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Iterative calculus on tangent floors

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Abstract

Tangent fibrations generate a "multi-floored tower", while raising from one of its floors to the next one, one practically reiterates the previously performed actions. In this way, the "tower" admits a ladder-shaped structure. Raising to the first floors suffices for iteratively performing the subsequent steps. The paper mainly studies the tangent functor. We describe the structure of multiple vector bundle which naturally appears on the floors, tangent maps, sector-forms, the lift of vector fields to upper floors. Further, we show how tangent groups of Lie groups lead to gauge theory, and explain in this context the meaning of covariant differentiation. Finally, we will point out within the floors special subbundles – the osculating bundles, which play an essential role in classical theories.

1 The tangent functor

The tangent functor T is a correspondence which attaches to a smooth manifold M, its tangent bundle TM (its *first floor*) and to a smooth mapping φ, its tangent map $T\varphi$. By applying k times the functor T to the manifold M, one obtains its k-th tangent space T^kM (k*-th floor of the manifold* M) and by applying it to the mapping f – the k-th tangent map T^kf - understood as a morphism between the k-th floors,

$$\left\{ \begin{array}{l} M \\ f \end{array} \right. \quad \overset{T}{\rightsquigarrow} \quad \left\{ \begin{array}{l} TM \\ Tf \end{array} \right. \quad \dots \quad \overset{T^k}{\rightsquigarrow} \quad \left\{ \begin{array}{l} T^kM \\ T^kf\,. \end{array} \right.$$

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1.1 Floors and projections

The natural projections from all the floors onto the previous floors π_1, π_2, π_3, ... and their tangent maps define on the k-th floor T^kM the structure of a k-fold vector bundle with the projections $\rho_1, \rho_2, \ldots, \rho_k$ on the floor $T^{k-1}M$,

$$\boxed{\rho_i \doteq T^{k-i}\pi_i : \ T^kM \longrightarrow T^{k-1}M, \ i = 1, 2, \ldots, k.}$$

We notice that the sequences $\rho_1, \rho_2, \ldots$ are different for different floors. From the first floor TM to the manifold M, we have one projection $\rho_1 = \pi_1$, while from the second floor to the first one, two projections $\rho_1 = T\pi_1$, $\rho_2 = \pi_2$, from the third one to the second one – three projections

$$\rho_1 = T^2\pi_1, \ \rho_2 = T\pi_2, \ \rho_3 = \pi_3 \,, \text{ and so on.}$$

$$\begin{array}{cccccccl} M & \xleftarrow{\pi_1} & TM & \xleftarrow{T\pi_1} & T^2M & \xleftarrow{T^2\pi_1} & T^3M & \| \ \rho_1 \\ & & TM & \xleftarrow{\pi_2} & T^2M & \xleftarrow{T\pi_2} & T^3M & \| \ \rho_2 \\ & & & & T^2M & \xleftarrow{\pi_3} & T^3M & \| \ \rho_3 \end{array}$$

$$\begin{aligned} k = 1 : &\quad \rho_1 = \pi_1 \,, \\ k = 2 : &\quad \rho_1 = T\pi_1 \,, \, \rho_2 = \pi_2 \,, \\ k = 3 : &\quad \rho_1 = T^2\pi_1 \,, \rho_2 = T\pi_2 \,, \, \rho_3 = \pi_3. \end{aligned}$$

For $k = 2$ and $k = 3$, we have indicated the projections on the corresponding commutative diagrams. For $k = 2$, the diagram has the shape of a rhombus:

$$\begin{tikzcd} & T^2M \arrow[dl, "\rho_1 = T\pi_1"'] \arrow[dr, "\rho_2 = \pi_2"] & \\ TM \arrow[dr, "\pi_1"'] & & TM \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

where there holds the equality[1]:

$$\pi_1\rho_1 = \pi_1\rho_2 \,.$$

For $k = 3$, the diagram has the shape of a 3-dimensional cube:

[1] We shall further omit the sign " $\circ$ " generally used for the composition of mappings; e.g., $\pi\rho_i \doteq \pi \circ \rho_i$.

$$
\begin{tikzcd}
 & & T^3M \arrow[dll, "\rho_1=T^2\pi_1"'] \arrow[d, dashed, "\rho_3=\pi_3"] \arrow[drr, "\rho_2=T\pi_2"] & & \\
T^2M \arrow[d, "\pi_2"'] \arrow[drr, "T\pi_1"] & & T^2M \arrow[dll, dashed, "T\pi_1"'] \arrow[drr, dashed, "\pi_2"] & & T^2M \arrow[dll, "T\pi_1"'] \arrow[d, "\pi_2"] \\
TM \arrow[drr, "\pi_1"] & & TM \arrow[d, "\pi_1"] & & TM \arrow[dll, "\pi_1"'] \\
 & & M & &
\end{tikzcd}
$$

where the following equalities hold:

$$
\begin{gathered}
\pi_1\pi_2\rho_1 \;=\; \pi_1\pi_2\rho_2 \;=\; \pi_1\pi_2\rho_3\,, \\
T\pi_1\rho_1 = T\pi_1\rho_2,\; T\pi_1\rho_3 = \pi_2\rho_1, \pi_2\rho_3 = \pi_2\rho_2,
\end{gathered}
$$

etc. This can be generalized, and in the general case T^kM, the diagram becomes a k-dimensional cube.

1.2 Tangent maps and differentials

The tangent map of a mapping f is defined as a pair

$$\boxed{Tf = (f \circ \pi, f_1)}\,,$$

where f is the mapping under discussion,

$$f : M_1 \;\longrightarrow\; M_2 : u \mapsto v = f(u),$$

and

$$f_1 \doteq df \;:\; T_uM_1 \longrightarrow T_vM_2 \;:\; du \;\mapsto\; dv = df \circ du,$$

is its differential, understood as a linear mapping between the tangent spaces[2]. The mappings f, f_1, together with the projection $\pi : TM_1 \to M_1$ are indicated in the following diagram by arrows:

$$
\begin{array}{ccc}
T_uM_1 & \xrightarrow{f_1} & T_vM_2 \\
{\scriptstyle\pi}\downarrow & & \downarrow \\
M_1 & \xrightarrow{f} & M_2
\end{array}
\qquad\qquad Tf = (f \circ \pi, f_1).
$$

[2] In local coordinates, the differential df is defined by the Jacobian matrix. The differentials du and dv, as column matrices, will be further respectively identified with the components of the vectors u_1 and v_1.

One can also discuss about higher order tangent maps, e.g., for a function or a system of functions $\varphi : T^{k-1}M \to \mathbb{R}^p$, we deal with the tangent maps $T\varphi$ and $T^2\varphi$:

$$\begin{array}{ccc} T^{k+1}M & \xrightarrow{\varphi_{k+1}} & T^2\mathbb{R}^p \\ {\scriptstyle \pi_{k+1}}\downarrow & & \downarrow \\ T^kM & \xrightarrow{\varphi_k} & T\mathbb{R}^p \\ {\scriptstyle \pi_k}\downarrow & & \downarrow \\ T^{k-1}M & \xrightarrow{\varphi} & \mathbb{R}^p \end{array} \qquad \begin{array}{l} T^2\varphi = (T\varphi \circ \pi_{k+1}, \varphi_{k+1}), \\ \\ T\varphi = (\varphi \circ \pi_k, \varphi_k). \end{array}$$

We further introduce a convenient *indexing system.* A usual scalar function $f : M \to \mathbb{R}$ admits, on distinct floors, distinct differentials:

$$f_1 = df, \;\; f_2 \doteq d(f \circ \pi_1), \;\; f_3 \doteq d(f \circ \pi_1\pi_2), \ldots$$

For higher order differentials, we shall use the following multi-index notation:

$$f_{12} \doteq d^2 f, \;\; f_{13} \doteq d(df \circ \pi_2), \;\; f_{23} \doteq d^2(f \circ \pi_1), \;\; f_{123} \doteq d^{\,3} f, \ldots$$

With these notations, for a given function f, we define the tangent maps Tf, T^2f and T^3f:

$$\begin{aligned}
k = 1 \;&:\; Tf = (f \circ \rho_1, f_1) \doteq \boxed{(f, f_1)}, \\
k = 2 \;&:\; T^2 f = \big((f \circ \pi_1, f_1) \circ \pi_2, (f \circ \pi_1, f_1)_2\big) = \\
&\qquad = (f \circ \pi_1\pi_2, f_1 \circ \pi_2, f_2 \circ T\pi_1, f_{12}) = \\
&\qquad = (f \circ \pi_1\pi_2 = f \circ \pi_2\rho_1, f_1 \circ \rho_2, f_2 \circ \rho_1, f_{12}) \doteq \boxed{(f, f_1, f_2, f_{12}\,)}, \\
k = 3 \;&:\; T^3 f = \big((f \circ \pi_1\pi_2, f_1 \circ \pi_2, f_2 \circ T\pi_1, f_{12}) \circ \pi_3, \\
&\qquad (f \circ \pi_1\pi_2, f_1 \circ \pi_2, f_2 \circ T\pi_1, f_{12})_3\big) = \\
&\qquad = (f \circ \pi_1\pi_2\rho_1 = f \circ \pi_1\pi_2\rho_2 = f \circ \pi_1\pi_2\rho_3, \\
&\qquad f_1 \circ \pi_2\rho_3 = f_1 \circ \pi_2\rho_2, f_2 \circ T\pi_1\rho_3 = f_2 \circ \pi_2\rho_1, f_{12} \circ \rho_3, \\
&\qquad f_3 \circ T\pi_1\rho_2 = f_3 \circ T\pi_1\rho_1, f_{13} \circ \rho_2, f_{23} \circ \rho_1, f_{123}) \doteq \\
&\qquad \doteq \boxed{(f, f_1, f_2, f_{12}, f_3, f_{13}, f_{23}, f_{123})}.
\end{aligned}$$

We shall use the following rule: *for denoting the tangent mapping T^kf, we write the symbols which define $T^{k-1}f$, and add the index k* – as a result, we obtain $2{\cdot}2^{k-1} = 2^k$ symbols in the writing of the mapping T^kf. Thus, symbols with the index i ($i = 1, 2, , \ldots, k$) will be related to the fiber of the bundle ρ_i and the other symbols, to the base of this bundle.

Remarks. The writing $f \circ \rho_1 \doteq f$ means that the function f is raised from the manifold M to the floor TM. The writing $f \circ \pi_1\pi_2\rho_1 = f \circ \pi_1\pi_2\rho_2 = f \circ \pi_1\pi_2\rho_3$ means that the symbol f is related to the common base of the bundles ρ_1, ρ_2 and ρ_3. The notation $f_2 \circ T\pi_1\rho_3 = f_2 \circ \pi_2\rho_1$ tells us that the symbol f_2 corresponds to the bases of the bundles ρ_1 and ρ_3 and to the fiber of the bundle ρ_2.

1.3 Coordinates and sector-forms

The coordinates on neighborhoods

$$U \xleftarrow{\pi_1} TU \xleftarrow{\pi_2} T^2U \ldots \xleftarrow{\pi_k} T^kU \ldots, \text{ where } \pi_k(T^kU) = T^{k-1}U, \;\; k = 1, 2, \ldots,$$

are automatically defined once the coordinate functions (u^i) are defined on the neighborhood U. Namely, if a mapping $\omega : U \to \mathbb{R}^n$ defines the coordinate functions on a neighborhood U, then on the neighborhood T^kU, coordinates are defined by the k-th tangent map $T^k\omega$,

$$\begin{aligned} \omega : U \; &\rightsquigarrow (u^i), \\ T\omega : TU &\rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}), \\ T^2\omega : T^2U &\rightsquigarrow (u^i, u^i_1, u^i_2, u^i_{12}, u^i_3, u^i_{13}, u^i_{23}, u^i_{123}), \\ \ldots \; &\ldots \end{aligned}$$

There holds the following rule: *in order to define the coordinate functions on the neighborhood T^kU, one appends their differentials to the coordinate functions on the neighborhood $T^{k-1}U$, i.e., the same coordinate functions as on the neighborhood $T^{k-1}U$, but with the extra index k.* As a result, one gets $2^k n$ coordinate functions[3] on the neighborhood T^kU. Here, the coordinates with the index i are fiber coordinates for the fibration ρ_i $(i = 1, 2, , \ldots, k)$, and the other ones are base ones.

