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Positive solutions of nth-order impulsive differential equations with integral boundary conditions

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Abstract

This paper is concerned with the existence and nonexistence of positive solutions of nth-order impulsive boundary value problem with integral boundary conditions. The fixed point theorem of cone expansion and compression is used to investigate the existence of at least one positive solution. Also, an example is given to illustrate the effective of our results.

1 Introduction

The theory of impulsive differential equations is adequate mathematical models for description of evolution processes characterized by the combination of a continuous and jumps change of their state. Impulsive differential equations have become an important area of research in recent years of the needs of modern technology, engineering, economic and physics. Moreover, impulsive differential equations are richer in applications compared to the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. For the introduction of the basic theory of impulsive equations, see [5, 6, 18, 22] and the references therein.

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At the same time, the theory of boundary value problems with integral boundary conditions for differential equations arises in different areas of applied mathematics and physics. For example, a class of boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [9], semiconductor problems [16], hydrodynamic problems [10]. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attentions. To indentify a few, we refer the reader to [1, 2, 8, 13, 19, 23, 24] and references therein. On the other hand, there are fewer results in the literature for higher-order differential equations with integral boundary conditions, see [3, 4, 20]. In particular, we would like to mention some results of Ahmad and Ntouyas [3, 4].

In [3], Ahmad and Ntouyas considered the following n-th order differential inclusion with four-point integral boundary conditions

$$\begin{cases} x^{(n)}(t) \in F(t, x(t)), \ 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad x'(0) = 0, \ x''(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_{\eta}^{1} x(s)ds, \quad 0 < \xi < \eta < 1. \end{cases}$$

The existence results were obtained by applying the nonlinear alternative of Leray-Schauder type and some suitable theorems of fixed point theory.

In [4], Ahmad and Ntouyas developed some existence results for the following nth-order boundary value problem with four-point nonlocal integral boundary conditions by using Krasnoselskii's fixed point theorem and Leray-Schauder degree theory

$$\begin{cases} x^{(n)}(t) = f(t, x(t)), \ 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad x'(0) = 0, \ x''(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ x(1) = \beta \displaystyle\int_{\eta}^{1} x(s)ds, \quad 0 < \xi < \eta < 1. \end{cases}$$

Recently, the existence and multiplicity of positive solutions for linear and nonlinear second-order impulsive differential equations with integral boundary conditions have been studied extensively. To identify a few, we refer to the reader to see [7, 11, 15, 17, 21, 26]. However, we can only see that the boundary value problems with integral boundary conditions for impulsive differential equations have been discussed in [12, 25]. So this paper fills the gap.

In [25], Zhang et al. investigated the existence of positive solutions of the following nth-order boundary value problem with integral boundary conditions by using the fixed point theory for strict set contraction operator

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), x''(t), \dots, x^{(n-2)}(t)) = \theta, \ t \in J, \ t \neq t_k, \\ \Delta x^{(n-2)}|_{t=t_k} = -I_k(x^{(n-2)}(t_k)), \ k = 1, 2, \dots, m, \\ x^{(i)}(0) = \theta, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(0) = x^{(n-2)}(1) = \displaystyle\int_0^1 g(t) x^{(n-2)}(t) dt. \end{cases}$$

In [12], Feng et al. studied the existence, nonexistence, and multiplicity of positive solutions for the following nth-order impulsive differential equations with integral boundary conditions

$$\begin{cases} x^{(n)}(t) + f(t, x(t)) = 0, \ t \in J, \ t \neq t_k, \\ -\Delta x^{(n-1)}|_{t=t_k} = I_k(x(t_k)), \ k = 1, 2, \dots, m, \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, \quad x(1) = \displaystyle\int_0^1 h(t) x(t) dt. \end{cases}$$

