair $\mathfrak{S} = \langle F, A \rangle$ is called a *fuzzy soft set* over U , where $A \subseteq E$ and F is a mapping given by $F : A \rightarrow \mathcal{F}(U)$.

The mapping $F : A \rightarrow \mathcal{F}(U)$ is called the *approximate function* of the fuzzy soft set $\langle F, A \rangle$. Given two fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over U , we say that $\langle F, A \rangle$ is a *fuzzy soft subset* of $\langle G, B \rangle$, denoted $\langle F, A \rangle \subset \langle G, B \rangle$, if $A \subseteq B$ and $F(a)$ is a fuzzy subset of $G(a)$ for all $a \in A$. By $\langle F, A \rangle \supset \langle G, B \rangle$, we mean that $\langle G, B \rangle \subset \langle F, A \rangle$. Moreover, $\langle F, A \rangle$ and $\langle G, B \rangle$ are said to be *equal*, denoted $\langle F, A \rangle = \langle G, B \rangle$, if $\langle F, A \rangle \subset \langle G, B \rangle$ and $\langle G, B \rangle \subset \langle F, A \rangle$.

A fuzzy soft set $\langle F, A \rangle$ is an (γ, δ) -fuzzy soft subset of $\langle G, B \rangle$, denoted by $\langle F, A \rangle \subset_{(\gamma, \delta)} \langle G, B \rangle$, if $A \subseteq B$ and for any $a \in A$, $F(a) \subseteq \vee q_{(\gamma, \delta)} G(a)$.

The following notions were initially introduced by Ali et al. [25] for Molodtsov's soft sets.

Definition 2. (see [23]) Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two fuzzy soft sets over U .

- (1) The *extended intersection* of $\langle F, A \rangle$ and $\langle G, B \rangle$ is defined as the the fuzzy soft set $\langle H, C \rangle = \langle F, A \rangle \cap_{\mathcal{E}} \langle G, B \rangle$ where $C = A \cup B$ and for all $c \in C$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A \setminus B, \\ G(c), & \text{if } c \in B \setminus A, \\ F(c) \cap G(c), & \text{if } c \in A \cap B. \end{cases}$$

- (2) The *restricted intersection* of $\langle F, A \rangle$ and $\langle G, B \rangle$ is defined as the fuzzy soft set $\langle H, C \rangle = \langle F, A \rangle \cap_{\mathcal{R}} \langle G, B \rangle$, where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

Let V be a subset of U and $A \subseteq E$. A fuzzy soft set $\langle F, A \rangle$ over V is called a *relative whole* (γ, δ) -fuzzy soft set (with respect to V and A), denoted by $\Sigma(V, A)$, if $F(a) = \chi_V^{(\gamma, \delta)}$ for all $a \in A$.

4 Fuzzy soft ideals over AG-groupoids

From now on, unless stated otherwise, let S denote an AG-groupoid.

Definition 3. Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be fuzzy soft sets over S .

- (1) The *extended product* of $\langle F, A \rangle$ and $\langle G, B \rangle$ is defined as the the fuzzy soft set $\langle H, C \rangle = \langle F, A \rangle \odot \langle G, B \rangle$, where $C = A \cup B$ and for all $c \in C$,

$$H(c) = \begin{cases} F(c), & \text{if } c \in A \setminus B, \\ G(c), & \text{if } c \in B \setminus A, \\ F(c) \circ G(c), & \text{if } c \in A \cap B. \end{cases}$$

- (2) The *restricted product* of $\langle F, A \rangle$ and $\langle G, B \rangle$ is defined as the fuzzy soft set $\langle H, C \rangle = \langle F, A \rangle \boxtimes \langle G, B \rangle$, where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \circ \langle G, B \rangle$ for all $c \in C$.

Proposition 1. Let $\langle F, A \rangle$ and $\langle G, A \rangle$ be fuzzy soft sets over S . Then

$$\langle F, A \rangle \odot \langle G, A \rangle = \langle F, A \rangle \boxtimes \langle G, A \rangle.$$

Proof. The proof is straightforward and thus omitted. \square

The above assertion says that the extended product operation \odot coincides with the restricted product \boxtimes operation, if the concerned fuzzy soft sets have the same parameter set. Thus for this particular case, we shall use only notation \odot to represent for both operations in what follows.

Definition 4. A fuzzy soft set $\langle F, A \rangle$ over an AG-groupoid S is called

- a fuzzy soft left (resp. right) ideal over S if $\Sigma \langle S, A \rangle \odot \langle F, A \rangle \subset \langle F, A \rangle$ (resp. $\langle F, A \rangle \odot \Sigma \langle S, A \rangle \subset \langle F, A \rangle$);
- a fuzzy soft bi-ideal over S if $\langle F, A \rangle \odot \langle F, A \rangle \subset \langle F, A \rangle$ and $[\langle F, A \rangle \odot \Sigma \langle S, A \rangle] \odot \langle F, A \rangle \subset \langle F, A \rangle$;
- a fuzzy soft generalized bi-ideal over S if $[\langle F, A \rangle \odot \Sigma \langle S, A \rangle] \odot \langle F, A \rangle \subset \langle F, A \rangle$;
- a fuzzy soft quasi-ideal over S if

$$[\langle F, A \odot \Sigma(S, A) \rangle] \cap_{\mathcal{E}} [\Sigma(S, A) \odot \langle F, A \rangle] \subset \langle F, A \rangle.$$

Definition 5. A fuzzy soft set $\langle F, A \rangle$ over an AG-groupoid S is called

- an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft left ideal over S if $\Sigma(S, A) \odot \langle F, A \rangle \subset_{(\gamma, \delta)} \langle F, A \rangle$;
- an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft generalized bi-ideal over S if $[\langle F, A \rangle \odot \Sigma(S, A)] \odot \langle F, A \rangle \subset_{(\gamma, \delta)} \langle F, A \rangle$;
- an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft bi-ideal over S if $\langle F, A \rangle \odot \langle F, A \rangle \subset_{(\gamma, \delta)} \langle F, A \rangle$, and $[\langle F, A \rangle \odot \Sigma(S, A)] \odot \langle F, A \rangle \subset_{(\gamma, \delta)} \langle F, A \rangle$;
- an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft quasi-ideal over S if

$$[\langle F, A \odot \Sigma(S, A) \rangle] \cap_{\mathcal{E}} [\Sigma(S, A) \odot \langle F, A \rangle] \subset_{(\gamma, \delta)} \langle F, A \rangle.$$

Example 1. Let $S = \{1, 2, 3\}$ and define a binary operation “.” on S as follows:

| · | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2 | 2 | 3 |
| 2 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 |

Then it can be shown that (S, \cdot) is an AG-groupoid.