In the following diagram, it is represented a cube with symbols attached to each vertex. Alltogether, these represent an element of the floor T^3M. If we adjoin to all the indices the upper index i, then we get the coordinates defined above. The endpoints of each of the three sides which are adjacent to the symbol u define a point of the floor T^2M, and the endpoints of the opposite sides define a tangent vector to T^2M at this point.

[3]Practically, to a point of the floor $T^{k-1}M$, one attaches a tangent vector to $T^{k-1}M$, at that point.

$$\begin{tikzcd}
& u_{123} \arrow[dl, "\rho_1"'] \arrow[d, dashed, "\rho_3"] \arrow[dr, "\rho_2"] & \\
u_{23} \arrow[d] \arrow[dr] & u_{12} \arrow[dl, dashed] \arrow[dr, dashed] & u_{13} \arrow[dl] \arrow[d] \\
u_2 \arrow[dr] & u_3 \arrow[d] & u_1 \arrow[dl] \\
& u &
\end{tikzcd}$$

A scalar function on the neighborhood T^kU, is called a (White, [17]) *sector-form* on T^kU, if it is linear and homogeneous on the fibers of all the bundles $\rho_1, \rho_2, \dots, \rho_k$.

Differentials of a function f from the manifold M are sector-forms on the corresponding neighborhoods, with partial derivatives as coefficients,

$$\begin{aligned}
&f,\\
f_1 &= f_i u_1^i,\\
f_{12} &= f_{ij} u_1^i u_2^i + f_k u_{12}^k\,,\\
f_{123} &= f_{ijk} u_1^i u_2^j u_3^k + f_{ij}(u_1^i u_{23}^j + u_2^i u_{13}^j + u_3^i u_{12}^j) + f_k u_{123}^k\,,\\
\dots &\quad \dots
\end{aligned}$$

We notice that the differential f_{12}, defined on the neighborhood T^2U, is a linear function both in the fiber coordinates (u_1^i, u_{12}^i) of the bundle ρ_1, and in the fiber coordinates (u_2^i, u_{12}^i) of the bundle ρ_2. This statement also holds true for its differentials of higher order f_{123} ….

The differentials may be considered for any sector-form. For instance, for the 1-form $\Phi = \varphi_i u_1^i$, which is a scalar function on the neighborhood TU, the differentials start by the index 2 :

$$\Phi = \varphi_i u_1^1,\ \Phi_2 = \partial_j \varphi_i u_1^i u_2^j + \varphi_k u_{12}^k,\ \partial_j \varphi_i \doteq \frac{\partial \varphi_i}{\partial u^j}\,, \dots$$

Actually, if in the expression of Φ_2, one performs the symmetrization and the skew-symmetrization of the coefficients $\partial_j \varphi_i = \partial_{(j}\varphi_{i)} + \partial_{[j}\varphi_{i]}$, then Φ_2 will include the exterior differential[4] $\mathrm{d}\Phi \doteq \partial_{[j}\varphi_{i]} u_1^i \wedge u_2^j$.

When lifting a function f from the neighborhood U to the floor TU, its differentials also start by the index 2:

$$f = f \circ \pi_1,\ f_2 = f_i u_2^i,\ f_{23} = f_{ij} u_2^i u_3^i + f_k u_{23}^k\ \dots$$

[4]Generally, the theory of sector-forms includes Cartan's theory of exterior forms.

Generally, a sector-form on the neighborhood T^2U is written in the following manner (where ψ_{ij}, ψ_k are arbitrary functions):

$$\Psi = \psi_{ij} u_1^i u_2^j + \psi_k u_{12}^k. \tag{1}$$

1.4 Lifts of a vector field

For every vector field X on the manifold M there exists a corresponding flow a_t, which may be understood as a 1-parameter group of (local) diffeomorphisms of the manifold M. The diffeomorphisms a_t are prolonged to the k-th floor T^kM, where the flow $T^k a_t$ induces a vector field $\overset{(k)}{X}$, called the *k-th order lift* of the vector field X,

$$a_t = \exp tX \;\rightsquigarrow\; T^k a_t = \exp t\overset{(1)}{X}\,.$$

On the neighborhood TU, one may associate to the coordinates (u^i, u_1^i) the natural frame and its dual coframe:

$$(\partial_i, \partial_i^1) \doteq \Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u_1^i}\Big), \quad d(u^i, u_1^i) \doteq (u_2^i, u_{12}^i).$$

Similarly, one may attach to the coordinates $(u^i, u_1^i, u_2^i, u_{12}^i)$ on the neighborhood T^2U, the natural frame and its dual coframe

$$\begin{aligned}
&(\partial_i, \partial_i^1, \partial_i^2, \partial_i^{12}) \doteq \Big(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u_1^i}, \tfrac{\partial}{\partial u_2^i}, \tfrac{\partial}{\partial u_{12}^i}\Big),\\
&d(u^i, u_1^i, u_2^i, u_{12}^i) \doteq (u_3^i, u_{13}^i, u_{23}^i, u_{123}^i), \quad \text{etc.}
\end{aligned}$$

The vector fields $X, \overset{(1)}{X}$ and $\overset{(2)}{X}$ are represented in the local frames in the form[5]:

$$\begin{aligned}
X &= \partial_i x^i,\\
\overset{(1)}{X} &= \partial_i x^i + \partial_i^1 x_1^i,\\
\overset{(2)}{X} &= \partial_i x^i + \partial_i^1 x_1^i + \partial_i^2 x_2^i + \partial_i^{12} x_{12}^i.
\end{aligned} \tag{2}$$

Main property: the operation of lifting vector fields is compatible with the Lie bracket:

$$\big[\overset{(k)}{X}, \overset{(k)}{Y}\big] = \overset{(k)}{[X, Y]}, \; k = 1, 2, \ldots \tag{3}$$

[5]When writing vector fields, we shall obey the following rule: *summation excludes differentiation.* Thus, the writing $\partial_i x^i$ means the linear combination of the operators ∂_i with the coefficients x^i, while the writing $\partial_j x^i$ indicates partial differentiation of the function x^i with respect to the operator ∂_j.

2 Tangent groups

2.1 The Leibniz rule

Consider a smooth mapping

$$\lambda : M_1 \times M_2 \longrightarrow M : (u,v) \longmapsto w = u \cdot v,$$

which attaches to any pair of points $u \in M_1$ and $v \in M_2$, a point $w \in M$. The tangent map

$$T\lambda : TM_1 \times TM_2 \longrightarrow TM : \big((u,u_1),(v,v_1)\big) \longmapsto (w,w_1)$$

attaches to a pair of vectors u_1 and v_1 at the points u and v, a vector w_1 at the point w. We write this as: $w = u \cdot v$ and $w_1 = (u \cdot v)_1$. The following relation can be called the *generalized Leibniz rule*:

$$\boxed{(u \cdot v)_1 = u_1 \cdot v + u \cdot v_1.} \tag{4}$$

Let us recall the case of the differential of a function of two variables:

$$z = z(x,y) \quad \rightsquigarrow \quad dz = \frac{\partial z}{\partial u}\, du + \frac{\partial z}{\partial v}\, dv.$$

In our case, we have to study the so-called *right* and *left translations* (in the first case, the element $v \in M_2$ is fixed, while in the second case, the fixed element is $u \in M_1$), and their tangent maps:

$$\lambda_v : M_1 \longrightarrow M : u \mapsto w, \quad \lambda_u : M_2 \longrightarrow M : v \mapsto w,$$

$$T\lambda_v : (u,u_1) \mapsto (w, u_1 \cdot v), \quad T\lambda_u : (v,v_1) \mapsto (w, u \cdot v_1).$$

The map $T\lambda_v$ brings the vector u_1 from the point u, to the vector $u_1 \cdot v$ at the point w, and $T\lambda_u$ brings the vector v_1 from the point v, to the vector $u \cdot v_1$ at the same point w. At w, the two vectors add according to the rule (4).

This rule can be extended to higher order differentials[6], for instance,

$$\boxed{(u \cdot v)_{12} = u_{12} \cdot v + u_2 \cdot v_1 + u_1 \cdot v_2 + u \cdot v_{12}.} \tag{5}$$

The tangent map $T^2\lambda$, on the floor T^2M generates the element

$$(w, w_1, w_2, w_{12}) = \big(\, u \cdot v, (u \cdot v)_1, (u \cdot v)_2, (u \cdot v)_{12}\big),$$

[6] These are generalizations of the following formulas of classical calculus:

$$(uv)' = u'v + uv',$$
$$(uv)'' = u''v + 2u'v' + uv''.$$

where w_1 and w_{12} are defined by relations (4), (5) and $w_2 = u_2 \cdot v + u \cdot v_2$.

In the following, it will be more convenient to represent $T\lambda$ and $T^2\lambda$ in matrix form:

$$T\lambda : \begin{pmatrix} w & 0 \\ w_1 & w \end{pmatrix} = \begin{pmatrix} u & 0 \\ u_1 & u \end{pmatrix} \cdot \begin{pmatrix} v & 0 \\ v_1 & v \end{pmatrix}, \tag{6}$$

$$T^2\lambda : \begin{pmatrix} w & 0 & 0 & 0 \\ w_1 & w & 0 & 0 \\ w_2 & 0 & w & 0 \\ w_{12} & w_2 & w_1 & w \end{pmatrix} = \begin{pmatrix} u & 0 & 0 & 0 \\ u_1 & u & 0 & 0 \\ u_2 & 0 & u & 0 \\ u_{12} & u_2 & u_1 & u \end{pmatrix} \cdot \begin{pmatrix} v & 0 & 0 & 0 \\ v_1 & v & 0 & 0 \\ v_2 & 0 & v & 0 \\ v_{12} & v_2 & v_1 & v \end{pmatrix}. \tag{7}$$

Then, the transition to higher order tangent maps $T^k\lambda$ is made *iteratively*. The transition from an element $u \in M$, first, to a 2×2-matrix, and then, to a 4×4-matrix is then made automatically:

$$u \quad \rightsquigarrow \quad \begin{pmatrix} u & 0 \\ u_1 & u \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} \begin{pmatrix} u & 0 \\ u_1 & u \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} u & 0 \\ u_1 & u \end{pmatrix}_2 & \begin{pmatrix} u & 0 \\ u_1 & u \end{pmatrix} \end{pmatrix}. \tag{8}$$

Remarks. The Leibniz rule is applicable in various situations. For instance, with its help, we can deduce the expressions of Lie derivatives.

By denoting the Lie derivatives with respect to some vector field X simply, by a prime mark, as: $\mathcal{L}_X \doteq (\ldots)'$, we have for the Lie derivative of a vector field Y:

$$(Yf)' = Y'f + Yf' \quad \rightsquigarrow \quad Y' = XY - YX = [XY].$$

For the Lie derivative of a 1-form Φ, we have:

$$\big(\Phi(Y)\big)' = \Phi'(Y) + \Phi(Y'),$$

and therefore, using the defining relations for the exterior derivative:

$$\mathrm{d}\Phi(X,Y) = X\big(\Phi(Y)\big) - Y\big(\Phi(X)\big) + \Phi([XY]),$$

it follows[7]:

$$\Phi' = \mathrm{d}\Phi(X,\cdot) + d\big(\Phi(X)\big),$$

for $\Phi = df$, from the latter equality, we get that $(df)' = df'$, i.e., *the Lie derivative commutes with differentiation*, etc.

2.2 Floors of a Lie group

Tangent groups of a Lie group are the floors of the group-manifold, with the induced group actions.

First of all, a Lie group G is a smooth manifold with a group composition law[8]

$$\gamma \,:\, G^2 \longrightarrow G \,:\, (a,b) \longmapsto c = ab,$$

[7]We notice that $\overset{(1)}{X}\Phi = \mathcal{L}_X \Phi$.

[8]We have denoted above the product of two elements by a dot: $u \cdot v$. For the product of group elements, we will omit this dot "$\cdot$" i.e., we will write: ab.

with the unity element e and inverse operation $a \rightsquigarrow a^{-1}$.