Motivated by the above results, in this study, we consider the following nth-order impulsive boundary value problem (BVP)

$$\begin{cases} u^{(n)}(t) + q(t) f(t, u(t)) = 0, \ t \in J := [0, 1], \ t \neq t_k, \\ \Delta u^{(n-2)}|_{t=t_k} = I_k(u^{(n-2)}(t_k)), \ k = 1, 2, \dots, l, \\ \Delta u^{(n-1)}|_{t=t_k} = -J_k(u^{(n-2)}(t_k)), \\ a u^{(n-2)}(0) - b u^{(n-1)}(0) = \displaystyle\int_0^1 g_1(s) u^{(n-2)}(s) ds, \\ c u^{(n-2)}(1) + d u^{(n-1)}(1) = \displaystyle\int_0^1 g_2(s) u^{(n-2)}(s) ds, \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n-3, \end{cases} \tag{1.1}$$

where $t_k \in (0, 1), \ k = 1, 2, \dots, l$ with $0 < t_1 < t_2 < \dots < t_l < 1$. $\Delta u^{(n-2)}|_{t=t_k}$ and $\Delta u^{(n-1)}|_{t=t_k}$ denote the jumps of $u^{(n-2)}(t)$ and $u^{(n-1)}(t)$ at $t = t_k$, i.e.,

$$\Delta u^{(n-2)}|_{t=t_k} = u^{(n-2)}(t_k^+) - u^{(n-2)}(t_k^-),$$
$$\Delta u^{(n-1)}|_{t=t_k} = u^{(n-1)}(t_k^+) - u^{(n-1)}(t_k^-),$$

where $u^{(n-2)}(t_k^+),\ u^{(n-1)}(t_k^+)$ and $u^{(n-2)}(t_k^-),\ u^{(n-1)}(t_k^-)$ represent the right-hand limits and left-hand limits of $u^{(n-2)}(t)$ and $u^{(n-1)}(t)$ at $t = t_k,\ k = 1, 2, \ldots, l$, respectively.

Throughout this paper we assume that following conditions hold:

$(C1)$ $a, b, c, d \in [0, +\infty)$ with $ac + ad + bc > 0$,

$(C2)$ $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty),\ q \in C([0,1], \mathbb{R}_+)$,

$(C3)$ $g_1,\ g_2 \in C([0,1], \mathbb{R}_+)$,

$(C4)$ $I_k \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $J_k \in C(\mathbb{R}_+, \mathbb{R}_+)$ are bounded functions such that $(c(1-t_k)+d)J_k(u(t_k)) > cI_k(u(t_k)),\ k = 1, 2, \ldots, l$.

By using the fixed point theorem of cone expansion and compression [14], we get the existence of at least one positive solution for the impulsive BVP (1.1).

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our main result.

2 Preliminaries

In this section, we present auxiliary lemmas which will be used later.

Throughout the rest of this paper, we assume that the points of impulse t_k are right dense for each $k = 1, 2, \ldots, l$. Let $J' = J \setminus \{t_1, t_2, \ldots, t_l\}$.

Set

$$
\begin{aligned}
PC(J) = \big\{&u : u \text{ is a map from } J \text{ into } \mathbb{R}_+ \text{ such that } u(t) \text{ is continuous at} \\
&t \neq t_k,\ u(t_k^+) \text{ and } u(t_k^-) \text{ exist and } u(t_k^-) = u(t_k),\ k = 1, 2, \ldots, l\big\}, \\
PC^1(J) = \big\{&u : u \text{ is a map from } J \text{ into } \mathbb{R}_+ \text{ such that } u'(t) \text{ is continuous at} \\
&t \neq t_k,\ u'(t_k^+) \text{ and } u'(t_k^-) \text{ exist and } u'(t_k^-) = u'(t_k),\ k = 1, 2, \ldots, l\big\}, \\
PC^{n-1}(J) = \big\{&u : u \text{ is a map from } J \text{ into } \mathbb{R}_+ \text{ such that } u^{(n-1)}(t) \text{ is} \\
&\text{continuous at } t \neq t_k,\ u^{(n-1)}(t_k^+) \text{ and } u^{(n-1)}(t_k^-) \text{ exist and} \\
&u^{(n-1)}(t_k^-) = u^{(n-1)}(t_k),\ k = 1, 2, \ldots, l\big\}.
\end{aligned}
$$

Obviously, $PC(J),\ PC^1(J)$ and $PC^{n-1}(J)$ are Banach spaces with the norms

$$
\|u\|_{PC} = \sup_{t \in J} \|u(t)\|,\ \|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\},
$$
$$
\|u\|_{PC^{n-1}} = \max\left\{\|u\|_{PC}, \|u'\|_{PC}, \ldots, \|u^{(n-1)}\|_{PC}\right\}.
$$