Let $A = \{0.35, 0.4\}$ and $\langle F, A \rangle$ be a fuzzy soft set over S such that

$$F(a)(x) = \begin{cases} 2a, & \text{if } x \in \{1, 2\}, \\ 0.4, & \text{otherwise,} \end{cases}$$

where $a \in A$ and $x \in S$. Then by definition, $\langle F, A \rangle$ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$ -fuzzy soft left ideal over S .

Next, let $B = \{0.7, 0.8\}$ and define a fuzzy soft set $\langle G, B \rangle$ over S as follows:

$$G(b)(x) = \begin{cases} b, & \text{if } x \in \{1, 2\}, \\ 0.4, & \text{otherwise,} \end{cases}$$

where $b \in B$ and $x \in S$. Then it is easy to verify that $\langle G, B \rangle$ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.4})$ -fuzzy soft bi-ideal over S .

Theorem 1. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) $B[a] \cap I[a] \cap L[a] \subseteq (B[a] I[a]) L[a]$, for some a in S .
- (iii) For any bi-ideal B , ideal I and left ideal L of S , $B \cap I \cap L \subseteq (BI) L$.
- (iv) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

- (v) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

- (vi) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft right ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal $\langle H, B \rangle$ of S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

Proof. (i) \Rightarrow (vi) Assume that (i) holds. Let $\langle F, A \rangle$, $\langle G, A \rangle$ and $\langle H, B \rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal over S , respectively. Let a be any element of S , $\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle = \langle K_1, A \cup B \rangle$ and $(\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle = \langle K_2, A \cup B \rangle$. For any $\epsilon \in A \cup B$. We consider the following cases,

Case 1: $\epsilon \in A - B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $K_2(\epsilon) = (F \circ G)(\epsilon)$ So, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$

Case 2: $\epsilon \in B - A$. Then $K_1(\epsilon) = H(\epsilon)$ and $K_2(\epsilon) = H(\epsilon) = K_1(\epsilon)$

Case 3: $\epsilon \in A \cap B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon) \cap H(\epsilon)$ and $K_2(\epsilon) = (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now we show that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now since S is regular AG-groupoid, so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$\begin{aligned} a &= (ax)a = [\{(ax)a\}x]a = (ax)\{(ax)a\} = [\{(ax)a\}x]\{(ax)a\} \\ &= \{(xa)(ax)\}\{(ax)a\} = [\{(ax)a\}(ax)](xa) \end{aligned}$$

Thus we have,

$$\begin{aligned} &\max \{((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma\} \\ &= \max \left\{ \bigvee_{a=pq} (F(\epsilon) \circ G(\epsilon))(p) \wedge H(\epsilon)(q), \gamma \right\} \\ &\geq \max \{ (F(\epsilon) \circ G(\epsilon))[\{(ax)a\}(ax)] \wedge H(\epsilon)(xa), \gamma \} \\ &= \max \left\{ \bigvee_{[\{(ax)a\}(ax)] = uv} (F(\epsilon)(u) \wedge G(\epsilon)(v)) \wedge H(\epsilon)(xa), \gamma \right\} \\ &\geq \max \{ F(\epsilon)\{(ax)a\} \wedge G(\epsilon)(ax) \wedge H(\epsilon)(xa), \gamma \} \\ &= \min \{ \max \{ F(\epsilon)\{(ax)a\}, \gamma \}, \max \{ G(\epsilon)(ax), \gamma \}, \max \{ H(\epsilon)(xa), \gamma \} \} \\ &\geq \min \{ \min \{ (F(\epsilon)(a), \delta), \min \{ G(\epsilon)(a), \delta \}, \min \{ H(\epsilon)(a), \delta \} \} \} \\ &= \min \{ (F(\epsilon) \wedge G(\epsilon) \wedge H(\epsilon))(a), \delta \} \\ &= \min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \end{aligned}$$

Thus $\min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \leq \max \{ ((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma \}$. This implies that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$.

Therefore in any case, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$. Hence

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

(vi) \Rightarrow (v) is obvious.

(v) \Rightarrow (iv) is obvious.

(iv) \Rightarrow (iii)

Assume that B , I and L are bi-ideal, ideal and left ideal of S , respectively, then $\Sigma(B, E)$, $\Sigma(I, E)$ and $\Sigma(L, E)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal over S , respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(B, E) \cap_\varepsilon \Sigma(I, E) \cap_\varepsilon \Sigma(L, E) \subset_{(\gamma, \delta)} (\Sigma(B, E) \odot \Sigma(I, E)) \odot \Sigma(L, E).$$

So,

$$\begin{aligned} \chi_{(B \cap I \cap L)}^{(\gamma, \delta)} &=_{(\gamma, \delta)} \chi_B^{(\gamma, \delta)} \cap \chi_I^{(\gamma, \delta)} \cap \chi_L^{(\gamma, \delta)} \\ &\subseteq \vee q_{(\gamma, \delta)} (\chi_B^{(\gamma, \delta)} \circ \chi_I^{(\gamma, \delta)}) \circ \chi_{\gamma L}^{(\gamma, \delta)} \\ &=_{(\gamma, \delta)} \chi_{BI}^{(\gamma, \delta)} \circ \chi_{\gamma L}^{(\gamma, \delta)} \\ &=_{(\gamma, \delta)} \chi_{(BI)L}^{(\gamma, \delta)}. \end{aligned}$$

Thus $B \cap I \cap L \subseteq (BI)L$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i)

Suppose that $B[a] = a \cup a^2 \cup (aS)a$, $I[a] = a \cup Sa \cup aS$ and $L[a] = a \cup Sa$ are principle bi-ideal, principle ideal and principle left ideal of S generated by a respectively. Thus by (ii), Lemma 3, left invertive law, paramedial law and using law $a(bc) = b(ac)$, we have,

$$\begin{aligned} &(a \cup a^2 \cup (aS)a) \cap (a \cup Sa \cup aS) \cap (a \cup Sa) \\ &\subseteq ((a \cup a^2 \cup (aS)a) (a \cup Sa \cup aS)) \\ &\quad (a \cup Sa) \\ &\subseteq \{S(a \cup Sa \cup aS)\} (a \cup Sa) \\ &\subseteq \{Sa \cup S(Sa) \cup S(aS)\} (a \cup Sa) \\ &= (Sa \cup aS) (a \cup Sa) \\ &= (Sa)a \cup (Sa)(Sa) \cup (aS)a \cup (aS)(Sa) \\ &= a^2S \cup a^2S \cup (aS)a \cup (aS)a \\ &= a^2S \cup (aS)a. \end{aligned}$$

Hence S is regular. □

Theorem 2. For an AG-groupoid S with left identity, the following are equivalent:

(i) S is regular.