The first floor TG of the manifold G becomes the *first tangent group* of the group G, with the composition law

$$T\gamma : \begin{pmatrix} c & 0 \\ c_1 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix} \begin{pmatrix} b & 0 \\ b_1 & b \end{pmatrix}, \tag{9}$$

having the zero vector at the point $e \in G$ as its unity element, and the inverse elements

$$\begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 \\ a_1^{-1} & a^{-1} \end{pmatrix}, \tag{10}$$

where

$$\boxed{a_1^{-1} = -a^{-1} a_1 a^{-1}.} \tag{11}$$

The second tangent group of a group G is the second floor T^2G of the manifold G, with the composition law

$$T^2\lambda : \begin{pmatrix} c & 0 & 0 & 0 \\ c_1 & c & 0 & 0 \\ c_2 & 0 & c & 0 \\ c_{12} & c_2 & c_1 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix} \begin{pmatrix} b & 0 & 0 & 0 \\ b_1 & b & 0 & 0 \\ b_2 & 0 & b & 0 \\ b_{12} & b_2 & b_1 & b \end{pmatrix}. \tag{12}$$

Its unity element is the zero vector at the unity element of the group TG, and the inverses of its elements are:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_{12} & a_2 & a_1 & a \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 & 0 \\ a_1^{-1} & a^{-1} & 0 & 0 \\ a_2^{-1} & 0 & a^{-1} & 0 \\ a_{12}^{-1} & a_2^{-1} & a_1^{-1} & a^{-1} \end{pmatrix}, \tag{13}$$

where, according[9] to (11), $a_1^{-1} = -a^{-1}a_1a^{-1}$, $a_2^{-1} = -a^{-1}a_2a^{-1}$, and

$$\boxed{a_{12}^{-1} = a^{-1}a_2a^{-1}a_1a^{-1} - a^{-1}a_{12}a^{-1} + a^{-1}a_1a^{-1}a_2a^{-1}}. \tag{14}$$

Raising to the following floors, we conclude that the *k-th tangent group* of a group G is the k-th floor T^kG.

Remark 2.2. If the group G is the general linear group $GL(n, \mathbb{R})$, then the diagonal blocks in the matrices (6), and further on, are regular matrices, i.e., elements of the group $GL(n, \mathbb{R})$, while the other blocks (with subscripts) are elements of the Lie algebra $gl(n, \mathbb{R})$.

[9] Formulas (11) and (14) generalize the formulas

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}, \quad \left(\frac{1}{u}\right)'' = \frac{2(u')^2 - u''u}{u^3}.$$

Other classical formulas can also be generalized, for instance, $\left(\frac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$ is generalized as: $(ab^{-1})_1 = a(a^{-1}a_1 - b^{-1}b_1)b^{-1}$ etc.

2.3 Representations of tangent groups

A smooth mapping

$$\lambda \ : \ M \times G \ \longrightarrow \ M \ : \ (u, a) \ \mapsto \ v = u \cdot a$$

defines a *right action* of the Lie group G on the space M, if all the maps

$$\lambda_a \ : \ M \to M \ : u \mapsto u \cdot a \ , \quad \forall a \in G \ ,$$

are transformations (diffeomorphisms) of M and the mapping $a \mapsto \lambda_a$ is a morphism between the group G and the group of transformations of the space M. The mapping

$$\lambda_u \ : \ G \to M \ : a \mapsto u \cdot a$$

defines, in the space M, the *orbit* $\lambda_u(G)$ of the point u.

The tangent map

$$T\lambda \ : \ TM \times TG \ \to \ TM \ : \big((u, u_1), (a, a_1)\big) \ \mapsto \ (v, v_1)$$

where

$$\begin{cases} v & = u \cdot a \, , \\ v_1 & = u_1 \cdot a + u \cdot a_1 = u_1 \cdot a + v \cdot (a^{-1} a_1), \end{cases}$$

defines a representation of the tangent group TG on the floor TM.

Let us notice two particular cases. For $a_1 = 0$, it is defined the action of the group G on the floor TM:

$$a_1 = 0 \quad \Rightarrow \quad u_1 \ \mapsto \ v_1 = u_1 \cdot a.$$

If $u_1 = 0$, then we have a linear mapping from the Lie algebra $T_e G$ to the tangent space $T_v M$, $\forall v \in M$:

$$u_1 = 0 \quad \Rightarrow \quad e_1 = a^{-1} a_1 \ \mapsto \ v_1 = u \cdot a_1 = v \cdot a^{-1} a_1 = v \cdot e_1.$$

Formula $v_1 = v \cdot a^{-1} a_1$ is known in the theory as the *fundamental equation* of the representation of the Lie group G.

Remarks. **2.3.** At every point $v \in M$ it is defined the vector $v_1 = v \cdot e_1$. This means that, in the space M, it is defined a vector field – *group operator*[10], tangent to orbits. Depending on the choice of the vector $e_1 \in T_e G$, in M there appear infinitely many operators of the group G.

[10]Sophus Lie called these operators *infinitesimal transformations* of the space M. It also makes sense to speak about *fundamental vector fields of the group* etc.

2.4. The fundamental equation $v_1 = v \cdot a^{-1}a_1$ defines on the group G a system of equations[11] $\vartheta^\alpha = \xi_i^\alpha \omega^i$, where ω^i is a left invariant cobasis on the Lie group G. At the same time, in a coordinate neighborhood of the point $v \in M$ we have a system of operators X_i:

$$X_i = \xi_i^\alpha \frac{\partial}{\partial v^\alpha}.$$

The forms ϑ^α and the operators X_i are not necessarily linearly independent, but the number of forms ϑ^α is equal to the dimension of the space M, while the number of operators X_i is equal to the dimension of the group G. The Pfaff system $\xi_i^\alpha \omega^i = 0$ defines for a fixed point $v \in M$ its *stabilizer* $H_v \subset G$.

2.5. The group G acts on itself by:

- left translations $l_b : a \mapsto b\,a$,
- right translations $r_b : a \mapsto ab$,
- inner automorphisms $A_b = l_b \circ r_b^{-1} : a \mapsto b\,ab^{-1}$ (conjugate representation).

A vector $e_1 \in T_eG$ is mapped by left translations l_b into a left invariant vector field $b\,e_1$, by right translations r_b into a right invariant vector field $e_1 b$ and by inner automorphisms A_b into the operator $b\,e_1 - e_1 b$.

2.6. The vector field $X \doteq b\,e_1$ (accordingly, $\tilde{X} \doteq e_1 b$) induces in the group G the flow of right (left) translations. The operator $\tilde{X} - X = e_1 b - b\,e_1$ induces the flow of inner automorphisms. If $e_1 \in T_eG$ is the tangent vector to a 1-parameter subgroup a_t of the group G, then

$$r_{a_t} = \exp tX, \quad l_{a_t} = \exp t\tilde{X}, \quad A_{a_t} = \exp t(\tilde{X} - X).$$

The left invariance of the operator X and the right invariance of the operator $\tilde{X}$ are a consequence of the fact that left and right translations commute[12]:

$$l_b\, r_{a_t} l_b^{-1} = r_{a_t}, \quad r_b\, l_{a_t} r_b^{-1} = l_{a_t}, \quad \forall b \in G.$$

For an arbitrary function f on G, we have the derivatives

$$Xf = (f \circ r_{a_t})'_{t=0}, \quad \tilde{X}f = (f \circ l_{a_t})'_{t=0}, \quad (\tilde{X} - X)f = (f \circ A_{a_t})'_{t=0}.$$

2.7. By considering the inverse $\kappa : a \to a^{-1}$, the vector fields X and $\tilde{X}$ will be related by the equality $\tilde{X} = -T\kappa X$, since $l_{a_t} = \kappa r_{a_t}^{-1} \kappa$.

2.8. The fundamental equations of left and right translations, as well as those of inner automorphisms on the floor TG, look as follows:

$$c_1 = (a_1 a^{-1})\, c, \qquad c_1 = c\,(a^{-1} a_1), \qquad c_1 = (a_1 a^{-1})\, c - c\,(a_1 a^{-1}).$$

[11] The matrix $\xi = (\xi_i^\alpha)$ plays in the local theory an exceptional role. For instance, in the formulation of Lie's theorems.

[12] Generally, with respect to a transformation b of the manifold $\mathcal{M}$, the flow a_t is transformed according to the following scheme:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{a_t} & \mathcal{M} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{M} & \xrightarrow{\tilde{a}_t} & \mathcal{M} \end{array} \qquad a_t \rightsquigarrow \tilde{a}_t = b\, a_t\, b^{-1}.$$

2.4 Gauge theory

2.4.1 Jacobian matrix

Consider a smooth mapping from an n-dimensional manifold $\mathcal{N}$ to an m-dimensional manifold $\mathcal{M}$,

$$\varphi \,:\, \mathcal{N} \;\longrightarrow\; \mathcal{M}.$$

The tangent map $T\varphi$ is understood as a morphism of floors :

$$\begin{array}{ccc} T\mathcal{N} & \xrightarrow{T\varphi} & T\mathcal{M} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{N} & \xrightarrow{\varphi} & \mathcal{M} \end{array}$$

For any pair of points $u \in \mathcal{N}$ and $v = \varphi(u) \in \mathcal{M}$, it is defined a linear transformation between the respective tangent spaces :

$$T_u\varphi \,:\, T_u\mathcal{N} \;\longrightarrow\; T_v\mathcal{M}.$$

In the coordinates u^i and v^α on the neighborhoods $U \subset \mathcal{N}$ and $V \subset \mathcal{M}$, the mapping φ is defined by m functions φ^α on the neighborhood U, which are φ-related to the coordinate functions v^α on the neighborhood V,

$$v^\alpha \circ \varphi = \varphi^\alpha.$$

The tangent map $T\varphi$ is defined by the differentials $d\varphi^\alpha = \varphi^\alpha_i du^i$, which are $T\varphi$-related with the cobasis dv^α on the neighborhood V,

$$dv^\alpha \circ T\varphi = d\varphi^\alpha.$$

The Jacobian matrix (φ^α_i) consists of the partial derivatives $\varphi^\alpha_i = \dfrac{\partial \varphi^\alpha}{\partial u^i}$ on the neighborhood U. At a fixed point $u \in U$, this is a numerical $(m \times n)$-matrix, which thus defines a linear transformation $T_u\varphi$.

The tangent maps $T\varphi$ and $T^2\varphi$ are defined by the system to the left (see below) and by the Jacobian matrix (to the right) respectively :

$$\left\{ \begin{array}{ll} v^\alpha \circ \varphi & = \varphi^\alpha\,, \\ v^\alpha_1 \circ T\varphi & = \varphi^\alpha_1\,, \end{array} \right. \qquad \begin{pmatrix} \varphi^\alpha_i & 0 \\ (\varphi^\alpha_i)_1 & \varphi^\alpha_i \end{pmatrix},$$

where $\varphi^\alpha_1 = \varphi^\alpha_i u^i_1$ and $(\varphi^\alpha_i)_1 = \varphi^\alpha_{ij} u^j_1$. At the point $u_{(1)} \doteq (u, u_1) \in TU$, the Jacobian matrix defines a linear mapping[13] :

$$T^2_{u_{(1)}}\varphi \,:\, T^2_{u_{(1)}}\mathcal{N} \longrightarrow T^2_{v_{(1)}}\mathcal{M}.$$

[13]More precisely : $T_{(u,u_1)}(T\varphi) \,:\, T_{(u,u_1)}(T\mathcal{N}) \longrightarrow T_{(v,v_1)}(T\mathcal{M})$.

Remarks.

2.9. Since the Jacobian matrix is defined on the whole neighborhood TU, it is also defined on any subset of this neighborhood, in particular, on the vector field X, regarded as a section of the bundle $TU \to U$, with the local components (x^i), namely:

$$\begin{pmatrix} \Phi & 0 \\ X\Phi & \Phi \end{pmatrix}, \quad \text{where} \quad \Phi \doteq (\varphi_i^\alpha), \;\; X\Phi \doteq (\varphi_{jk}^i x^k).$$

2.10. Taking iterations of the tangent functor $T\varphi \rightsquigarrow T^2\varphi \rightsquigarrow T^3\varphi \rightsquigarrow \ldots$ the staircase structure of the Jacobian matrix is preserved:

$$\Phi \;\rightsquigarrow \begin{pmatrix} \Phi & 0 \\ X\Phi & \Phi \end{pmatrix} \rightsquigarrow \begin{pmatrix} \Phi & 0 & 0 & 0 \\ X\Phi & \Phi & 0 & 0 \\ Y\Phi & 0 & \Phi & 0 \\ YX\Phi & Y\Phi & X\Phi & \Phi \end{pmatrix} \rightsquigarrow \quad \ldots$$

2.4.2 Gauge group

We shall give in the following an invariant (coordinate-free) definition of the gauge group[14].