We shall reduce problem (1.1) to an integral equation in $PC^1(J)$. To this goal, firstly by means of the transformation

$$u^{(n-2)}(t) = y(t), \tag{2.1}$$

and the boundary conditions $u^{(j)}(0) = 0,\ j = 1, 2, \dots, n-3$, one can obtain that

$$u^{(j)}(t) = \int_0^t \frac{(t-s)^{n-3-j}}{(n-3-j)!} y(s)ds,\ j = 0, 1, 2, \dots, n-3. \tag{2.2}$$

Thus, under the transformation (2.1), we obtain the following BVP,

$$\begin{cases} -y''(t) = q(t)f\left(t, u(t)\right) = 0,\ t \in J,\ t \neq t_k, \\ \Delta y|_{t=t_k} = I_k(y(t_k)),\ k = 1, 2, \dots, l, \\ \Delta y'|_{t=t_k} = -J_k(y(t_k)), \\ ay(0) - by'(0) = \displaystyle\int_0^1 g_1(s)y(s)ds, \\ cy(1) + dy'(1) = \displaystyle\int_0^1 g_2(s)y(s)ds. \end{cases} \tag{2.3}$$

Note that, the nth order BVP (1.1) has a solution if and only if the second order BVP (2.3) has a solution.

Set

$$\triangle := \begin{vmatrix} -\displaystyle\int_0^1 g_1(s)\left(b + as\right)ds & \rho - \displaystyle\int_0^1 g_1(s)\left(d + c(1-s)\right)ds \\ \rho - \displaystyle\int_0^1 g_2(s)\left(b + as\right)ds & -\displaystyle\int_0^1 g_2(s)\left(d + c(1-s)\right)ds \end{vmatrix}, \tag{2.4}$$

and

$$\rho := ad + ac + bc. \tag{2.5}$$

Lemma 2.1. *Let $(C1)$-$(C4)$ hold. Assume that $\triangle \neq 0$. Then $y(t)$ is a solution of the BVP (2.3) if and only if $y(t)$ is a solution of the following integral equation*

$$\begin{aligned} y(t) \quad = \quad & \int_0^1 G\left(t, s\right) q(s) f\left(s, u(s)\right) ds + \sum_{k=1}^{l} W_k\left(t, t_k\right) \\ & + A(f)(b + at) + B(f)(d + c(1-t)), \end{aligned} \tag{2.6}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(b+as)(d+c(1-t)), & s\le t,\\ (b+at)(d+c(1-s)), & t\le s,\end{cases} \tag{2.7}$$

$$W_k(t,t_k)=\frac{1}{\rho}\begin{cases}(b+at)\left(-cI_k(y(t_k))+(d+c(1-t_k))J_k(y(t_k))\right), & t<t_k,\\ (d+c(1-t))\left(aI_k(y(t_k))+(b+at_k)J_k(y(t_k))\right), & t_k\le t,\end{cases} \tag{2.8}$$

$$A(f)=\frac{1}{\triangle}\begin{vmatrix}\int_0^1 g_1(s)H(s)ds, & \rho-\int_0^1 g_1(s)(d+c(1-s))ds\\ \int_0^1 g_2(s)H(s)ds & -\int_0^1 g_2(s)(d+c(1-s))ds\end{vmatrix}, \tag{2.9}$$

$$B(f)=\frac{1}{\triangle}\begin{vmatrix}-\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s)H(s)ds\\ \rho-\int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s)H(s)ds\end{vmatrix}, \tag{2.10}$$

and

$$H(s)=\int_0^1 G(s,r)\,q(r)f(r,u(r))\,dr+\sum_{k=1}^{l}W_k(s,t_k). \tag{2.11}$$

Proof. Let y satisfies the integral equation (2.6), then we will show that y is a solution of the impulsive BVP (2.3). Since y satisfies equation (2.6), then we have

$$\begin{aligned} y(t) &= \int_0^1 G(t,s)\,q(s)f(s,u(s))\,ds+\sum_{k=1}^{l}W_k(t,t_k)\\ &\quad +A(f)(b+at)+B(f)(d+c(1-t)), \end{aligned}$$