(ii) $L[a] \cap I[a] \cap Q[a] \subseteq (L[a]I[a])Q[a]$, for some a in S .

(iii) For any left ideal L , ideal I and quasi-ideal Q of S , $L \cap I \cap Q \subseteq (LI)Q$

(iv) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

(v) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft right ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

Proof. (i) \Rightarrow (v) Assume that $\langle F, A \rangle$, $\langle G, A \rangle$ and $\langle H, B \rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal of a regular AG-groupoid S , respectively. Let a be any element of S , $\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \tilde{\cap} \langle H, B \rangle = \langle K_1, A \cup B \rangle$ and $(\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle = \langle K_2, A \cup B \rangle$. For any $\epsilon \in A \cup B$. We consider the following cases,

Case 1: $\epsilon \in A - B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $K_2(\epsilon) = (F \circ G)(\epsilon)$. So, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$

Case 2: $\epsilon \in B - A$. Then $K_1(\epsilon) = H(\epsilon)$ and $K_1(\epsilon) = H(\epsilon) = K_2(\epsilon)$

Case 3: $\epsilon \in A \cap B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon) \cap H(\epsilon)$ and $K_2(\epsilon) = (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now we show that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now since S is regular, so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)a = [\{(ax)a\}x]a = \{(xa)(ax)\}a.$$

Thus

$$\begin{aligned} & \max \{((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma\} \\ = & \max \left\{ \bigvee_{a=pq} (F(\epsilon) \circ G(\epsilon))(p) \wedge H(\epsilon)(q), \gamma \right\} \\ \geq & \max \{ (F(\epsilon) \circ G(\epsilon))[(xa)(ax)] \wedge H(\epsilon)(a), \gamma \} \\ = & \max \left\{ \bigvee_{(xa)(ax)=uv} (F(\epsilon)(u) \wedge G(\epsilon)(v)) \wedge H(\epsilon)(a), \gamma \right\} \\ \geq & \max \{ F(\epsilon)(xa) \wedge G(\epsilon)(ax) \wedge H(\epsilon)(a), \gamma \} \\ = & \min \{ \max \{ F(\epsilon)(xa), \gamma \}, \max \{ G(\epsilon)(ax), \gamma \}, \max \{ H(\epsilon)(a), \gamma \} \} \\ \geq & \min \{ \min \{ (F(\epsilon)(a), \delta \}, \min \{ G(\epsilon)(a), \delta \}, \min \{ H(\epsilon)(a), \delta \} \} \\ = & \min \{ (F(\epsilon) \wedge G(\epsilon) \wedge H(\epsilon))(a), \delta \} \\ = & \min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \end{aligned}$$

Thus we deduce

$$\min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \leq \max \{ ((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma \},$$

which implies that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$.

Therefore in any case, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$. Hence

$$\langle F, A \rangle \cap_{\mathcal{E}} \langle G, A \rangle \cap_{\mathcal{E}} \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

(v) \Rightarrow (iv) is obvious.

(iv) \Rightarrow (iii)

Assume that L , I and Q are left ideal, ideal and quasi-ideal of S respectively, then $\Sigma(L, E)$, $\Sigma(I, E)$ and $\Sigma(Q, E)$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft left ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft ideal and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft quasi-ideal over S , respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(L, E) \cap_{\mathcal{E}} \Sigma(I, E) \cap_{\mathcal{E}} \Sigma(Q, E) \subset_{(\gamma, \delta)} (\Sigma(L, E) \odot \Sigma(I, E)) \odot \Sigma(Q, E).$$

So, we get

$$\begin{aligned} \chi_{(L \cap I \cap Q)}^{(\gamma, \delta)} &=_{(\gamma, \delta)} \chi_L^{(\gamma, \delta)} \cap \chi_I^{(\gamma, \delta)} \cap \chi_Q^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)}(\chi_L^{\delta} \circ \chi_I^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} \\ &=_{(\gamma, \delta)} (\chi_{LI}^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(LI)Q}^{(\gamma, \delta)}. \end{aligned}$$

Therefore $L \cap I \cap Q \subseteq (LI)Q$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow i

$L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are left ideal, ideal and quasi-ideal of S generated a respectively. Thus by (ii), Lemma 3 and medial law we have,

$$\begin{aligned} & (a \cup Sa) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\ & \subseteq ((a \cup Sa)(a \cup Sa \cup aS)) \\ & \quad (a \cup (Sa \cap aS)) \\ & \subseteq \{(a \cup Sa)S\}(a \cup aS) \\ & = \{aS \cup (Sa)S\}(a \cup aS) \\ & = (aS)(a \cup aS) \\ & = (aS)a \cup (aS)(aS) \\ & = (aS)a \cup a^2S. \end{aligned}$$

Hence S is regular. \square

Theorem 3. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) $B[a] \cap I[a] \cap Q[a] \subseteq (B[a] I[a]) Q[a]$, for some a in S .
- (iii) For any bi-ideal B , ideal I and quasi-ideal Q of S , $B \cap I \cap Q \subseteq (BI) Q$.
- (iv) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

- (v) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal $\langle F, A \rangle$, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal $\langle G, A \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle H, B \rangle$ over S , we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

Proof. (i) \Rightarrow (v) Assume that $\langle F, A \rangle$, $\langle G, A \rangle$ and $\langle H, B \rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal of a regular AG-groupoid S , respectively. Let a be any element of S , $\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \tilde{\cap} \langle H, B \rangle = \langle K_1, A \cup B \rangle$ and $(\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle = \langle K_2, A \cup B \rangle$. For any $\epsilon \in A \cup B$. We consider the following cases,

Case 1: $\epsilon \in A - B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $K_2(\epsilon) = (F \circ G)(\epsilon)$. So, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$

Case 2: $\epsilon \in B - A$. Then $K_1(\epsilon) = H(\epsilon)$ and $K_2(\epsilon) = H(\epsilon) = K_1(\epsilon)$

Case 3: $\epsilon \in A \cap B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon) \cap H(\epsilon)$ and $K_2(\epsilon) = (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. First we show that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now since S is regular, so for $a \in S$ there exist $x \in S$ such that by using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax) a = (((ax) a) x) a = ((xa) (ax)) a = [a\{(xa) x\}]a.$$