Definition[15]. Let M be a smooth manifold of dimension n. The *gauge group* at a point $u_{(k)}$ of the k-th floor T^kM is the group $\mathcal{G}_k$ of linear transformations of the tangent space $T^{k+1}_{(u_k)}M$, induced on this space by diffeomorphisms of the manifold M.

Proposition 2.1. *The gauge group $\mathcal{G}_k$ is isomorphic to the k-th tangent group of the linear group $GL(n,\mathbb{R})$,*

$$\mathcal{G}_k \;\sim\; T^k\big(GL(n,\mathbb{R})\big), \quad k = 0, 1, 2, \ldots$$

P r o o f. Transformations of the tangent space are defined independently from local coordinates, but, in the natural bases, they are defined by Jacobian matrices. At the first steps, we have:

$k = 0 \;\Rightarrow\; \mathcal{G} \sim GL(n,\mathbb{R})$ – the linear group is generated by the Jacobian matrices $\mathfrak{a} = (a^i_j)$ of diffeomorphisms a at the point $u \in M$,

$k = 1 \;\Rightarrow\; \mathcal{G}_1 \sim T\big(GL(n,\mathbb{R})\big)$ – the first tangent group is generated by the Jacobian matrices $\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}$ of diffeomorphisms Ta at the point $(u, u_1) \in TM$, by means of the block $\mathfrak{a}_1 = (a^i_{jk} u^k_1)$,

---

[14] See also [1].

[15] In the case $k = 0$, we set:

$$u_{(0)} = u\,, \;\; T^0M = M, \;\; \mathcal{G}_0 = \mathcal{G}\,, \;\; T^{0+1}_{(u_0)}M = T_uM, \;\; \text{etc.}, \; u_{(1)} = (u, u_1), \ldots$$

$k=2 \;\Rightarrow\; \mathcal{G}_2 \sim T^2\big(GL(n,\mathbb{R})\big)$ – the second tangent group is defined by the Jacobian matrices $\begin{pmatrix} \mathfrak{a} & 0 & 0 & 0 \\ \mathfrak{a}_1 & \mathfrak{a} & 0 & 0 \\ \mathfrak{a}_2 & 0 & \mathfrak{a} & 0 \\ \mathfrak{a}_{12} & \mathfrak{a}_2 & \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}$ of diffeomorphisms $T^2 a$ at the point $(u, u_1, u_2, u_{12}) \in T^2 M$, by means of the blocks

$$\mathfrak{a}_1 = (a^i_{jk} u^k_1), \;\; \mathfrak{a}_2 = (a^i_{jk} u^k_2), \;\; \mathfrak{a}_{12} = (a^i_{jkl} u^k_1 u^l_2 + a^i_{jk} u^k_{12}).$$

Further, we proceed iteratively. □

Remarks.

2.11. The actions in the group $\mathcal{G}_2$ (and in the following groups $\mathcal{G}_k$), i.e., multiplication and inverse operation, are defined by:

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{b} & 0 \\ \mathfrak{b}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}\mathfrak{b} & 0 \\ (\mathfrak{a}\mathfrak{b})_1 & \mathfrak{a}\mathfrak{b} \end{pmatrix}, \quad (\mathfrak{a}\mathfrak{b})_1 = \mathfrak{a}_1 \mathfrak{b} + \mathfrak{a}\mathfrak{b}_1 \,,$$

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}^{-1} = \begin{pmatrix} \mathfrak{a}^{-1} & 0 \\ \mathfrak{a}^{-1}_1 & \mathfrak{a}^{-1} \end{pmatrix}, \quad \mathfrak{a}^{-1}_1 = -\mathfrak{a}^{-1}\mathfrak{a}_1\mathfrak{a}^{-1}.$$

2.12. To a vector field X on the neighborhood TU it corresponds a linear pseudo-group, with the multiplication and inversion of matrices :

$$\begin{pmatrix} \mathfrak{a} & 0 \\ X\mathfrak{a} & \mathfrak{a} \end{pmatrix} \cdot \begin{pmatrix} \mathfrak{b} & 0 \\ X\mathfrak{b} & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}\mathfrak{b} & 0 \\ X(\mathfrak{a}\mathfrak{b}) & \mathfrak{a}\mathfrak{b} \end{pmatrix},$$

$$\begin{pmatrix} \mathfrak{a} & 0 \\ X\mathfrak{a} & \mathfrak{a} \end{pmatrix}^{-1} = \begin{pmatrix} \mathfrak{a}^{-1} & 0 \\ -\mathfrak{a}^{-1} X\mathfrak{a}\, \mathfrak{a}^{-1} & \mathfrak{a}^{-1} \end{pmatrix}.$$

There appears a natural question: what is the relation between the above invariant definition of the gauge group and the jet bundle approach?

Let us consider the group $\mathfrak{G}_k$ of k-jets of diffeomorphisms of the manifold M at a point $u \in M$.

Proposition 2.2. *There exists a homomorphism from the jet group* $\mathfrak{G}_k$ *onto the k-th gauge group* $\mathfrak{G}_k \to \mathcal{G}_k$. *The kernel of this homomorphism is the stabilizer* H_{k-1} *of the point* $u_{(k-1)} \in T^{k-1}M$. *The gauge group* $\mathcal{G}_k$ *is isomorphic to the quotient group* $\mathfrak{G}_k / H_{k-1}$.

P r o o f. The case $k=1$ is trivial. In the case $k=2$, the group $\mathfrak{G}_2$ is generated by 2-jets $(a^i_j \,, a^i_{jk})$ at the point $u \in M$. In the group $\mathfrak{G}_2$ the actions (multiplication and inversion of its elements) are defined by:

$$(a^i_j \,, a^i_{jk}) \cdot (b^i_j \,, b^i_{jk}) = (a^i_l b^l_j \,, a^i_{jl} b^l_i + a^i_l b^k_{ij}),$$

$$(a^i_j \,, a^i_{jk})^{-1} = (\bar{a}^i_j \,, -\bar{a}^i_l a^l_{sj} \bar{a}^s_k)\,, \quad \text{where} \quad (\bar{a}^i_j) = (a^i_j)^{-1}.$$

The mapping

$$\mathfrak{G}_2 \longrightarrow \mathcal{G}_2 \,:\, (a^i_j, a^i_{jk}) \;\rightsquigarrow\; \begin{pmatrix} a^i_j & 0 \\ a^i_{jk} u^k_1 & a^i_j \end{pmatrix}$$

is homomorphic. The stabilizer H_1 of the element $(u, u_1) \in TM$ generates a 2-jet which is transformed into the identity matrix:

$$H_1 = \{(a^i_j, a^i_{jk}) \,|\, a^i_j = \delta^i_j\,,\ a^i_{jk}u^k_1 = 0\}.$$

According to the theorem of homomorphisms, the isomorphism $\mathcal{G}_2 \sim \mathfrak{G}_2/H_1$ holds.

In the case $k > 2$, the reasoning is similar. □

3 Elements of the theory of connections

3.1 The structure $\triangle_h \oplus \triangle_v$

A *connection* in a bundle $\pi : M_1 \to M$ with the n-dimensional base M and r-dimensional fibers is defined as a structure $\triangle_h \oplus \triangle_v$, where $\triangle_v = \ker T\pi$ is the vertical distribution and $\triangle_h$ is the horizontal one, supplementary to the distribution $\triangle_v$.

On a domain of local chart $U \subset M_1$ with coordinates (u^i, u^α), where u^i are base coordinates and u^α are fiber ones ($i = 1, 2, \ldots, n$; $\alpha = n+1, \ldots, n+r$), a basis (frame + coframe) can be *adapted* to the structure $\triangle_h \oplus \triangle_v$,

$$\boxed{(X_i, X_\alpha) = (\partial_j\,, \partial_\beta\,) \cdot \begin{pmatrix} \delta^j_i & 0 \\ \Gamma^\beta_i & \delta^\beta_\alpha \end{pmatrix}, \quad \begin{pmatrix} \omega^j \\ \omega^\beta \end{pmatrix} = \begin{pmatrix} \delta^j_i & 0 \\ -\Gamma^\beta_i & \delta^\beta_\alpha \end{pmatrix} \cdot \begin{pmatrix} du^i \\ du^\alpha \end{pmatrix}.} \tag{15}$$

The horizontal vector fields $X_i = \partial_i + \Gamma^\alpha_i \partial_\alpha$ generate a basis of the distribution Δ_h, and the forms $\omega^\alpha = du^\alpha - \Gamma^\alpha_i du^i$ vanish on X_i. The forms ω^i vanish on the vertical vector fields $X_\alpha = \partial_\alpha$ generating a basis for Δ_v.

Remark.

3.1. It is a known fact that an n-dimensional subspace on an $(n+r)$-dimensional vector space is defined up to nr parameters; for the adapted basis, these parameters are the nr quantities Γ^α_i. Generally, they depend both on the base and on the fiber coordinates (u^i, u^α).

3.2. A Pfaff system $\omega^\alpha = 0$ is equivalent to a system of ordinary differential equations (ODE's)

$$\omega^\alpha = du^\alpha - \Gamma^\alpha_i du^i = 0 \quad \Longleftrightarrow \quad \frac{\partial u^\alpha}{\partial u^i} = \Gamma^\alpha_i. \tag{16}$$

It is thus established a link between the structure $\triangle_h \oplus \triangle_v$ and the given differential equations. If the DE's cannot be brought into the form (16), then for the quantity Γ^α_i, the relations overlap and in the definition of the distribution Δ_h there might exist some arbitrariness. Using this arbitrariness, the distribution Δ_h can be changed in such a way as to find the solutions of the DE's.

3.3. In a vector bundle, one can define a *linear connection*. In this case, the quantities Γ^α_i are linear and homogeneous on the fibers: $\Gamma^\alpha_i = \Gamma^\alpha_{i\beta}u^\beta$. The coefficients $\Gamma^\alpha_{i\beta}$ depend on the base coordinates. The system (16) consists of linear DE's. The transport of fibers along a path is done by means of linear transformations.

3.4. A classical *affine connection* on the manifold M is equivalent to a linear connection on the floor TM. Then, the quantities Γ_i^α define on the neighborhood $U \subset M$ a 1-form with values in the Lie algebra $gl(n,\mathbb{R})$: $\Gamma_i^\alpha \rightsquigarrow -\Gamma_{jk}^i du^k$. On the neighborhood $TU \subset TM$ it is defined the adapted basis[16]:

$$(X_i, X_i^1) = (\partial_j\,, \partial_j^1\,) \cdot \begin{pmatrix} \delta_i^j & 0 \\ -\Gamma_{ik}^j u_1^k & \delta_i^j \end{pmatrix}, \begin{pmatrix} U_2^j \\ U_{12}^j \end{pmatrix} = \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_{ik}^j u_1^k & \delta_i^j \end{pmatrix} \cdot \begin{pmatrix} u_2^i \\ u_{12}^i \end{pmatrix}. \tag{17}$$

3.5. With respect to coordinate transformations on the neighborhood U, the natural basis is transformed as:

$$\begin{cases} \tilde{u}^i & = a^i(u^j), \\ \tilde{u}^\alpha & = a^\alpha(u^j, u^\beta), \end{cases}$$

$$(\tilde{\partial}_i\,, \tilde{\partial}_\alpha) = (\partial_j\,, \partial_\beta) \cdot \begin{pmatrix} \bar{a}_i^j & 0 \\ \bar{a}_i^\beta & \bar{a}_\alpha^\beta \end{pmatrix}, \quad \begin{pmatrix} d\tilde{u}^i \\ d\tilde{u}^\alpha \end{pmatrix} = \begin{pmatrix} a_j^i & 0 \\ a_j^\alpha & a_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} du^j \\ du^\beta \end{pmatrix}, \tag{18}$$

$$a_j^i = \frac{\partial a^i}{\partial u^j}, \ a_j^\alpha = \frac{\partial a^\alpha}{\partial u^j}, \ a_\beta^\alpha = \frac{\partial a^\alpha}{\partial u^\beta},$$
$$a_k^i \bar{a}_j^k = \delta_j^i\,, \ a_\gamma^\alpha \bar{a}_\beta^\gamma = \delta_\beta^\alpha\,, \ a_j^\alpha \bar{a}_i^j + a_\beta^\alpha \bar{a}_i^\beta = 0,$$

and the adapted basis (15) is transformed as follows:

$$(\tilde{X}_i\ \tilde{X}_\alpha) = (X_j\ X_\beta) \cdot \begin{pmatrix} \bar{a}_i^j & 0 \\ 0 & \bar{a}_\alpha^\beta \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\omega}^j \\ \tilde{\omega}^\beta \end{pmatrix} = \begin{pmatrix} a_i^j & 0 \\ 0 & a_\alpha^\beta \end{pmatrix} \cdot \begin{pmatrix} \omega^i \\ \omega^\alpha \end{pmatrix}. \tag{19}$$