i.e.,

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\rho}(b+as)(d+c(1-t))q(s)f\left(s,u(s)\right)ds \\
&\quad + \int_t^1 \frac{1}{\rho}(b+at)(d+c(1-s))q(s)f\left(s,u(s)\right)ds \\
&\quad + \frac{1}{\rho}\sum_{0<t_k<t}(d+c(1-t))\left(aI_k(y(t_k))+(b+at_k)J_k\left(y(t_k)\right)\right) \\
&\quad + \frac{1}{\rho}\sum_{t<t_k<1}(b+at)\left(-cI_k(y(t_k))+(d+c(1-t_k))J_k\left(y(t_k)\right)\right) \\
&\quad + A(f)(b+at)+B(f)(d+c(1-t)), \\
y'(t) &= -\int_0^t \frac{c}{\rho}(b+as)q(s)f\left(s,u(s)\right)ds \\
&\quad + \int_t^1 \frac{a}{\rho}(d+c(1-s))q(s)f\left(s,u(s)\right)ds \\
&\quad - \frac{1}{\rho}\sum_{0<t_k<t}c\left(aI_k(y(t_k))+(b+at_k)J_k\left(y(t_k)\right)\right) \\
&\quad + \frac{1}{\rho}\sum_{t<t_k<1}a\left(-cI_k(y(t_k))+(d+c(1-t_k))J_k\left(y(t_k)\right)\right) \\
&\quad + A(f)a-B(f)c.
\end{aligned}$$

So that

$$\begin{aligned}
y''(t) &= \frac{1}{\rho}\left(-c(b+at)-a(d+c(1-t))\right)q(t)f\left(t,u(t)\right) \\
&= \frac{1}{\rho}\left(-(ad+ac+bc)\right)q(t)f\left(t,u(t)\right) = -q(t)f\left(t,u(t)\right),
\end{aligned}$$

$$y''(t)+q(t)f(t,u(t))=0.$$

Since

$$\begin{aligned}
y(0) &= \int_0^1 \frac{b}{\rho}(d+c(1-s))q(s)f\left(s,u(s)\right)ds \\
&\quad + \frac{1}{\rho}\sum_{k=1}^{l} b\left(-cI_k(y(t_k))+(d+c(1-t_k))J_k\left(y(t_k)\right)\right) \\
&\quad + A(f)b+B(f)(d+c),
\end{aligned}$$

$$\begin{aligned} y'(0) &= \int_0^1 \frac{a}{\rho}(d+c(1-s))q(s)f\left(s,u(s)\right)ds + A(f)a - B(f)c \\ &+ \frac{1}{\rho}\sum_{k=1}^{l} a\left(-cI_k(y(t_k)) + (d+c(1-t_k))J_k\left(y(t_k)\right)\right), \end{aligned}$$

we have that

$$\begin{aligned} ay(0)-by'(0) &= B(f)\left(ad+ac+bc\right) \\ &= \int_0^1 g_1(s)\Bigg[\int_0^1 G\left(s,r\right)q(r)f\left(r,u(r)\right)dr + \sum_{k=1}^{l} W_k\left(s,t_k\right) \\ &\quad + A(f)(b+as) + B(f)(d+c(1-s))\Bigg]ds. \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} y(1) &= \int_0^1 \frac{d}{\rho}(b+a(s))q(s)f\left(s,u(s)\right)ds + A(f)(b+a) + B(f)d \\ &+ \frac{1}{\rho}\sum_{k=1}^{l} d\left(aI_k(y(t_k)) + (b+at_k)J_k\left(y(t_k)\right)\right), \\ y'(1) &= -\int_0^1 \frac{c}{\rho}(b+a(s))q(s)f\left(s,u(s)\right)ds \\ &- \frac{1}{\rho}\sum_{k=1}^{l} c\left(aI_k(y(t_k)) + (b+at_k)J_k\left(y(t_k)\right)\right) + A(f)a - B(f)c, \end{aligned}$$

we have that

$$\begin{aligned} cy(1)+dy'(1) &= A(f)\left(ad+ac+bc\right) \\ &= \int_0^1 g_2(s)\Bigg[\int_0^1 G\left(s,r\right)q(r)f\left(r,u(r)\right)dr + \sum_{k=1}^{l} W_k\left(s,t_k\right) \\ &\quad + A(f)(b+as) + B(f)(d+c(1-s))\Bigg]ds. \end{aligned} \tag{2.13}$$

From (2.6), (2.12) and (2.13), we get that

$$\begin{cases} \left[-\int_0^1 g_1(s)(b+as)ds\right]A(f)+\left[\rho-\int_0^1 g_1(s)(d+c(1-s))ds\right]B(f) \\ =\int_0^1 g_1(s)H(s)ds, \\ \left[\rho-\int_0^1 g_2(s)(b+as)ds\right]A(f)+\left[-\int_0^1 g_2(s)(d+c(1-s))ds\right]B(f) \\ =\int_0^1 g_2(s)H(s)ds, \end{cases}$$

which implies that $A(f)$ and $B(f)$ satisfy (2.9) and (2.10), respectively. Then $y(t)$ satisfies all the conditions of (2.3), hence $y(t)$ is a solution of (2.3).