Thus

$$\begin{aligned}
& \max \{((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma\} \\
= & \max \left\{ \bigvee_{a=xy} (F(\epsilon) \circ G(\epsilon))(x) \wedge H(\epsilon)(y), \gamma \right\} \\
\geq & \max \{ (F(\epsilon) \circ G(\epsilon))[a\{(xa) x\}] \wedge H(\epsilon)(a), \gamma \} \\
= & \max \left\{ \bigvee_{[a\{(xa)x\}] = pq} (F(\epsilon)(p) \wedge G(\epsilon)(q)) \wedge H(\epsilon)(a), \gamma \right\} \\
\geq & \max \{ F(\epsilon)(a) \wedge G(\epsilon)\{(xa) x\} \wedge H(\epsilon)(a), \gamma \} \\
= & \min \{ \max \{ F(\epsilon)(a), \gamma \}, \max \{ G(\epsilon)\{(xa) x\}, \gamma \}, \max \{ H(\epsilon)(a), \gamma \} \} \\
\geq & \min \{ \min \{ (F(\epsilon)(a), \delta \}, \min \{ G(\epsilon)(a), \delta \}, \min \{ H(\epsilon)(a), \delta \} \} \\
= & \min \{ (F(\epsilon) \wedge G(\epsilon) \wedge H(\epsilon))(a), \delta \} \\
= & \min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \}
\end{aligned}$$

Thus we have

$$\min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \leq \max \{ ((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma \},$$

which implies that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$.

Therefore in any case, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$. Hence

$$\langle F, A \rangle \cap_{\mathcal{E}} \langle G, A \rangle \cap_{\mathcal{E}} \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$

Assume that B , I and Q are bi-ideal, ideal and quasi-ideal of regular AG-groupoid of S respectively, then $\Sigma(B, E)$, $\Sigma(I, E)$ and $\Sigma(Q, E)$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft bi-ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft ideal and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft quasi-ideal over S , respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(B, E) \cap_{\mathcal{E}} \Sigma(I, E) \cap_{\mathcal{E}} \Sigma(Q, E) \subset_{(\gamma, \delta)} (\Sigma(B, E) \odot \Sigma(I, E)) \odot \Sigma(Q, E).$$

Thus it follows that

$$\begin{aligned}
\chi_{(B \cap I \cap Q)}^{(\gamma, \delta)} &= {}_{(\gamma, \delta)} \chi_L^{(\gamma, \delta)} \cap \chi_I^{(\gamma, \delta)} \cap \chi_Q^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)} (\chi_B^{\delta} \circ \chi_I^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} \\
&= {}_{(\gamma, \delta)} (\chi_{BI}^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} = {}_{(\gamma, \delta)} \chi_{(BI)Q}^{(\gamma, \delta)}.
\end{aligned}$$

Therefore $B \cap I \cap Q \subseteq (BI)Q$.

$(iii) \Rightarrow (ii)$ is obvious.

(ii) \Rightarrow (i)

Since $B[a] = a \cup a^2 \cup (aS)a$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are principle bi-ideal, principle ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii) and Lemma 3, using law $a(bc) = b(ac)$, medial law and left invertive law, we have,

$$\begin{aligned}
 & (a \cup a^2 \cup (aS)a) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
 \subseteq & ((a \cup a^2 \cup (aS)a) (a \cup Sa \\
 & \cup aS)) (a \cup (Sa \cap aS)) \\
 \subseteq & (S(a \cup Sa \cup aS)) (a \cup aS) \\
 = & (Sa \cup S(Sa) \cup S(aS)) (a \cup aS) \\
 = & (Sa \cup S(Sa) \cup S(aS)) (a \cup aS) \\
 = & (aS \cup Sa) (a \cup aS) \\
 = & (aS)a \cup (aS)(aS) \cup (Sa)a \cup (Sa)(aS) \\
 = & (aS)a \cup a^2S \cup a(aS).
 \end{aligned}$$

Hence S is regular. \square

Theorem 4. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) $I[a] \cap I[a] \cap I[a] \subseteq (I[a] I[a]) I[a]$, for some a in S .
- (iii) For any ideals I_1, I_2 and I_3 of S , $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.
- (iv) If $\langle F, A \rangle$, $\langle G, A \rangle$ and $\langle H, B \rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideals over S , then

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

Proof. (i) \Rightarrow (iv)

Assume that $\langle F, A \rangle$, $\langle G, A \rangle$ and $\langle H, B \rangle$ are any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideals over a regular AG-groupoid S , respectively. Let a be any element of S , $\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle = \langle K_1, A \cup B \rangle$ and $(\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle = \langle K_2, A \cup B \rangle$. For any $\epsilon \in A \cup B$. We consider the following cases,

Case 1: $\epsilon \in A - B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $K_2(\epsilon) = (F \circ G)(\epsilon)$. So we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $K_1(\epsilon) = H(\epsilon)$ and $K_1(\epsilon) = H(\epsilon) = K_2(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon) \cap H(\epsilon)$ and $K_2(\epsilon) = (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. First we show that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now since S is regular, so for $a \in S$ there exist $x \in S$ such that by using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)a = [\{(ax)a\}x]a = ((xa)(ax))a.$$

Thus

$$\begin{aligned}
& \max \{((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma\} \\
= & \max \left\{ \bigvee_{a=uv} (F(\epsilon) \circ G(\epsilon))(u) \wedge H(\epsilon)(v), \gamma \right\} \\
\geq & \max \{ (F(\epsilon) \circ G(\epsilon))[(xa)(ax)] \wedge H(\epsilon)(a), \gamma \} \\
= & \max \left\{ \bigvee_{(xa)(ax)=bc} (F(\epsilon)(b) \wedge G(\epsilon)(c)) \wedge H(\epsilon)(a), \gamma \right\} \\
\geq & \max \{ F(\epsilon)(xa) \wedge G(\epsilon)(ax) \wedge H(\epsilon)(a), \gamma \} \\
= & \min \{ \max \{ F(\epsilon)(xa), \gamma \}, \max \{ G(\epsilon)(ax), \gamma \}, \max \{ H(\epsilon)(a), \gamma \} \} \\
\geq & \min \{ \min \{ (F(\epsilon)(a), \delta \}, \min \{ G(\epsilon)(a), \delta \}, \min \{ H(\epsilon)(a), \delta \} \} \\
= & \min \{ (F(\epsilon) \wedge G(\epsilon) \wedge H(\epsilon))(a), \delta \} \\
= & \min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \}
\end{aligned}$$

Thus $\min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \leq \max \{ ((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma \}$. This implies that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$.

Therefore in any case, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$. Hence

$$\langle F, A \rangle \cap_{\mathcal{E}} \langle G, A \rangle \cap_{\mathcal{E}} \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

(iv) \Rightarrow (iii) is obvious.