We obtain a transformation of the quantities $\Gamma_i^\alpha \rightsquigarrow \tilde{\Gamma}_i^\alpha$

$$\tilde{\Gamma}_i^\alpha \circ a = (a_\beta^\alpha\, \Gamma_j^\beta + a_j^\alpha) \bar{a}_i^j\,. \tag{20}$$

In the case of linear connections, when

$$\Gamma_i^\alpha = \Gamma_{i\beta}^\alpha u^\beta, \ \tilde{\Gamma}_i^\alpha = \tilde{\Gamma}_{i\beta}^\alpha \tilde{u}^\beta, \ \tilde{u}^\alpha = a^\alpha = a_\beta^\alpha u^\beta, \ a_j^\alpha = a_{j\beta}^\alpha u^\beta,$$

formula (20) defines a transformation $\Gamma_{i\beta}^\alpha \rightsquigarrow \tilde{\Gamma}_{i\beta}^\alpha$

$$\tilde{\Gamma}_{i\beta}^\alpha \circ a = (a_\sigma^\alpha\, \Gamma_{j\gamma}^\sigma + a_{j\gamma}^\alpha) \bar{a}_i^j \bar{a}_\beta^\gamma\,. \tag{21}$$

3.2 Covariant differentiation

A tensor field of type (p,q) is split in the presence of the structure $\triangle_h \oplus \triangle_v$ into $2^{\,p+q}$ invariant blocks. In the adapted basis (15), taking into account relations (19), these blocks have a tensorial character. When on some floor, one performs the usual differentiation, in the formulas corresponding to the natural bases, there appear partial derivatives, while in the formulas corresponding to the adapted bases, instead of partial derivatives, there appear covariant ones.

[16] The sign "$-$" is put in order to make our formulas correspond to the ones in tensor analysis.

3.2.1 Decomposition of a vector field

A vector field, as a tensor field of type (1,0), is decomposed with respect to the structure $\triangle_h \oplus \triangle_v$ into 2 invariant blocks. Let us consider the vector field (2) on the floor TM and decompose this field, in matrix writing, in the natural and in the adapted frames, see (17):

$$\overset{(1)}{X} = (\,\partial_i\,,\partial_i^1)\cdot\begin{pmatrix} x^i \\ x^i_j u^j_1 \end{pmatrix} = (\,X_i\,,X_i^1)\cdot\begin{pmatrix} x^i \\ x^i_{,j} u^j_1 \end{pmatrix}.$$

The partial derivatives from the natural frame $x^i_j = \partial_j x^i$ are replaced, in the adapted basis, by covariant derivatives:

$$\boxed{x^i_{,j} = \partial_j x^i + \Gamma^i_{kj} x^k.} \tag{22}$$

3.2.2 Decomposition of sector-forms

A sector form, as a tensor field of type (0,1), is also decomposed in the structure $\triangle_h \oplus \triangle_v$ into 2 invariant blocks. We consider the sector-form (1) on the floor TM and decompose it (in matrix writing), in the natural and in the adapted coframes, see (17) :

$$\Psi = (\,\psi_{ij} u^i_1\,,\psi_j)\cdot\begin{pmatrix} u^j_2 \\ u^j_{12} \end{pmatrix} = (\,\tilde{\psi}_{ij} u^i_1\,,\psi_j)\cdot\begin{pmatrix} U^j_2 \\ U^j_{12} \end{pmatrix}.$$

There appears a transformation

$$\psi_{ij} \quad \rightsquigarrow \quad \tilde{\psi}_{ij} = \psi_{ij} - \psi_k \Gamma^k_{ji}.$$

If $\Psi = \Phi_2 = \partial_j \varphi_i u^i_1 u^j_2 + \varphi_j u^j_{12}$ is the differential of a 1-form $\Phi = \varphi_i u^i_1$ on the manifold M, partial derivatives $\partial_j \varphi_i$ are replaced by covariant ones :

$$\boxed{\varphi_{i,j} = \partial_j \varphi_i - \varphi_k \Gamma^k_{ji}\,.} \tag{23}$$

3.2.3 Decomposition of affinor fields

A tensor field of type (1,1), i.e., an affinor field, is decomposed in the structure $\triangle_h \oplus \triangle_v$ into 4 invariant blocks.

Let us return to the gauge group. When saying that the Jacobian matrix

$$\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix} = \begin{pmatrix} a^i_j & 0 \\ a^i_{jk} u^k_1 & a^i_j \end{pmatrix}$$

defines on the neighborhood TU a transformation of the tangent spaces $T^2_{(u_1)}M$, we understand that this happens in the natural basis $(\partial_i\,,\partial^1_i\,;u^j_2\,,u^j_{12})$ and it can be represented as a vector-valued form

$$\mathcal{A}=(\partial_i\,,\partial^1_i\,)\cdot\begin{pmatrix} a^i_j & 0 \\ a^i_{jk}u^k_1 & a^i_j \end{pmatrix}\cdot\begin{pmatrix} u^j_2 \\ u^j_{12} \end{pmatrix}.$$

In the adapted basis (17), the vector-valued form $\mathcal{A}$ is written as

$$\mathcal{A}=(X_i\,,X^1_i\,)\cdot\begin{pmatrix} a^i_j & 0 \\ a^i_{j\,,k}\,u^k_1 & a^i_j \end{pmatrix}\cdot\begin{pmatrix} U^j_2 \\ U^j_{12} \end{pmatrix},$$

where

$$\boxed{a^i_{j\,,k}=a^i_{jk}-a^i_l\Gamma^l_{jk}+\Gamma^i_{lk}a^l_j} \tag{24}$$

is the covariant derivative of the Jacobian matrix (a^j_j). We can convince ourselves of this if we multiply the matrices below. It is the way the matrix of a linear map is transformed when passing from a basis to another one:

$$\begin{pmatrix} \delta^i_s & 0 \\ \Gamma^i_{sk}u^k_1 & \delta^i_s \end{pmatrix}\cdot\begin{pmatrix} a^s_t & 0 \\ a^s_{tk}\,u^k_1 & a^s_t \end{pmatrix}\cdot\begin{pmatrix} \delta^t_j & 0 \\ -\Gamma^t_{jk}u^k_1 & \delta^t_j \end{pmatrix}.$$

In the adapted basis, all the blocks of the vector-valued form $\mathcal{A}$ are tensors.

As a conclusion, considering that the gauge group $\mathcal{G}_2$ is generated by the group of matrices $\begin{pmatrix} \mathfrak{a} & 0 \\ \mathfrak{a}_1 & \mathfrak{a} \end{pmatrix}$ then, depending on the basis in which the transformations T^2a are represented (natural or adapted ones), the block $\mathfrak{a}_1$ has the form $\mathfrak{a}_1=(a^i_{jk}u^k_1)$, or $\mathfrak{a}_1=(a^i_{j\,,k}u^k_1)$.

3.3 Basic formulas of the theory of connections

3.3.1 Morphism of bundles with connections

The following commutative diagram defines a *morphism* between the bundles π_1 and π_2:

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{F} & \mathcal{M}_2 \\ \pi_1\downarrow & & \downarrow\pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array} \tag{25}$$

Consider on each of the bundles π_1 and π_2 connections, i.e., structures $\Delta_h\oplus\Delta_v$ and $\tilde{\Delta}_h\oplus\tilde{\Delta}_v$ on the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

On the neighborhoods $U \subset \mathcal{M}_1$ and $F(U) \subset \mathcal{M}_2$ with coordinates (u^i, u^α) and (v^a, v^λ), we have the natural and the adapted bases (to the right we have indicated the quantities Γ_i^β and Λ_b^λ),

$$\begin{array}{rlll} U: & (\partial_i, \partial_\alpha\,; du^j, du^\beta), & (X_i, X_\alpha\,; \omega^j, \omega^\beta), & (\Gamma_i^\beta), \\ F(U): & (\partial_a, \partial_\lambda\,; dv^b, dv^\mu), & (X_a, X_\lambda\,; \theta^{\,b}, \theta^{\,\mu}), & (\Lambda_b^\lambda). \end{array}$$

The mapping F is locally defined by the functions (f^a, f^λ),

$$\begin{cases} v^a \circ F = f^a, \\ v^\lambda \circ F = f^\lambda. \end{cases}$$

The functions f^a are π-related to the functions $\bar{f}^a$, which define the mapping f on the neighborhood $\pi_1(U)$, thus, $f^a = \bar{f}^a \circ \pi_1$. The tangent map TF is defined in the natural bases by the Jacobian matrix (to the left) and in the adapted bases, by the same matrix, in which we modify the "south-western" block:

$$\begin{pmatrix} f_i^a & 0 \\ f_i^\lambda & f_\alpha^\lambda \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} f_i^a & 0 \\ F_i^\lambda & f_\alpha^\lambda \end{pmatrix},$$

where $f_i^a \doteq \dfrac{\partial f^a}{\partial u^i}$, $f_i^\lambda \doteq \dfrac{\partial f^\lambda}{\partial u^i}$, $f_\alpha^\lambda \doteq \dfrac{\partial f^\lambda}{\partial u^\alpha}$, and

$$\boxed{F_i^\lambda = f_i^\lambda + f_\beta^\lambda \Gamma_i^\beta - (\Lambda_b^\lambda \circ F) f_i^b.} \tag{26}$$

In the case of vector bundles equipped with linear connections, the quantities (26) are linear functions on the fibers:

$$F_i^\lambda = F_{i\alpha}^\lambda\, u^\alpha,$$

with the coefficients

$$\boxed{F_{i\alpha}^\lambda = \partial_i f_\alpha^\lambda - f_\beta^\lambda \Gamma_{i\alpha}^\beta + (\Lambda_{a\mu}^\lambda \circ f) f_i^a f_\alpha^\mu.} \tag{27}$$

Remarks.

3.6. The block (26) appears in the process of transformation of the (Jacobian) matrix of the linear map TF with respect to the change of bases, i.e., when passing from the natural bases to the adapted ones:

$$\begin{pmatrix} f_j^b & 0 \\ f_j^\mu & f_\beta^\mu \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} \delta_b^a & 0 \\ -\Lambda_b^\lambda & \delta_\mu^\lambda \end{pmatrix}_{\circ F} \cdot \begin{pmatrix} f_j^b & 0 \\ f_j^\mu & f_\beta^\mu \end{pmatrix} \cdot \begin{pmatrix} \delta_i^j & 0 \\ \Gamma_i^\beta & \delta_\alpha^\beta \end{pmatrix}.$$

The block F_i^λ is a mixed tensor by virtue of relations (19). It is different from the block f_i^λ of the Jacobian matrix.

3.7. The coefficients (27) appear in the formula (26), if we take into account the linearity of the functions $f^\lambda = f^\lambda_\alpha u^\alpha$. The quantities $F^\lambda_{i\alpha}$ generate a mixed tensor.