Conversely, if $y(t)$ is a solution of the BVP (2.3), by integrating one can easily show that $y(t)$ is in the form (2.6).

Lemma 2.2. *Let* $(C1)$-$(C4)$ *hold. Assume*

$(C5)$ $\triangle<0,\ \rho-\int_0^1 g_2(s)(b+as)ds>0,\ a-\int_0^1 g_1(s)ds>0.$

Then for $y\in PC^1(J)\cap C^2(J')$ *with* $f,q\geq 0$*, the solution* y *of the problem* (2.3) *satisfies*

$$y(t)\geq 0 \text{ for } t\in J.$$

Proof. It is an immediate consequence of the facts that $G\geq 0$ on $[0,1]\times[0,1]$ and $A(f)\geq 0,\ B(f)\geq 0$.

Lemma 2.3. *Let* $(C1)$-$(C5)$ *hold. Assume that*

$(C6)$ $c-\int_0^1 g_2(s)ds<0.$

Then the solution $y\in PC^1(J)\cap C^2(J')$ *of the problem* (2.3) *satisfies* $y'(t)\geq 0$ *for* $t\in J$.

Proof. Assume that the inequality $y'(t)<0$ holds. Since $y'(t)$ is nonincreasing on J, one can verify that

$$y'(1)\leq y'(t),\ t\in J.$$

From the boundary conditions of the problem (2.3), we have

$$-\frac{c}{d}y(1)+\frac{1}{d}\int_0^1 g_2(s)y(s)ds\leq y'(t)<0.$$

The last inequality yields

$$-cy(1) + \int_0^1 g_2(s)y(s)ds < 0.$$

Therefore, we obtain that

$$\int_0^1 g_2(s)y(1)ds < \int_0^1 g_2(s)y(s)ds < cy(1),$$

i.e.,

$$\left(c - \int_0^1 g_2(s)ds\right) y(1) > 0.$$

According to Lemma 2.2, we have that $y(1) \geq 0$. So, $c - \int_0^1 g_2(s)ds > 0$. However, this contradicts to condition $(C6)$. Consequently, $y'(t) \geq 0$ for $t \in J$.

Note that by Lemmas 2.1 and 2.2, we know that if $(C1)-(C5)$ are satisfied, then the solutions of the BVPs (1.1) and (2.3) are both positive. Therefore, we only need to deal with the existence of the positive solutions of (2.3).

To establish the existence of positive solutions in $PC^1(J) \cap C^2(J')$, we construct a cone $\mathcal{P}$ in $PC(J)$ by

$$\begin{aligned}\mathcal{P} \quad = \quad & \Big\{y \in PC(J) : y \text{ is nonnegative, nondecreasing on } J \text{ and } y' \text{ is} \\ & \text{nonincreasing on } J,\ ay(0) - by'(0) = \int_0^1 g_1(s)y(s)ds\Big\}.\end{aligned}$$

Obviously, $\mathcal{P}$ is a cone in $PC(J)$. We note that, for each $y \in \mathcal{P}$, $\|y\|_{PC} = \sup_{t\in J} |y(t)| = y(1)$.

Lemma 2.4. *The Green's function $G(t, s)$ defined by (2.7) satisfies*

$$\min_{t\in J} G(t, s) \geq \gamma G(s, s),\ \forall s \in J, \tag{2.14}$$

where $\gamma = \min\left\{\frac{d}{d+c}, \frac{b}{b+a}\right\}$.