Assume that I_1, I_2 and I_3 are any ideals of regular AG-groupoid of S respectively, then $\Sigma(I_1, E), \Sigma(I_2, E)$ and $\Sigma(I_3, E)$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft ideals over S , respectively. Now we have assume that (iv) holds, then we have

$$\Sigma(I_1, E) \cap_{\mathcal{E}} \Sigma(I_2, E) \cap_{\mathcal{E}} \Sigma(I_3, E) \subset_{(\gamma, \delta)} (\Sigma(I_1, E) \odot \Sigma(I_2, E)) \odot \Sigma(I_3, E).$$

So, we get

$$\begin{aligned}
\chi_{(I_1 \cap I_2 \cap I_3)}^{(\gamma, \delta)} &= (\gamma, \delta) \chi_{I_1}^{(\gamma, \delta)} \cap \chi_{I_2}^{(\gamma, \delta)} \cap \chi_{I_3}^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)} (\chi_{I_1}^{(\gamma, \delta)} \circ \chi_{I_2}^{(\gamma, \delta)}) \circ \chi_{I_3}^{(\gamma, \delta)} \\
&= (\gamma, \delta) (\chi_{I_1 I_2}^{(\gamma, \delta)}) \circ \chi_{I_3}^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(I_1 I_2) I_3}^{(\gamma, \delta)}.
\end{aligned}$$

Therefore $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i)

Since $I[a] = a \cup Sa \cup aS$ is a principle ideal of S generated by a . Thus by

(ii), Lemma 3, left invertive law, medial law and paramedial law, we have

$$\begin{aligned}
& (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \\
& \subseteq ((a \cup Sa \cup aS) (a \cup Sa \cup aS)) (a \cup Sa \cup aS) \\
& \subseteq \{(a \cup Sa \cup aS) S\} (a \cup Sa \cup aS) \\
& = \{aS \cup (Sa) S \cup (aS) S\} (a \cup Sa \cup aS) \\
& = \{aS \cup Sa\} (a \cup Sa \cup aS) \\
& = (aS) a \cup (aS) (Sa) \cup (aS) (aS) \cup (Sa) a \\
& \quad \cup (Sa) (Sa) \cup (Sa) (aS) \\
& = (aS) a \cup a^2 S.
\end{aligned}$$

Hence S is regular. \square

Theorem 5. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) $Q[a] \cap I[a] \cap Q[a] \subseteq (Q[a] I[a]) Q[a]$, for some a in S .
- (iii) For any quasi-ideals Q_1, Q_2 and ideal I of S , $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$.
- (iv) If $\langle F, A \rangle$ and $\langle H, B \rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideals over S , and $\langle G, A \rangle$ is a $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal over S , then we have

$$\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \cap_\varepsilon \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

Proof. (i) \Rightarrow (iv) Assume that $\langle F, A \rangle$ and $\langle H, B \rangle$ are any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideals and $\langle G, A \rangle$ is any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft ideal of a regular AG-groupoid S , respectively. Let a be any element of S , $\langle F, A \rangle \cap_\varepsilon \langle G, A \rangle \tilde{\cap} \langle H, B \rangle = \langle K_1, A \cup B \rangle$ and $(\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle = \langle K_2, A \cup B \rangle$. For any $\epsilon \in A \cup B$. We consider the following cases,

Case 1: $\epsilon \in A - B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $K_2(\epsilon) = (F \circ G)(\epsilon)$. Hence we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $K_1(\epsilon) = H(\epsilon)$ and $K_2(\epsilon) = H(\epsilon) = K_2(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $K_1(\epsilon) = F(\epsilon) \cap G(\epsilon) \cap H(\epsilon)$ and $K_2(\epsilon) = (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. First we show that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)} (F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$. Now since S is regular, so for $a \in S$ there exist $x \in S$ such that by using left invertive law and also using law $a(bc) = b(ac)$, we have

$$a = (ax) a = [\{(ax) a\} x] a = \{(xa) (ax)\} a = [a \{(xa) x\}] a.$$

Thus

$$\begin{aligned}
& \max \{((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma\} \\
= & \max \left\{ \bigvee_{a=pq} (F(\epsilon) \circ G(\epsilon))(p) \wedge H(\epsilon)(q), \gamma \right\} \\
\geq & \max \{ (F(\epsilon) \circ G(\epsilon))[a\{(xa)x\}] \wedge H(\epsilon)(a), \gamma \} \\
= & \max \left\{ \bigvee_{[a\{(xa)x\}] = bc} (F(\epsilon)(b) \wedge G(\epsilon)(c)) \wedge H(\epsilon)(a), \gamma \right\} \\
\geq & \max \{ F(\epsilon)(a) \wedge G(\epsilon)\{(xa)x\} \wedge H(\epsilon)(a), \gamma \} \\
= & \min \{ \max \{ F(\epsilon)(a), \gamma \}, \max \{ G(\epsilon)\{(xa)x\}, \gamma \}, \max \{ H(\epsilon)(a), \gamma \} \} \\
\geq & \min \{ \min \{ (F(\epsilon)(a), \delta), \min \{ G(\epsilon)(a), \delta \}, \min \{ H(\epsilon)(a), \delta \} \} \\
= & \min \{ (F(\epsilon) \wedge G(\epsilon) \wedge H(\epsilon))(a), \delta \} \\
= & \min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \}
\end{aligned}$$

Thus $\min \{ (F(\epsilon) \cap G(\epsilon) \cap H(\epsilon))(a), \delta \} \leq \max \{ ((F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon))(a), \gamma \}$. This implies that $F(\epsilon) \cap G(\epsilon) \cap H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ G(\epsilon)) \circ H(\epsilon)$.