3.8. A smooth mapping

$$f : M_1 \longrightarrow M_2$$

gives rise to a morphism of the tangent bundles (of the first floors)

$$\begin{array}{ccc} TM_1 & \xrightarrow{Tf} & TM_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

The coordinates (u^i) and (v^a), given on the neighborhoods $U \subset M_1$ and $\widetilde{U} \subset M_2$, induce on the neighborhoods $TU \subset TM_1$ and $T\widetilde{U} \subset TM_2$ the coordinates (u^i, u^i_1) and (v^α, v^α_1). The tangent map Tf is defined by the system

$$\begin{cases} v^\alpha = f^\alpha, \\ v^\alpha_1 = f^\alpha_1 = f^\alpha_i u^i_1, \qquad f^\alpha_i \doteq \dfrac{\partial f^\alpha}{\partial u^i}. \end{cases}$$

Consider on the manifolds M_1 and M_2 the affine connections with coefficients Γ^k_{ij} and $\Lambda^\alpha_{\beta\gamma}$ on the neighborhoods U and $\widetilde{U}$, and assume that on the neighborhoods TU and $T\widetilde{U}$ we have both natural bases and adapted ones. The second tangent map $T^2 f$ is defined in the natural and in the adapted bases by matrices with different lower-left blocks:

$$\begin{pmatrix} f^\alpha_j & 0 \\ f^\alpha_{jk} u^k_1 & f^\alpha_j \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} f^\alpha_j & 0 \\ F^\alpha_{jk} u^k_1 & f^\alpha_j \end{pmatrix}.$$

The block $(f^\alpha_j)_1 = f^\alpha_{jk} u^k_1$ of the Jacobian matrix is replaced by the block

$$F^\alpha_i = F^\alpha_{ij} u^j_1, \quad \text{where}$$

$$\boxed{F^\alpha_{ij} = f^\alpha_{ij} - f^\alpha_k \Gamma^k_{ij} + (\Lambda^\alpha_{\beta\gamma} \circ f) f^\beta_i f^\gamma_j,} \quad f^\alpha_i \doteq \frac{\partial f^\alpha}{\partial u^i}, \; f^\alpha_{ij} \doteq \frac{\partial^2 f^\alpha}{\partial u^i \partial u^j}. \tag{28}$$

Formulas (26), (27) and (28) represent *fundamental formulas* of the theory of connections. These objects appear quite frequently in a way or another in differential-geometric constructions. The question is where to consider these connections. Riemannian geometry and its dual (co-Riemannian geometry) give a univocal answer to this question. In the Cartan method the connections appear by the use of nonholonomic bases.

Remarks.

3.9. Riemannian geometry. In the case when f is an immersion into the Euclidean space $\tilde{M}$, the relation (28) becomes *Gauss' formula* from the theory of surfaces.

Assume that f is an immersion from an n-dimensional smooth manifold M into the $(n+r)$-dimensional Euclidean space $\tilde{M}$. The image $f(M) \subset \tilde{M}$ represents an *n-dimensional surface*, locally given by the parametric equations

$$u^I = f^I(t^i), \quad i = 1, ..., n, \; I = 1, ..., n+r.$$

The linearly independent columns of the Jacobian matrix $f_i^I = \dfrac{\partial f^I}{\partial t^i}$ generate in the tangent plane to the surface $f(M)$ a vector basis. The Gram matrix consisting of the scalar products of these vectors is regular and thus, invertible:

$$g_{ij} \doteq f_i^I f_j^I \qquad \rightsquigarrow \qquad (g_{ij})^{-1} \doteq (g^{ij}).$$

On the surface $f(M)$, it is defined the *metric* and the *first fundamental form.* The quantities Γ_{ij}^k and Λ_{JK}^I are fixed as follows. First of all, we set $\Lambda_{JK}^I = 0$ – this property, in the Euclidean space $\tilde{M}$, is an invariant one. Second, we impose the condition $f_k^I F_{ij}^I = 0$. Therefore, we get the expressions of the *Christoffel symbols*:

$$f_k^I F_{ij}^I = 0 \quad \rightsquigarrow \quad f_k^I f_{ij}^I - g_{kl}\Gamma_{ij}^l = 0 \quad \rightsquigarrow \quad \Gamma_{ij}^k = f_l^I f_{ij}^I g^{kl}. \tag{29}$$

The vectors F_{ij}^I (with fixed indices i and j) belong to the normal plane to the surface $f(M)$ and they can be expressed in the vector basis as (n_α^I), $\alpha = 1, \ldots, r$:

$$F_{ij}^I = n_\alpha^I h_{ij}^\alpha \ .$$

The coefficients h_{ij}^α define the *second fundamental form* of the surface $f(M)$ with values in the normal plane. We obtain the famous *Gauss' formula* in the theory of surfaces:

$$f_{ij}^I = f_k^I \Gamma_{ij}^k + n_\alpha^I h_{ij}^\alpha.$$

On the surface $f(M)$ it is thus defined a *Riemannian geometry.*

3.10. Co-Riemannian geometry. Let $\varphi : \tilde{M} \to M$ be a submersion from the $(n+r)$-dimensional Euclidean space $\tilde{M}$ to an r-dimensional smooth manifold M. The space $\tilde{M}$ is fibered into an r-parameter family of n-dimensional fibers. Locally, φ is defined by the system

$$v^\alpha = \varphi^\alpha(u^I), \quad \alpha = 1, ..., r, \ I = 1, ..., n+r.$$

The lines of the Jacobian matrix $\varphi_I^\alpha = \dfrac{\partial \varphi^\alpha}{\partial u^I}$ are linearly independent gradient vectors, transversal to the fibers. The Gram matrix consisting of their scalar products is a regular (invertible) one:

$$g^{\alpha\beta} \doteq \varphi_I^\alpha \varphi_I^\beta \qquad \rightsquigarrow \qquad (g^{\alpha\beta})^{-1} \doteq (g_{\alpha\beta}).$$

In the bundle φ it is defined the so-called *co-metric*. We notice that the quantities $g^{\alpha\beta}$ are defined on the space $\tilde{M}$, but, by their indices, they belong to the manifold M and, with respect to coordinate changes on M, they transform as the components of a tensor.

Let us rewrite the object (28) in a different form:

$$\Phi_{IJ}^\alpha = \varphi_{IJ}^\alpha - \varphi_K^\alpha \Gamma_{IJ}^K + \Lambda_{\beta\gamma}^\alpha \varphi_I^\beta \varphi_J^\gamma.$$

We set $\Gamma_{IJ}^K = 0$, which has in the space $\tilde{M}$ an invariant meaning. Second, we impose the condition $\Phi_{IJ}^\alpha \varphi_I^\beta \varphi_J^\gamma = 0$, and define the coefficients $\Lambda_{\beta\gamma}^\alpha \circ \varphi$ (we will not indicate explicitly the composition $\circ\,\varphi$), as follows:

$$\Phi_{IJ}^\alpha \varphi_I^\beta \varphi_J^\gamma = 0 \quad \rightsquigarrow \quad \varphi_{IJ}^\alpha \varphi_I^\beta \varphi_J^\gamma + \Lambda_{\lambda\mu}^\alpha g^{\lambda\beta} g^{\mu\gamma} = 0 \quad \rightsquigarrow \quad \Lambda_{\beta\gamma}^\alpha = -g_{\lambda\beta} g_{\mu\gamma} \varphi_{IJ}^\alpha \varphi_I^\lambda \varphi_J^\mu. \tag{30}$$

These are the so-called *co-Christoffel symbols* $\Lambda_{\beta\gamma}^\alpha$ which compose a tensor object:

$$\Phi_{IJ}^\alpha = \varphi_{IJ}^\alpha - g_{\lambda\beta} g_{\mu\gamma} \varphi_{KL}^\alpha \varphi_K^\lambda \varphi_L^\mu \varphi_I^\beta \varphi_J^\gamma.$$

These are the bases of *co-Riemannian geometry* [17], in which the object of study are families of surfaces in the Euclidean space $\tilde{M}$, as the fibers of the submersion φ.

3.11. Geodesics and co-geodesics. The equality $F^{\alpha}_{ij} = 0$ provides, for $\dim M_1 = 1$, the equations of geodesic lines, and for $\dim M_2 = 1$, the equation of a co-geodesic field.

3.12. Cartan's test. A smooth mapping

$$f : M \to \tilde{M},$$

is represented on the neighborhoods $U \subset M$ and $\tilde{U} = f(U) \subset \tilde{M}$, with coordinates (u^i) and (v^α) by the following system:

$$v^\alpha \circ f = f^\alpha, \quad i = 1, 2, \ldots, \dim M; \ \alpha = 1, 2, , \ldots, \dim \tilde{M}.$$

Consider on the manifolds M and $\tilde{M}$ the nonholonomic bases (X_i, ω^j) and (Y_α, θ^β). On U and $\tilde{U}$, these bases are defined with respect to the natural bases by the matrices A, A^{-1} and B, B^{-1}:

$$\begin{aligned} (X_i, \omega^j) \quad &\rightsquigarrow \quad X_i = \partial_j \bar{A}^j_i, \quad \omega^j = A^j_i du^i, \\ (Y_\alpha, \theta^\beta) \quad &\rightsquigarrow \quad Y_\alpha = \partial_\beta \bar{B}^\beta_\alpha, \quad \theta^\beta = B^\beta_\alpha dv^\alpha, \end{aligned}$$

A linear map between the tangent spaces

$$T_u f : T_u M \to T_v \tilde{M}, \quad v = f(u),$$

is defined in the natural bases (∂_i, du^j) and $(\partial_\alpha, dv^\beta)$ by means of the Jacobian matrix (f^α_i), and in the nonholonomic bases (X_i, ω^j), (Y_α, θ^β), by the matrix (F^α_i), or by the vector-valued form

$$\mathcal{F} = \partial_\alpha \otimes dv^\alpha = Y_\alpha \otimes \theta^\alpha\,, \quad \text{where:}$$

$$dv^\alpha \circ Tf = f^\alpha_i du^i, \qquad f^\alpha_i \doteq \frac{\partial f^\alpha}{\partial u^i}, \tag{31}$$

$$\theta^\alpha \circ Tf = F^\alpha_i \omega^i, \qquad F^\alpha_i \doteq (B^\alpha_\beta \circ f) f^\beta_j \bar{A}^j_i. \tag{32}$$

For the sake of simplicity, we will represent equation (31) in the form $dv^\alpha = f^\alpha_i du^i$, and equation (32), in the form $\theta^\alpha = F^\alpha_i \omega^i$. Actually, this is the differential equation of a single mapping f, but, in the first case, it is represented in local form in the natural bases and, in the second case, in the nonholonomic bases, not depending on the coordinates.

Cartan's test consists of the following. Using the structure equations for the forms ω^i and θ^α (see line 1), one calculates the exterior derivative of the equation $\theta^\alpha = F^\alpha_i \omega^i$ (line 2) and, in the result (line 3), one uses Cartan's lemma (line 4):

$$\mathrm{d}\,\omega^i = -\frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k, \qquad \mathrm{d}\,\theta^\alpha = -\frac{1}{2} \tilde{c}^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma\,,$$

$$-\frac{1}{2} (\tilde{c}^\alpha_{\beta\gamma} \circ f) F^\beta_i F^\gamma_j \omega^i \wedge \omega^j = dF^\alpha_j \wedge \omega^j - \frac{1}{2} F^\alpha_k c^k_{ij} \omega^i \wedge \omega^j,$$

$$\Big\{ dF^\alpha_j - \frac{1}{2} F^\alpha_k c^k_{ij} \omega^i + \frac{1}{2} (\tilde{c}^\alpha_{\beta\gamma} \circ f) F^\beta_i F^\gamma_j \omega^i \Big\} \wedge \omega^j = 0\,,$$

$$dF^\alpha_j = \Big\{ \frac{1}{2} F^\alpha_k c^k_{ij} - \frac{1}{2} (\tilde{c}^\alpha_{\beta\gamma} \circ f) F^\beta_i F^\gamma_j + \lambda^\alpha_{ij} \Big\} \omega^i\,.$$

[17] A joke by M. Spivak: the analogues of the Gauss-Codazzi equations appearing in the dual case were called by him the *Dazzi-Cogauss equations*.