Proof. Applying (2.7), we have that for $t,\ s \in J$

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{b+at}{b+as}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{d+c(1-t)}{d+c(1-s)}, & 0 \le s \le t \le 1, \end{cases}$$
$$\ge \begin{cases} \dfrac{b}{b+a}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{d}{d+c}, & 0 \le s \le t \le 1, \end{cases} \ge \gamma.$$

Thus, we have

$$\min_{t\in J} G(t,s) \ge \gamma G(s,s).$$

Lemma 2.5. *Let $y \in \mathcal{P}$. Then $\min\limits_{t\in J} y(t) \ge \Gamma \|y\|_{PC}$, where*

$$\Gamma = \frac{\displaystyle\int_0^1 g_1(s)sds}{a - \displaystyle\int_0^1 g_1(s)(1-s)ds}.$$

Proof. For $y \in \mathcal{P}$, since $y'(t)$ is nonincreasing on J one arrives at

$$\frac{y(t)-y(0)}{t} \ge \frac{y(1)-y(0)}{1},$$

i.e., $y(t) \ge ty(1) + (1-t)y(0)$. Hence,

$$\int_0^1 g_1(s)y(s)ds \ge \int_0^1 g_1(s)sy(1)ds + \int_0^1 g_1(s)(1-s)y(0)ds.$$

By $ay(0) - by'(0) = \displaystyle\int_0^1 g_1(s)y(s)ds$, we get

$$y(0) \ge \frac{\displaystyle\int_0^1 g_1(s)sds}{a - \displaystyle\int_0^1 g_1(s)(1-s)ds} y(1).$$

So, the proof of Lemma 2.5 is completed.

Now define an operator T by

$$(Ty)(t)=\int_0^1 G(t,s)\,F(s,y(s))q(s)ds+\sum_{k=1}^{l} W_k(t,t_k)$$

$$+A(f)(b+at)+B(f)(d+c(1-t)), \tag{2.15}$$

where $F(t,y(t))=f\left(t,\int_0^t \frac{(t-r)^{n-3}}{(n-3)!}y(r)dr\right)$, $G,\ W_k,\ A(f)$ and $B(f)$ are defined as in (2.7), (2.8), (2.9) and (2.10), respectively.

Lemma 2.6. *(i) If $u\in PC^{n-1}(J)\cap C^n(J')$ is a positive solution of BVP (1.1), then $y(t)=u^{(n-2)}(t)\in PC^1(J)\cap C^2(J')$ is a fixed point of T;*

(ii) If $y\in PC^1(J)\cap C^2(J')$ is a fixed point of T, then $u(t)=\int_0^t \frac{(t-s)^{n-3}}{(n-3)!}y(s)ds\in PC^{n-1}(J)\cap C^n(J')$ is a positive solution of BVP (1.1).

Proof. From Lemmas 2.1 and 2.2, the proof can be easily seen.

Lemma 2.7. *Let $(C1)$-$(C6)$ hold. Then $T:\mathcal{P}\to\mathcal{P}$ is completely continuous.*

Proof. For all $y\in\mathcal{P}$, Lemmas 2.1, 2.2, 2.3 and the definition of T, we have

$$(Ty)(t)\geq 0,\ (Ty)'(t)\geq 0,\ \text{and } (Ty)'(t) \text{ is nonincreasing on } J,$$
$$a(Ty)(0)-b(Ty)'(0)=\int_0^1 g_1(s)(Ty)(s)ds.$$

Then $Ty\in\mathcal{P}$. So T is an operator from $\mathcal{P}$ to $\mathcal{P}$. By Arzela-Ascoli theorem, we can prove that operator T is completely continuous.

The following fixed point theorem is fundamental and important to the proof of the existence of at least one positive solution in the next section.

Lemma 2.8. *([14]) Let Ω_1 and Ω_2 be two bounded open sets in Banach space E, such that $0\in\Omega_1$ and $\bar{\Omega}_1\subset\Omega_2$. Let $\mathcal{P}$ be a cone in E and let operator $T:\mathcal{P}\cap\left(\bar{\Omega}_2\setminus\Omega_1\right)\to\mathcal{P}$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) $Ty\ngeq y,\ \forall y\in\mathcal{P}\cap\partial\Omega_1;\ Ty\nleq y,\ \forall y\in\mathcal{P}\cap\partial\Omega_2$;

(ii) $Ty\nleq y,\ \forall y\in\mathcal{P}\cap\partial\Omega_1;\ Ty\ngeq y,\ \forall y\in\mathcal{P}\cap\partial\Omega_2$.

Then, T has at least one fixed point in $\mathcal{P}\cap\left(\bar{\Omega}_2\setminus\Omega_1\right)$.

3 Main Results

We are now ready to apply the fixed point theorem of cone expansion and compression [14] to the operator T in order to get sufficient conditions for the existence of at least one positive solution to the problem (1.1).