Therefore in any case, we have $K_1(\epsilon) \subseteq \vee q_{(\gamma, \delta)} K_2(\epsilon)$. Hence

$$\langle F, A \rangle \cap_{\mathcal{E}} \langle G, A \rangle \cap_{\mathcal{E}} \langle H, B \rangle \subset_{(\gamma, \delta)} (\langle F, A \rangle \odot \langle G, A \rangle) \odot \langle H, B \rangle.$$

(iv) \Rightarrow (iii) Assume that Q_1 and Q_2 are quasi-ideals and I is an ideal of a regular AG-groupoid S , then $\Sigma(Q_1, E)$ and $\Sigma(Q_2, E)$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft quasi-ideals and, $\Sigma(I, E)$ is a $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft ideal over S . Now we have assume that (iv) holds, then we have

$$\Sigma(Q_1, E) \cap_{\mathcal{E}} \Sigma(I, E) \cap_{\mathcal{E}} \Sigma(Q_2, E) \subset_{(\gamma, \delta)} (\Sigma(Q_1, E) \odot \Sigma(I, E)) \odot \Sigma(Q_2, E).$$

So, we get

$$\begin{aligned}
\chi_{(Q_1 \cap I \cap Q_2)}^{(\gamma, \delta)} &=_{(\gamma, \delta)} \chi_{Q_1}^{(\gamma, \delta)} \cap \chi_I^{(\gamma, \delta)} \cap \chi_{Q_2}^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)} (\chi_{Q_1}^{(\gamma, \delta)} \circ \chi_I^{(\gamma, \delta)}) \circ \chi_{Q_2}^{(\gamma, \delta)} \\
&=_{(\gamma, \delta)} (\chi_{Q_1 I}^{(\gamma, \delta)}) \circ \chi_{Q_2}^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(Q_1 I) Q_2}^{(\gamma, \delta)}.
\end{aligned}$$

Thus $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i) Since $Q[a] = a \cup (Sa \cap aS)$ and $I[a] = a \cup Sa \cup aS$ are principle quasi-ideal and principle ideal of S generated by a respectively. Thus by (ii),

left invertive law, medial law and Lemma 3 , we have,

$$\begin{aligned}
 & (a \cup (Sa \cap aS)) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
 \subseteq & ((a \cup (Sa \cap aS)) (a \cup Sa \cup aS)) (a \cup (Sa \cap aS)) \\
 \subseteq & \{(a \cup aS) S\} (a \cap aS) \\
 = & \{aS \cup (aS) S\} (a \cap aS) \\
 = & (aS \cup Sa) (a \cap aS) \\
 = & \{(aS) a \cup (aS) (aS) \cup (Sa) a \cup (Sa) aSa \\
 = & (aS) a \cup a^2 S \cup a (aS) .
 \end{aligned}$$

Hence S is regular. \square

Theorem 6. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) For any principle bi-ideal $B[a]$, $B[a] = (B[a]S) B[a]$.
- (iii) For any bi-ideal B , $B = (BS) B$.
- (iv) For any generalized bi-ideal B , $B = (BS) B$.
- (v) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal $\langle F, B \rangle$ over S , we have

$$\langle F, B \rangle =_{(\gamma, \delta)} ((\langle F, B \rangle \odot \Sigma(S, E)) \odot \langle F, B \rangle).$$

- (vi) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal $\langle F, B \rangle$ over S , we have

$$\langle F, B \rangle =_{(\gamma, \delta)} ((\langle F, B \rangle \odot \Sigma(S, E)) \odot \langle F, B \rangle).$$

Proof. (i) \Rightarrow (vi) Assume that $\langle F, B \rangle$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal over a regular AG-groupoid S . Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx)b$. First we have to show that $F(\epsilon) \subseteq \vee q_{(\gamma, \delta)}((F(\epsilon) \circ$

$\chi_S^{(\delta)} \circ F(\epsilon)$. Then we have

$$\begin{aligned}
& \max \left\{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ F(\epsilon)(b), \gamma \right\} \\
= & \max \left\{ \bigvee_{b=pq} \left\{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)})(p) \wedge F(\epsilon)(q) \right\}, \gamma \right\} \\
\geq & \max \{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)})(bx) \wedge F(\epsilon)(b), \gamma \} \\
= & \max \left\{ \bigvee_{bx=cd} (F(\epsilon)(c) \wedge \chi_S^{(\gamma, \delta)}(d)) \wedge F(\epsilon)(b), \gamma \right\} \\
\geq & \max \{ F(\epsilon)(b) \wedge \chi_S^{(\gamma, \delta)}(x) \wedge F(\epsilon)(b), \gamma \} \\
= & \min \{ \max \{ F(\epsilon)(b), \gamma \}, 1, \max \{ F(\epsilon)(b), \gamma \} \} \\
\geq & \min \{ \min \{ F(\epsilon)(b), \delta \}, \min \{ F(\epsilon)(b), \delta \} \} \\
= & \min \{ F(\epsilon)(b) \wedge F(\epsilon)(b), \delta \} \\
= & \min \{ F(\epsilon)(b), \delta \}
\end{aligned}$$

Thus $\min \{ F(\epsilon)(b), \delta \} \leq \max \left\{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ F(\epsilon)(b), \gamma \right\}$. This implies that $F(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ F(\epsilon)$. That is $H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ F(\epsilon)$. Thus in any case, we have

$$H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ F(\epsilon).$$

Therefore,

$$\langle F, B \rangle \subset_{(\gamma, \delta)} (\langle F, B \rangle \odot \Sigma(S, E)) \odot F, B.$$

Since $\langle F, B \rangle$ is any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal over S , we have

$$(\langle F, B \rangle \odot \Sigma(S, E)) \odot F, B \subset_{(\gamma, \delta)} \langle F, B \rangle.$$

Thus

$$\langle F, B \rangle =_{(\gamma, \delta)} (\langle F, B \rangle \odot \Sigma(S, E)) \odot F, B.$$

(vi) \Rightarrow (v) is obvious.

(v) \Rightarrow (iv) Assume that B is a bi-ideal of S . Assume that the given condition (v) holds, then $\Sigma(B, E)$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal over S . Now by the assumption, we have

$$\Sigma(B, E) =_{(\gamma, \delta)} (\Sigma(B, E) \odot \Sigma(S, E)) \odot \Sigma(B, E).$$

Hence we have

$$\begin{aligned}
\chi_B^{(\gamma, \delta)} &=_{(\gamma, \delta)} (\chi_B^{(\gamma, \delta)} \circ \chi_S^{(\gamma, \delta)}) \circ \chi_B^{(\gamma, \delta)} \\
&=_{(\gamma, \delta)} (\chi_{BS}^{(\gamma, \delta)}) \circ \chi_B^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(BS)B}^{(\gamma, \delta)}.
\end{aligned}$$

Therefore $B = (BS)B$.