There appears an object λ^α_{ij}, which is symmetric in its lower indices. This object defines the symmetric part of the differential dF^α_i. On the other hand, we can apply to F^α_i the operator $d = X_i \otimes \omega^i$, define the differential dF^α_i and distinguish directly its antisymmetric and its symmetric parts:

$$dF^\alpha_j = X_i F^\alpha_j \omega^i = (X_{[i}F^\alpha_{j]} + X_{(i}F^\alpha_{j)})\omega^i,$$

Comparing the two expressions for dF^α_i, we conclude:

$$\lambda^\alpha_{ij} = X_{(i}F^\alpha_{j)} = B^\alpha_\beta F^\beta_{kl} \bar{A}^k_i \bar{A}^l_j\,,$$

where

$$F^\alpha_{ij} = f^\alpha_{ij} - f^\alpha_k \Gamma^k_{ij} + (\Lambda^\alpha_{\beta\gamma} \circ f) f^\beta_i f^\gamma_j, \tag{33}$$

with the connection coefficients

$$\Gamma^k_{ij} = \bar{A}^k_s \partial_{(i} A^s_{j)}, \quad \Lambda^\gamma_{\alpha\beta} = \bar{B}^\gamma_\sigma \partial_{(\alpha} B^\sigma_{\beta)}. \tag{34}$$

This way, the second tangent map

$$T^2 f : T^2 M \to T^2 \tilde{M}$$

is defined in the natural bases, as usually, by the Jacobian matrix with the lower-left block $(f^\alpha_i)_1 = f^\alpha_{ij} u^j_1$, and in the adapted ones – by the matrix with lower-left block $F^\alpha_{ij} u^j_1$, with the connections (33):

$$\begin{pmatrix} f^\alpha_i & 0 \\ (f^\alpha_i)_1 & f^\alpha_i \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} f^\alpha_i & 0 \\ F^\alpha_{ij} u^j_1 & f^\alpha_i \end{pmatrix}.$$

By the choice of the nonholonomic bases, Cartan's method anticipates the link between the frames and the structure of the manifold under study. For instance, in the theory of surfaces, the Darboux frame is related to principal directions and, in projective geometry – to the Wiltschinsky directrices. This is how the frame is related to congruences and to complex lines in line geometry etc.

G.F. Laptev called objects (33), appearing in the process of Cartan differential prolongations, the *fundamental objects of the mapping,* [9].

4 Subbundles of the floors

4.1 Osculating bundles

On the k-th floor $T^k M$ of an n-dimensional manifold M, the equality of the projections[18]

$$\rho_1 = \rho_2 = \ldots = \rho_k \tag{35}$$

defines a kn-dimensional subbundle $\mathrm{Osc}^{k-1} M$,

$$\boxed{\mathrm{Osc}^{k-1} M \subset T^k M,\ k = 2, 3, \ldots} \tag{36}$$

This subbundle of the k-th floor $T^k M$ is called the *osculating bundle of order* $k-1$ *of the manifold* M.

According to the definition, the osculating bundle $\mathrm{Osc}^{k-1} M$ consists precisely of those elements of the floor $T^k M$, which have a common image through all projections (35).

[18]In the same way as on the plane xy, the equality of the coordinate functions $x = y$ defines a straight line.

In a local chart, the coordinates with the same number of lower indices are equal. Thus, on a domain of local chart of the second floor $T^2U \subset T^2M$ the elements belonging to the bundle Osc$M$ are those which satisfy the equality: $u_1^i = u_2^i$. On a neighborhood of the third floor $T^3U \subset T^3M$, the elements belonging to the osculating bundle Osc^2M are those for which $u_1^i = u_2^i = u_3^i$ and $u_{12}^i = u_{13}^i = u_{23}^i$, etc. It is clear, judging by the number of coordinates, that

$$\mathrm{dimOsc}M = 3n, \quad \mathrm{dimOsc}^2M = 4n \quad \text{etc.}$$

The coordinates on the osculating bundle can be denoted either by $u^i, u_1^i, u_{11}^i, u_{111}^i, \ldots$, or by $u^i, \overset{(1)}{u}{}^i, \overset{(2)}{u}{}^i, \ldots \overset{(k)}{u}{}^i, \ldots$, but the notation with differentials $u^i, du^i, d^2u^i, d^3u^i, \ldots$ is not appropriate, since the meaning of higher order differentials on the floors is a different one.

We notice that the subbundle $\mathrm{Osc}M \subset T^2M$ is an integral surface of a $3n$-dimensional distribution – the linear span of the operators

$$\langle\, \partial_i,\, \partial_i^1 + \partial_i^2,\, \partial_i^{12}\, \rangle.$$

The functions $(u_1^i - u_2^i)$ are invariants of these operators.

Similarly, the subbundle $\mathrm{Osc}^2M \subset T^3M$ is the integral surface of a $4n$-dimensional distribution, namely, the linear span of the operators

$$\langle\, \partial_i,\, \partial_i^1 + \partial_i^2 + \partial_i^3,\, \partial_i^{12} + \partial_i^{23} + \partial_i^{13},\, \partial_i^{123}\, \rangle.$$

For these operators, the functions $(u_1^i - u_2^i,\, u_1^i - u_3^i,\, u_{12}^i - u_{23}^i,\, u_{13}^i - u_{23}^i)$ are invariants.

Remark 4.1. A vector field $\overset{(2)}{X} = \partial_i x^i + \partial_i^1 x_1^i + \partial_i^2 x_2^i + \partial_i^{12} x_{12}^i$ on the floor T^2M, with equal components $x_1^i = x_2^i$ is tangent to the surface Osc$M$,

$$(x_1^i = x_2^i) \;\Rightarrow\; \overset{(2)}{X} = \partial_i x^i + (\partial_i^1 + \partial_i^2) x_1^i + \partial_i^{12} x_{12}^i.$$

4.2 Lagrange – Hamilton

A considerable contribution to the development of analytic mechanics was brought by Lagrange and Hamilton[19]. Their approaches are different, but, as is well known, the Legendre transformation allows us to transform the Hamilton system into the Lagrange equations. We could say that Hamilton's theory, which is built on the $4n$-dimensional second floor T^2M of the manifold M, reduces, on the $3n$-dimensional osculating bundle Osc$M$, to Lagrange's theory. Hamiltonian theory is a generalization of Lagrange's one.

A scalar function $H = H(u, u_1)$ defined on the floor TM is called a *Hamiltonian.* To a Hamiltonian H, it is associated on the floor TM a vector field X,

$$X = \sum_i H_{u_1^i} \partial_i - \sum_i H_{u^i} \partial_i^1, \quad H_i \doteq \frac{\partial H}{\partial u^i}, \quad H_{u_1^i} \doteq \frac{\partial H}{\partial u_1^i}. \tag{37}$$

[19] J.L.Lagrange (1736-1813), W.R.Hamilton (1805-1865), A.M.Legendre (1752-1833).

With respect to the vector field X, the function H and the *symplectic form* $\Omega = du^i \wedge du_1^i$ are invariant: $XH = 0$, $\mathcal{L}_X \Omega = 0$.

The flow $a_t = \exp tX$ is defined by the system

$$\begin{cases} \dot{u}^i = H_{u_1^i}\,, \\ \dot{u}_1^i = -H_{u^i}\,, \end{cases} \qquad \dot{u}^i \doteq \frac{du^i}{dt}\,,\ \dot{u}_1^i \doteq \frac{du_1^i}{dt}\,. \tag{38}$$

The system (38), called the *Hamiltonian system*, defines a section of the second floor (indices of the coordinates are omitted):

$$\pi : T^2M \to TM : (u, u_1, u_2, u_{12}) \rightsquigarrow (u, u_1), \begin{cases} u_2 = \dot{u}(u, u_1)\,, \\ u_{12} = \dot{u}_1(u, u_1). \end{cases} \tag{39}$$

Proposition 4.1. *The Hamiltonian system* (38) *reduces on the osculating bundle* Osc$M \subset T^2M$ *to the Lagrange system*

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{u}^i}\Big) - \frac{\partial L}{\partial u^i} = 0. \tag{40}$$

P r o o f. *The Legendre transformation* provides the transition from the Hamiltonian $H = H(u, u_1)$ to the *Lagrangian* $L = L(u, u_2)$ on the floor T^2M, under the condition

$$H(u, u_1) - \sum_i u_1^i u_2^i + L(u, u_2) = 0.$$

Generally, this transition is not possible on the floor T^2M:

$$d(H - \sum_i u_1^i u_2^i + L) \equiv 0 \iff (H_{u^i} + L_{u^i} = 0,\ H_{u_1^i} = u_2^i,\ L_{u_2^i} = u_1^i),$$

since, by hypothesis, the function H does not depend on the coordinates u_2 and the function L does not depend on the coordinates u_1. The transition $H \rightsquigarrow L$ is only possible under the assumption that $u_1^i = u_2^i = \dot{u}$, i.e., on the osculating bundle Osc$M$. Under this condition, the system (38) is indeed reduced to the system (40).

Let us add that the system (39) defines a $2n$-dimensional section of the bundle T^2M, while the system (40) defines a $2n$-dimensional section of the bundle Osc$M$. Consequently, Hamilton geometry on the floor T^2M completely reduces to Lagrange geometry on the bundle Osc M. □

4.3 Jacobi equation and connections on T^2M

On a Riemannian manifold (M, g), when studying the first variation of the arc length, one naturally works on the first floor TM – and, as a result, it is determined a connection (called the *canonical* or *Cartan* connection, [16]) on this space, with coefficients

$$N_j^i = \gamma_{jk}^i(u) u_1^k.$$

In the notations of the previous sections, N_j^i is actually $-\Gamma_j^i$. The main property of this connection is that its autoparallel curves coincide with the geodesics of g.

Similarly, the geodesic deviation equation "lives" on the second floor T^2M and thus, it will naturally give rise to connections on this bundle. We can immediately realize this by the presence in it of two vector fields – the *velocity* vector field and the *deviation* vector field.

As shown above, T^2M has the structure of a 2-fold linear bundle, see Section 1.1, with fibrations

$$T^2M \underset{\rho_2}{\overset{\rho_1}{\rightrightarrows}} TM \text{ and } T^2M \xrightarrow{\pi} TM, \text{ where } \pi := \pi_1\rho_1 = \pi_1\rho_2. \tag{41}$$

If $(u, u_1, u_2, u_{12}) =: (u^i, u^i_1, u^i_2, u^i_{12})$ are the local coordinates of a point $p \in T^2M$, then:

$$\rho_1(p) = (u, u_2), \quad \rho_2(p) = (u, u_1), \quad \pi(p) = u.$$

To the linear mappings ρ_1 and ρ_2, there correspond two vertical distributions: $\Delta_{v_1} = \ker T\rho_1$ and $\Delta_{v_2} = \ker T\rho_2$ of dimension $2n$, $n = dimM$, with the n-dimensional intersection $\Delta_{12} = \Delta_{v_1} \cap \Delta_{v_2}$ and the $3n$-dimensional sum $\Delta_{v_1} + \Delta_{v_2} = kerT\pi$.

We define a *connection* on the second floor T^2M as a splitting

$$\Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12}, \tag{42}$$

where $\Delta_{v_1} = \Delta_2 \oplus \Delta_{12}$, $\Delta_{v_2} = \Delta_1 \oplus \Delta_{12}$. The horizontal distributions for the three fibrations are:

$$\Delta_{h_i} = \Delta \oplus \Delta_i \text{ for } \rho_i,\ i = 1, 2 \text{ and } \Delta_h = \Delta \text{ for } \pi.$$

Each of the distributions $\Delta, \Delta_1, \Delta_2, \Delta_{12}$ has the dimension n.

To the coordinate functions on the neighborhood T^2U, it corresponds the *adapted basis*, see (17) – the frame

$$(X_i, X^1_i, X^2_i, X^{12}_i)$$

and the dual coframe

$$(U^i_3, U^i_{13}, U^i_{23}, U^i_{123}).$$

The coframe is defined as in (15); in matrix writing,

$$\begin{pmatrix} U^i_3 \\ U^i_{13} \\ U^i_{23} \\ U^i_{123} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ N^{i_1}_j & \delta^i_j & 0 & 0 \\ N^{i_2}_j & 0 & \delta^i_j & 0 \\ M^{i_{12}}_j & N^{i_{12}}_{j_1} & N^{i_{12}}_{j_2} & \delta^i_j \end{pmatrix} \cdot \begin{pmatrix} u^j_3 \\ u^j_{13} \\ u^j_{23} \\ u^j_{123} \end{pmatrix} \tag{43}$$

where $N^{i_1}_{\ j} = N^{i_1}_{\ j}(u, u_1)$, $N^{i_2}_{\ j} = N^{i_2}_{\ j}(u, u_2)$ and $N^{i_{12}}_{\ j_1}, N^{i_{12}}_{\ j_2}, M^{i_{12}}_{\ j}$ may depend[20] on all the variables u, u_1, u_2, u_{12}.