Theorem 3.1. *Assume that the conditions* $(C1)$-$(C6)$ *are satisfied. In addition, letting* f, I_k *and* J_k *satisfy the following conditions:*

$(C7)$ *There exists* $\rho_1 > 0$ *such that* $F(t, y) \geq L_1 y$ *for all* $(t, y) \in J \times [0, \rho_1]$, *where* $L_1 > \left[\gamma\Gamma \int_0^1 G(s,s)q(s)ds\right]^{-1}$;

$(C8)$ *There exists* $\rho_2 > \rho_1 > 0$ *such that* $F(t, y) \leq L_2\rho_2$, $I_k(y) \leq \epsilon_k\rho_2$, $J_k(y) \leq \bar{\epsilon}_k\rho_2$, $k = 1, 2, \dots, l, \forall\ 0 \leq y \leq \rho_2$, $t \in J$ *where* ϵ_k, $\bar{\epsilon}_k \geq 0$ *and* $L_2 \geq 0$ *satisfies*

$$L_2 \int_0^1 G(s,s)\, q(s) ds + \frac{1}{\rho}(c+d)(2a+b)\sum_{k=1}^{l} \eta_k + A(b+a) + B(d+c) < 1.$$

Here,

$$A = \frac{1}{\triangle}\begin{vmatrix} \int_0^1 g_1(s)Hds, & \rho - \int_0^1 g_1(s)(d+c(1-s))ds \\ \int_0^1 g_2(s)Hds & -\int_0^1 g_2(s)(d+c(1-s))ds \end{vmatrix},$$

$$B = \frac{1}{\triangle}\begin{vmatrix} -\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s)Hds \\ \rho - \int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s)Hds \end{vmatrix},$$

$$\eta_k = \max\{\epsilon_k, \bar{\epsilon}_k\},$$

and

$$H = L_2 \int_0^1 G(r,r)q(r)dr + \frac{1}{\rho}(c+d)(2a+b)\sum_{k=1}^{l} \eta_k.$$

Then the BVP (1.1) *has at least one positive solution.*

Proof. Considering $(C7)$, we show that

$$Ty \nleq y,\ y \in \mathcal{P},\ \|y\|_{PC} = \rho_1. \tag{3.1}$$

In fact, if there exists $y_0 \in \mathcal{P},\ \|y_0\|_{PC} = \rho_1$ such that $Ty_0 \leq y_0$, then with Lemmas 2.4, 2.5, we have

$$\begin{aligned} y_0(t) &\geq (Ty_0)(t) \geq \int_0^1 G(t,s)\, F(s, y_0(s))\, q(s)ds \\ &\geq \gamma \int_0^1 G(s,s)\, F(s, y_0(s))\, q(s)ds \\ &\geq \gamma L_1 \int_0^1 G(s,s)\, y_0(s) q(s)ds \\ &\geq \gamma \Gamma L_1 \int_0^1 G(s,s)\, q(s)ds\, \|y_0\|_{PC} \\ &\geq \gamma \Gamma L_1 \int_0^1 G(s,s)\, q(s)ds\ \rho_1 \\ &> \rho_1. \end{aligned}$$

Hence, $\rho_1 = \|y_0\|_{PC} \geq \|Ty_0\|_{PC} > \rho_1 = \|y_0\|_{PC}$, which is a contradiction. So, (3.1) holds.

Now, we prove that

$$Ty \ngeq y,\ y \in \mathcal{P},\ \|y\|_{PC} = \rho_2. \tag{3.2}$$

In fact, if there exists $y_1 \in \mathcal{P},\ \|y_1\|_{PC} = \rho_2$ such that $Ty_1 \geq y_1$, then we have

$$\begin{aligned} y_1(t) &\leq (Ty_1)(t) \\ &= \int_0^1 G(t,s)\, F(s, y_1(s)) q(s)ds + \sum_{k=1}^{l} W_k(t, t_k) + A(f)(b+at) \\ &\quad + B(f)(d + c(1-t)) \\ &\leq \Bigg(L_2 \int_0^1 G(s,s)\, q(s)ds + \frac{1}{\rho}(c+d)(2a+b) \sum_{k=1}^{l} \eta_k + A(b+a) \\ &\quad + B(d+c) \Bigg) \rho_2 \\ &< \rho_2, \end{aligned}$$

that is, $\|y_1\|_{PC} < \|y_1\|_{PC}$, which is a contradiction. Hence, (3.2) holds.