(iv) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i)

Since $B[a] = a \cup a^2 \cup (aS)a$ is a principle bi-ideal of S generated by a respectively. Thus by (ii) and Lemma 3, we have,

$$\begin{aligned}
 a \cup a^2 \cup (aS)a &= [\{a \cup a^2 \cup (aS)a\}S] (a \cup a^2 \cup (aS)a) \\
 &= [aS \cup a^2S \cup \{(aS)a\}S] (a \cup a^2 \cup (aS)a) \\
 &= (aS \cup a^2S \cup a(aS)) (a \cup a^2 \cup (aS)a) \\
 &= (aS)a \cup (aS)a^2 \cup (aS)((aS)a) \\
 &\quad \cup (a^2S)a \cup (a^2S)a^2 \cup (a^2S)((aS)a) \\
 &\quad \cup (a(aS))a \cup (a(aS))a^2 \cup (a(aS))((aS)a) \\
 &= (aS)a \cup a^2S \cup (aS)a \cup a^2S \cup a^2S \cup a^2S \\
 &\quad \cup (aS)a \cup (aS)a \cup (aS)a \\
 &= a^2S \cup (aS)a.
 \end{aligned}$$

Hence S is regular □

Theorem 7. For an AG-groupoid S with left identity, the following are equivalent:

- (i) S is regular.
- (ii) $B[a] \cap Q[a] \subseteq (B[a]S)Q[a]$, for some a in S .
- (iii) For any bi-ideal B and quasi-ideal Q of S , $B \cap Q \subseteq (BS)Q$.
- (iv) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft bi-ideal $\langle F, B \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle G, Q \rangle$ over S , we have

$$\langle F, B \rangle \cap_\varepsilon \langle G, Q \rangle \subset_{(\gamma, \delta)} (\langle F, B \rangle \odot \Sigma(S, E)) \odot \langle G, Q \rangle.$$

- (v) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal $\langle F, B \rangle$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal $\langle G, Q \rangle$ over S , we have

$$\langle F, B \rangle \cap_\varepsilon \langle G, Q \rangle \subset_{(\gamma, \delta)} (\langle F, B \rangle \odot \Sigma(S, E)) \odot \langle G, Q \rangle.$$

Proof. (i) \Rightarrow (v) Assume that $\langle F, B \rangle$ and $\langle G, Q \rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft quasi-ideal over a regular AG-groupoid S , respectively. Let a be any element of S , $\langle F, B \rangle \cap_\varepsilon \langle G, Q \rangle = \langle K_1, B \cup Q \rangle$. For any $\epsilon \in B \cup Q$. We consider the following cases,

Case 1: $\epsilon \in B - Q$. Then $H(\epsilon) = F(\epsilon) = (F \circ G)(\epsilon)$.

Case 2: $\epsilon \in Q - B$. Then $H(\epsilon) = G(\epsilon) = (F \circ G)(\epsilon)$.

Case 3: $\epsilon \in B \cap Q$. Then $H(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $(F \circ G)(\epsilon) = F(\epsilon) \circ G(\epsilon)$. First we show that $F(\epsilon) \cap G(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ G(\epsilon)$. Now since S is regular, for $a \in S$ there exist $x \in S$ such that $a = (ax)a = [\{(ax)a\}x]a$ by left invertive law and the law $a(bc) = b(ac)$.

Thus we deduce that

$$\begin{aligned}
 & \max \left\{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ G(\epsilon)(a), \gamma \right\} \\
 = & \max \left\{ \bigvee_{a=pq} \left\{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)})(p) \wedge G(\epsilon)(q) \right\}, \gamma \right\} \\
 \geq & \max \{ (F(\epsilon) \circ \chi_S^{(\gamma, \delta)})[\{(ax)a\}x] \wedge G(\epsilon)(a), \gamma \} \\
 = & \max \left\{ \bigvee_{[\{(ax)a\}x]=cd} (F(\epsilon)(c) \wedge \chi_S^{(\gamma, \delta)}(d)) \wedge G(\epsilon)(a), \gamma \right\} \\
 \geq & \max \{ F(\epsilon)\{(ax)a\} \wedge \chi_S^{(\gamma, \delta)}(x) \wedge G(\epsilon)(a), \gamma \} \\
 = & \min \{ \max \{ F(\epsilon)\{(ax)a\}, \gamma \}, 1, \max \{ G(\epsilon)(a), \gamma \} \}. \\
 \geq & \min \{ \min \{ F(\epsilon)(a), \delta \}, \min \{ G(\epsilon)(a), \delta \} \} \\
 = & \min \{ F(\epsilon)(a) \wedge G(\epsilon)(a), \delta \} \\
 = & \min \{ (F(\epsilon) \cap G(\epsilon))(a), \delta \}
 \end{aligned}$$

Hence $F(\epsilon) \cap G(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ G(\epsilon)$. That is,

$$H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ G(\epsilon).$$

Thus in any case, we have

$$H(\epsilon) \subseteq \vee q_{(\gamma, \delta)}(F(\epsilon) \circ \chi_S^{(\gamma, \delta)}) \circ G(\epsilon).$$

Therefore,

$$\langle F, B \rangle \cap_{\mathcal{E}} \langle G, Q \rangle \subset_{(\gamma, \delta)} (\langle F, B \rangle \odot \Sigma(S, E)) \odot \langle G, Q \rangle.$$

(v) \Rightarrow (iv) is obvious.

(iv) \Rightarrow (iii)

Assume that B and Q are bi-ideal and quasi-ideal of regular AG-groupoid of S respectively, then $\Sigma(B, E)$ and $\Sigma(Q, E)$ be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft bi-ideal and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy soft quasi-ideal $\langle G, Q \rangle$ over S . Now by the assumption, we have

$$\Sigma(B, E) \cap_{\mathcal{E}} \Sigma(Q, E) =_{(\gamma, \delta)} (\Sigma(B, E) \odot \Sigma(S, E)) \odot \Sigma(Q, E).$$

So we have,

$$\begin{aligned}\chi_{(B \cap Q)}^{(\gamma, \delta)} &=_{(\gamma, \delta)} \chi_B^{(\gamma, \delta)} \cap \chi_S^{(\gamma, \delta)} \cap \chi_Q^{(\gamma, \delta)} \subseteq \vee q_{(\gamma, \delta)}(\chi_B^{(\gamma, \delta)} \circ \chi_S^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} \\ &=_{(\gamma, \delta)} (\chi_{BS}^{(\gamma, \delta)}) \circ \chi_Q^{(\gamma, \delta)} =_{(\gamma, \delta)} \chi_{(BS)Q}^{(\gamma, \delta)}.\end{aligned}$$

Therefore $B \cap Q \subseteq (BS)Q$.

(iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i)

Since $B[a] = a \cup a^2 \cup (aS)a$ and $Q[a] = a \cup (Sa \cap aS)$ are principle bi-ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii), Eq. (1), Lemma 3, medial law and left invertive law, we have

$$\begin{aligned}& \{a \cup a^2 \cup (aS)a\} \cap \{a \cup (Sa \cap aS)\} \\ & \subseteq \{(a \cup a^2 \cup (aS)a)S\} (a \cup (Sa \cap aS)) \\ & \subseteq \{aS \cup a^2S \cup ((aS)a)S\} (a \cup Sa) \\ & = \{aS \cup a^2S \cup (Sa)(aS)\} (a \cup Sa) \\ & = \{(aS)a \cup (a^2S)a \cup \{(Sa)(aS)\}a \cup (aS)(Sa) \\ & \quad \cup (a^2S)(Sa) \cup (Sa)(aS)(Sa)\} \\ & \subseteq (aS)a \cup Sa^2 \cup (aS)a \cup (aS)a \cup Sa^2 \cup (aS)a \\ & = (aS)a \cup a^2S\end{aligned}$$

Hence S is regular as required. \square

5 Conclusions

We have applied the theory of fuzzy soft sets to algebraic ideal theory of regular AG-groupoids. We introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy soft left ideals (right ideals, bi-ideals and quasi-ideals) over AG-groupoids. Moreover, we presented some characterizations for regular AG-groupoids based on these newly introduced fuzzy soft ideals over them. The natural extension of this research work is to study intra-regular AG-groupoids using fuzzy soft ideals.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Program No. 11301415), Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2013JQ1020), Shaanxi Provincial Research Plan for Young Scientific and Technological New Stars (Program No. 2014KJXX-73), Scientific Research Program Funded by Shaanxi Provincial

Education Department of China (Program No. 2013JK1098) and New Star Team of Xi'an University of Posts and Telecommunications. The Pakistani authors are thankful to the Higher Education Commission of Pakistan for financial support.

References

- [1] L.A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965) 338–353.
- [2] Z. Pawlak, Rough sets, *International Journal of Information and Computer Sciences* 11 (1982) 341–356.
- [3] M.B. Gorzalzany, A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems* 21 (1987) 1–17.
- [4] D. Molodtsov, Soft set theory—First results, *Computer and Mathematics with Applications* 37 (1999) 19–31.
- [5] M. Akram, B. Davvaz, F. Feng, Intuitionistic Fuzzy Soft K-Algebras, *Mathematics in Computer Science*, 2013, 7 (3), 353–365.
- [6] M. Akram, F. Feng, Soft intersection Lie algebras, Quasigroups and Related Systems, 2013, 21 (1), 11–18.
- [7] M. Akram, N.O. Alshehri, R.S. Alghamdi, Fuzzy soft K-algebras, *Utilitas Mathematica* 90 (2013) 307–325.
- [8] N.O. Alshehri, M. Akram, R.S. Al-ghamdi, Applications of Soft Sets in K-Algebras, *Advances in Fuzzy Systems*, Volume 2013 (2013), Article ID 319542, 8 pages, 2013.
- [9] V. Leoreanu-Fotea, B. Davvaz, F. Feng, C. Chiper, Join spaces, soft join spaces and lattices, *Analele Stiintifice Ale Universitatii Ovidius Constanta-Seria Matematica* 22(1), 155–167, 2014.
- [10] X. Liu, D. Xiang, J. Zhan, K.P. Shum, Isomorphism theorems for soft rings, *Algebra Colloquium* 19(4) (2012) 649–656.
- [11] X. Liu, D. Xiang, J. Zhan, Fuzzy isomorphism theorems of soft rings, *Neural Computing and Applications* 21(2) (2012) 391–397.
- [12] X. Ma, H.S. Kim, (M,N)-Soft Intersection BL-Algebras and Their Congruences, *The Scientific World Journal*, vol. 2014, Article ID 461060, 6 pages, 2014.

- [13] G. Muhiuddin, F. Feng, Y.B. Jun, Subalgebras of BCK/BCI-algebras based on cubic soft sets, *The Scientific World Journal*, vol. 2014, Article ID 458638, 9 pages.
- [14] C.H. Park, Y.B. Jun, M.A. Öztürk, Soft WS-algebras, *Communications of Korean Mathematical Society* 23 (2008) 313–324.
- [15] A. Sezgin, A.O. Atagun, E. Aygun, A note on soft near-rings and idealistic soft near-rings, *Filomat* 25 (2011) 53–68.
- [16] A. Sezgin, A.O. Atagun, Soft groups and normalistic soft groups, *Comput. Math. Appl.* 62 (2011) 685–698.
- [17] S.Z. Song, H.S. Kim, Y.B. Jun, Ideal Theory in Semigroups Based on Intersectional Soft Sets, *The Scientific World Journal*, vol. 2014, Article ID 136424, 7 pages, 2014.
- [18] D.R.P. Williams, A.B. Saeid, Fuzzy soft ideals in subtraction algebras, *Neural Computing and Applications* 21 (2012) 159–169.
- [19] X. Xin and W. Li, Soft Congruence Relations over Rings, *The Scientific World Journal*, vol. 2014, Article ID 541630, 9 pages, 2014.
- [20] Y. Yang, X. Xin, P. He, Applications of Soft Union Sets in the Ring Theory, *Journal of Applied Mathematics*, vol. 2013, Article ID 474890, 9 pages, 2013.
- [21] J. Zhan, Y.B. Jun, Soft BL-algebras based on fuzzy sets, *Computers and Mathematics with Applications* 59 (2010) 2037–2046.
- [22] P. V. Protić and N. Stevanović, On Abel-Grassmann's groupoids, *Proc. Math. Conf. Priština*, (1994), 31–38.
- [23] Y.Q. Yin and J. Zhan, Characterization of ordered semigroups in terms of fuzzy soft ideals, *Bull. Malays. Math. Sci. Soc.* 35 (2012), 997–1015.
- [24] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, *J. Fuzzy Math.* 9(3) (2001) 589–602.
- [25] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.* 57 (2009) 1547–1553.

Feng FENG,
 Department of Applied Mathematics, School of Science,
 Xi'an University of Posts and Telecommunications,
 Xi'an 710121, China.
 Email: fengnix@hotmail.com

Madad KHAN,
Department of Mathematics,
COMSATS Institute of Information Technology,
Abbottabad 22060, Pakistan.
Email: madadmath@yahoo.com

Violeta LEOREANU-FOTEA,
Faculty of Mathematics,
“Alexandru Ioan Cuza” University,
700506 Iași, Romania.
Email: leoreanu2002@yahoo.com

Saima ANIS,
Department of Mathematics,
COMSATS Institute of Information Technology,
Abbottabad 22060, Pakistan.
Email: saimaanispk@gmail.com

Nadeem AJAIB,
Department of Mathematics,
COMSATS Institute of Information Technology,
Abbottabad 22060, Pakistan.
Email: nadeem.ajaib@yahoo.com