The frame is defined by the inverse matrix

$$\begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ -N^{i_1}_{\ j} & \delta^i_j & 0 & 0 \\ -N^{i_2}_{\ j} & 0 & \delta^i_j & 0 \\ -\tilde{M}^{i_{12}}_{\ j} & -N^{i_{12}}_{\ j_1} & -N^{i_{12}}_{\ j_2} & \delta^i_j \end{pmatrix}, \ M^{i_{12}}_{\ j} - \tilde{M}^{i_{12}}_{\ j} = N^{i_{12}}_{\ i_1} N^{i_1}_{\ j} + N^{i_{12}}_{\ i_2} N^{i_2}_{\ j}.$$

Thus, a vector field X and a 1-form θ on T^2M are split into invariant blocks as

$$X = x^i X_i + x^i_1 X^1_i + x^i_2 X^2_i + x^i_{12} X^{12}_i, \ \theta = \theta_i U^i_3 + \theta^1_i U^i_{13} + \theta^2_i U^i_{23} + \theta^{12}_i U^i_{123}.$$

Remark 4.2. With respect to coordinate changes on T^2M, the connection coefficients transform as:

$$N^{i'_\alpha}_{\ j'_\beta} = a^{i'}_{\ i}(a^j_{\ j'} N^{i_\alpha}_{\ j_\beta} + a^{i_\alpha}_{\ j'_\beta}), \tag{44}$$

$$M^{i'_{12}}_{\ j'} = a^{i'}_{\ i}(a^j_{j'} M^{i_{12}}_{\ j} + a^{j_1}_{j'} N^{i_{12}}_{\ j_1} + a^{j_2}_{j'} N^{i_{12}}_{\ j_2} + a^{i_{12}}_{\ j'}), \quad \alpha, \beta \in \{1, 2, (12)\}, \tag{45}$$

where indices designated by the same letter have the same numerical values (and are subject to Einstein summation convention) if and only if they correspond to the same local chart, e.g., $i = i_\alpha$ (and we perform summation by these), but i is not equal to i' (and no summation is performed). Conversely, if the functions $N^{i_\alpha}_{j_\beta}$ and $M^{i_{12}}_j$ obey the rules (44), (45), they define a connection on T^2M.

Consider now:

- a smooth curve $u : [0,1] \to M$ and
- a variation $\alpha : [0,1] \times (-\varepsilon_0, \varepsilon_0) \to M$ of u, with fixed endpoints.

The variation α determines a 2-dimensional surface $\tilde{\alpha}$ in T^2M:

$$\tilde{\alpha}(t, \varepsilon) = (\alpha(t, \varepsilon), \alpha'_t(t, \varepsilon)), \alpha'_\varepsilon(t, \varepsilon), \alpha''_{t\varepsilon}(t, \varepsilon)). \tag{46}$$

Its coordinate curve $\varepsilon = 0$ is a curve on T^2M, which we will call the *lift of the variation* α. Along this curve, we have:

$$u = u(t), \ u_1 = \frac{du}{dt}, \quad u_2 = v, \quad u_{12} = \frac{dv}{dt},$$

where $v = \alpha'_\varepsilon(t, 0)$ is the deviation vector field.

Remark 4.3. In particular, if $\alpha^i(t, \varepsilon) = u^i(t) + \varepsilon v^i(t)$, the deviation vector field $v = \tilde{\alpha}'_\varepsilon|_{\varepsilon=0}$ coincides with the velocity vector $\dot{u}$. The lift to T^2M of the variation is $\tilde{u}(t) =$

[20] These conditions are to insure that the vector fields X_i are projectable onto TM (with respect to both fibrations ρ_1 and ρ_2).

$(u(t), \dot{u}(t), \dot{u}(t), \ddot{u}(t))$, which can be identified with the curve $t \mapsto (u(t), \dot{u}(t), \ddot{u}(t))$ on the osculating bundle Osc$M$. This way, the lift of u to Osc$M \subset T^2M$ is a particular case of a lift of a variation of u to T^2M.

Raising to T^2M, geodesics parametrized by the arclength $t = s$ of the base manifold M are described by

$$\frac{du_1^i}{dt} + \gamma^i{}_{jk} u_1^j u_1^k = 0. \tag{47}$$

The geodesic deviation equation:

$$\frac{\nabla^2 v^i}{dt^2} = R_j{}^i{}_{kl} \dot{u}^j v^k \dot{u}^l,$$

is written, in terms of partial derivatives (and substituting $\dot{u}^i = u_1^i, v^i = u_2^i$), as:

$$\begin{aligned} du_{12}^i + (\gamma^i_{lk} u_2^k) du_1^l + (\gamma^i{}_{lk} u_1^k) du_2^l + \\ + \left[\frac{d}{dt}(\gamma^i{}_{kl} u_2^k) + (\gamma^i{}_{kh} u_1^h)(\gamma^k_{jl} u_2^j) - u_2^j u_1^h R_h{}^i{}_{jl} \right] du^l = 0. \end{aligned} \tag{48}$$

This suggests us to define a linear connection:

$$N^{i_1}_{\ l} = N^{i_{12}}_{\ l_2} := \gamma^i{}_{lk} u_1^k, \ \ N^{i_2}_{\ l} = N^{i_{12}}_{\ l_1} := \gamma^i{}_{lk} u_2^k, \tag{49}$$

$$M^{i_{12}}_{\ l} = C_1(N^{i_2}_{\ l}) + N^{i_1}_{\ k} N^{k_2}_l - u_2^j u_1^h R^{\ i}_{hj\,l}\,, \tag{50}$$

where C_1 is an arbitrary vector field of the form:

$$C_1 = u_1^k\,\partial_k + u_{12}^k\,\partial_k^2 + G_1^k\,\partial_k^1 + G_{12}^k\,\partial_k^{12}.$$

For the functions $N^{i_2}_{\ l} = N^{i_2}_{\ l}(u, u_2)$, we have, actually,

$$C_1 N^{i_2}_{\ l} = u_1^k\,\partial_k N^{i_2}_{\ l} + u_{12}^k\,\partial_k^2 N^{i_2}_{\ l}.$$

These functions obey the rules of transformation (44), (45), hence, they define a connection on T^2M.

In terms of this connection, the geodesic equation is written $U^i_{13}(\dot{\tilde{u}}) = 0$, while the 1-forms U^i_{123} serve to describe the Jacobi equation. More precisely,

Proposition 4.2. *Let $c : u = u(t)$ be a curve on M, v be a vector field along u and the following curve on T^2M :*

$$\tilde{u}(t) := \big(\, u = u(t),\, u_1 = \frac{du}{dt}(t),\, u_2 = v(t),\, u_{12} = \frac{dv}{dt}\,\big).$$

Then:

a) c is a geodesic if and only if $U^i_{13}(\dot{\tilde{u}}) = 0$;

b) if c is a geodesic, then v is a Jacobi field along c iff $U^i_{123}(\dot{\tilde{u}}) = 0$.

5 Historical background

Subsequent differentiation, or the so-called differential prolongation defined an important problem in the geometry of the last century. We shall mention here three directions. First of all, if to the coordinate functions (u^i) one successively adjoins the differentials $du^i, d^2u^i, \ldots$, then the dimension of the space is each time increased by their number: $n, 2n, 3n, \ldots$ In this direction, it was done a substantial work (E. Bompiani, V. Vagner) and, in particular, by the Romanian school of geometry (R. Miron, Gh. Atanasiu, [3, 10]). Second, French mathematicians promoted the so-called jet bundle approach (Ch. Ehresmann, A. Roux, [5, 14]), which was adopted also in other countries (I. Kolař, W.F. Pohl [11]). Third, differential prolongations were remarked in the iterations of the tangent functor. This is the approach that we have adopted here. Its cornerstone is the fact that the tangent bundles (floors) have a structure of multiple vector bundles. French mathematicians were also the first to pay attention to this fact (Ch. Ehresmann, J. Pradines, [7, 12]). Cl. Godbillon noticed the fundamental role of the second floor in the interpretation of Hamiltonian systems, [8]. A more detailed analysis was made more recently (W. Bertram, J.T. White, [5, 17], the second author started to study floors from 1962, [13]). From a technical point of view, it was necessary to introduce new notations and a convenient indexing, in order to obtain a comfortable description of iterative structures. It was thus born a new theory: the theory of sector-forms, which is added to jet structures. Moreover, floors include the theory of higher order motions, in particular, of the interactions between fields and flows, a domain which has been little studied up to now.

We have insisted a little more on the subject of connections in bundles. The structure $\triangle_h \oplus \triangle_v$ (Ch. Ehresmann, [6]) is a generalization of the idea of line integral, where the transport of fibers depends on the choice of the path. Still, a deeper meaning is hidden in the interaction of non-commuting fields and of curvature of the space. One had to give up holonomic reference frames and the first decisive step in this direction was made by J.A. Schouten, by the use of nonholonomic bases and of nonholonomy objects, [15]. Specializing the basis in the structure $\triangle_h \oplus \triangle_v$, it is designed a scheme which includes a series of classical theories: Lie groups, representations, symmetries, movements, curvature of the space, morphisms of bundles with connections, invariants of mappings, Cartan's test etc. One can also speak about higher order connections, given by structures of the type $\triangle_h \oplus \triangle_{hv} \oplus \triangle_v$ and $\triangle \oplus \triangle_1 \oplus \triangle_2 \oplus \triangle_{12}$, in which it is performed a specialization of the basis of the required type, [2, 4].

The beauty of the category-theoretic approach consists of the fact that the structures are defined in an invariant manner, without resorting to any coordinate system and that the transition from a floor to the next one is a repetitive process – i.e., it is thus created an iterative calculus.

References

[1] Asanov G.S., *Fibered Generalization of the Gauge Field Theory. Finslerian and Jet Gauge Fields* (in Russian), Univ. Moscow Eds., Moscow, 1989.

[2] Atanasiu Gh., Balan V., Brînzei N., Rahula M., *Differential-Geometric Structures. Tangent Bundles, Connections in Fiber Bundles, Exponential Law and Jet Spaces* (in Russian), Librokom Eds., Moscow, 2010.

[3] Atanasiu Gh., Balan V., Brînzei N., Rahula M., *Differential Geometry of Second Order. Miron-Atanasiu Theory* (in Russian), Librokom Eds., Moscow, 2010.

[4] Balan V., Rahula M., Voicu N., *Tangent Structures in Geometry and Their Applications*, KRASAND, Moscow, 2013.

[5] Bertram W., *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings*, Memoirs of AMS 900, 2008.

[6] Ehresmann Ch., *Les connexions infinitésimales dans un espace fibré différentiable*, Coll. de Topologie, Bruxelles, CBRM (1950), 29-55; Œuvres Complètes, t. I, 28, 179-204.

[7] Ehresmann Ch., *Catégories doubles et catégories structurées*, C.R. Acad. Sci., Paris, 256(1958), 1198-1201.

[8] Godbillon Cl., *Géométrie Différentielle et Mécanique Analytique*, Hermann, Paris, 1969.

[9] Laptev G.F., *About invariant analytic theory of differentiable mappings* (in Russian), Tr. Geom. Sem., VINITI, 6(1974), 37-42.

[10] Miron R., *The Geometry of Higher Order Finsler Spaces*, Hadronic Press, USA, 1998.

[11] Pohl W.F., *Differential geometry of higher order*, Topology, 1(1962), 169-211.

[12] Pradines J., *Suites exactes vectorielles doubles et connexions*, C.R. Acad. Sci., Paris, 278(1974), 1587-1590.

[13] Rahula M., *Consecutive derivations on a variety* (in Russian), Utch. Zapiski Tartu-skogo Un-ta., 129(1962), 37-61.

[14] Roux A., *Jets et connexions*, Pub. Dep. Math., 7(1970), no. 4, 1-43.

[15] Schouten J.A., *Über nicht-holonome Übertragungen in einer L_n*, Math. Zeitschrift, 30, S, 1929, 149-172.

[16] Shen, Z., *Lectures on Finsler Geometry*, World Scientific, 2001.

[17] White J.T., *The Method of Iterated Tangents with Applications in Local Riemannian Geometry*, Pitman Publ., 1982.

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