Applying (ii) of Lemma 2.8 to (3.1) and (3.2) yields that T has a fixed point $y^* \in \bar{\mathcal{P}}_{\rho_1,\rho_2} = \{y^* \in \mathcal{P} : \rho_1 \leq \|y^*\|_{PC} \leq \rho_2\}$. Thus it follows that BVP

(2.3) has one positive solution y^* with $\rho_1 \leq \|y^*\|_{PC} \leq \rho_2$. Then the nth-order BVP (1.1) has at least one positive solution

$$u(t) = \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y^*(r)dr.$$

Our last result corresponds to the case when problem (1.1) has no positive solution. Write

$$\Lambda = \left[\gamma\Gamma \int_0^1 G(s,s)\, q(s)ds\right]^{-1} \tag{3.3}$$

Theorem 3.2. *Assume that* $(C1)$-$(C6)$ *hold. If* $F(t,y) > \Lambda y,\ t \in J,\ y > 0$, *then problem* (1.1) *has no positive solution.*

Proof. Assume to the contrary that problem (1.1) has a positive solution, that is, T has a fixed point y. Then $y \in \mathcal{P},\ y > 0$ for $t \in J$, and

$$\begin{aligned} y(t) &\geq \int_0^1 G(t,s)\, F(s, y(s))\, q(s)ds \\ &> \Lambda\gamma \int_0^1 G(s,s)\, y(s) q(s)ds \\ &\geq \Lambda\gamma\Gamma \int_0^1 G(s,s)\, q(s)ds\, \|y\|_{PC} \\ &= \|y\|_{PC}, \end{aligned}$$

which is a contradiction, and this completes the proof.

To illustrate how our main results can be used in practice we present an example.

4 Example

Example 4.1 Consider the following problem

$$\begin{cases} u^{(6)}(t) + 90f(t, u(t)) = 0, \ t \in J, \ t \neq \dfrac{1}{2}, \\ \Delta u^{(4)}|_{t=\frac{1}{2}} = I_1\left(u^{(4)}\left(\dfrac{1}{2}\right)\right), \\ \Delta u^{(5)}|_{t=\frac{1}{2}} = -J_1\left(u^{(4)}\left(\dfrac{1}{2}\right)\right), \\ 2u^{(4)}(0) - u^{(5)}(0) = \displaystyle\int_0^1 u^{(4)}(s)ds, \\ \dfrac{1}{3}u^{(4)}(1) + 2u^{(5)}(1) = \displaystyle\int_0^1 u^{(4)}(s)ds, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \end{cases} \tag{4.1}$$

where

$$F(t, y) = 0.0006y + 0.15$$

$$I_1(y) = J_1(y) = \frac{y}{250}, \ y \geq 0.$$

By simple calculation, we get $\rho = 5, \ \triangle = -\dfrac{25}{6}, \ \gamma = \Gamma = \dfrac{1}{3}$, and

$$G(t, s) = \frac{1}{15} \begin{cases} (1 + 2s)(7 - t), & s \leq t, \\ (1 + 2t)(7 - s), & t \leq s. \end{cases}$$

Taking $\epsilon_1 = \dfrac{1}{200}, \ \bar{\epsilon}_1 = \dfrac{1}{210}, \ L_1 = 0.12, \ L_2 = 0.001, \ \rho_1 = 1, \rho_2 = 1000$, we can obtain that $\eta_1 = \dfrac{1}{200}, \ A = B = \dfrac{127386}{15000}$. It is clear that $(C1)$-$(C6)$ are satisfied. Next, we show that $(C7)$ and $(C8)$ are also satisfied.

For $(t, y) \in J \times [0, 1]$, we have $F(t, y) \geq 0.15 \geq 0.12 \geq L_1 y$.
So $(C7)$ is satisfied.

For $(t, y) \in [0, 1] \times [0, 1000]$, we get $F(t, y) \leq 0.75 \leq L_2\rho_2 = 1$, $I_1(y) \leq 4 \leq \epsilon_1\rho_2 = 5$ and $J_1(y) \leq 4 \leq \bar{\epsilon}_1\rho_2 = \dfrac{1000}{210}$. Hence $(C8)$ holds.

Then all conditions of Theorem 3.1 hold. Therefore, BVP (4.1) has at least one positive solution.

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