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On the efficiency of some p-groups

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Abstract

Let p be a prime number. In this paper, we work on the efficiency of the p-groups G_1 and G_2 defined by the presentations,

$$\mathcal{P}_{G_1} = \left\langle a,\ b,\ c\ ;\ ab = bac,\ bc = cb,\ ac = ca,\ a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ c^{p^\gamma} = 1 \right\rangle$$

where $\alpha \geq \beta > \gamma \geq 1$ and

$$\mathcal{P}_{G_2} = \left\langle a,\ b\ ;\ ab = ba^{1+p^{\alpha-\gamma}},\ a^{p^\alpha} = 1,\ b^{p^\beta} = 1 \right\rangle$$

where $\alpha \geq 2\gamma$, $\beta > \gamma \geq 1$ and $\alpha + \beta > 3$. For example, if we let $p = 2$, then by [1], the groups defined by these presentations becomes 2-groups. It is known that these groups play an important role in the theory of groups of nilpotency class 2.

1 Introduction

Let G be a finitely presented group with a presentation

$$\mathcal{P} = \langle \mathbf{x}\ ;\ \mathbf{r} \rangle\,. \tag{1}$$

Then the deficiency of this presentation is defined by $|\mathbf{r}| - |\mathbf{x}|$, and is denoted by $def(\mathcal{P})$. Moreover, the group deficiency of a finitely presented group G is given by

$$def_G(G) = min\{def(\mathcal{P})\ :\ \mathcal{P}\ is\ a\ finite\ group\ presentation\ for\ G\}.$$

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One can apply similar definitions for the semigroup deficiency of a finitely presented semigroup S, $def_S(S)$. Let us consider the second integral homology $H_2(G)$ of a finite group G. It is well known that the group G (or semigroup S) is efficient as a group (or as a semigroup), if we have $def_G(G) = rank(H_2(G))$ (or $def_S(S) = rank(H_2(S^1))$ where S^1 is obtained from S by adjoining an identity). We can refer to the reader [2, 3, 8, 9, 10] for more details.

One of the most effective way to show efficiency for the group G is to use *spherical pictures* ([7, 18]) over $\mathcal{P}$. These geometric configurations are the representative elements of the second homotopy group $\pi_2(\mathcal{P})$ of $\mathcal{P}$ which is a left $\mathbb{Z}G$-module. They are denoted by $\mathbb{P}$.

Suppose $\mathbf{Y}$ is a collection of spherical pictures over $\mathbb{P}$. Then, by [18], one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *equivalence (rel* $\mathbf{Y}$*) of spherical pictures*. Then, again in [18], Pride proved that *the elements* $\langle \mathbb{P} \rangle$ $(\mathbb{P} \in \mathbf{Y})$ *generate* $\pi_2(\mathcal{P})$ *as a module if and only if every spherical picture is equivalent (rel* $\mathbf{Y}$*) to the empty picture*. Therefore one can easily say that if the elements $\langle \mathbb{P} \rangle$ $(\mathbb{P} \in \mathbf{Y})$ generate $\pi_2(\mathcal{P})$, then $\mathbf{Y}$ *generates* $\pi_2(\mathcal{P})$.

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$, is the number of discs of $\mathbb{P}$ labeled by R minus the number of discs labeled by R^{-1}. We remark that if any two pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent then $exp_R(\mathbb{P}_1) = exp_R(\mathbb{P}_2)$, for all $R \in \mathbf{r}$. Let n be a non-negative integer. Then $\mathcal{P}$ is said to be *n-Cockcroft* if $exp_R(\mathbb{P}) \equiv 0 \pmod{n}$, (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. Then a group G is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation. To verify that the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a set of generating pictures. The case $n = 0$ is just called Cockcroft. One can refer [11], [13], [14], [15] and [17] for the Cockcroft property and [9], [17] for the n-Cockcroft property.

The subject *efficiency*, for the presentation $\mathcal{P}$ as in (1) and so for the group G, is related to the q-Cockcroft property (see Theorem 1.1 below). We can refer, for example, [4] and [10] for the definition and applications of efficiency. We then have the following result.

Theorem 1.1 ([12, 17])**.** *Let $\mathcal{P}$ be as in* (1)*. Then $\mathcal{P}$ is efficient if and only if it is q-Cockcroft for some prime q.*

2 Main results

In [6], Bacon and Kappe worked on two-generator p-groups of nilpotency class 2 where $p \neq 2$. Also, in [16], Kappe, Sarmin and Visscher worked on two-generator 2-groups of nilpotency class 2. Also let us consider the following semigroups defined by the presentations:

$$\Big\langle a,\ b,\ c\ ;\ ab = bac,\ bc = cb,\ ac = ca,\ a^{p^{\alpha}+1} = a,\ b^{p^{\beta}+1} = b,\ c^{p^{\gamma}+1} = c\Big\rangle \quad (2)$$

where $\alpha \geq \beta \geq \gamma \geq 1$ and

$$\Big\langle a,\ b\ ;\ ab = ba^{1+p^{\alpha-\gamma}},\ a^{p^{\alpha}+1} = a,\ b^{p^{\beta}+1} = b\Big\rangle \quad (3)$$

where $\alpha \geq 2\gamma$, $\beta \geq \gamma \geq 1$ and $\alpha + \beta > 3$. In [1], the authors showed that the semigroups defined by the presentations (2) and (3) have the orders

$$p^{\alpha+\beta+\gamma} + p^{\alpha} + p^{\beta} + p^{\gamma} + p^{\alpha+\beta} + p^{\beta+\gamma} + p^{\alpha+\gamma} \quad \text{and} \quad p^{\alpha+\beta} + p^{\alpha} + p^{\beta},$$

respectively.

Now let us again think the following presentations for the groups G_1 and G_2 which are given in abstract

$$\mathcal{P}_{G_1} = \Big\langle a,\ b,\ c\ ;\ ab = bac,\ bc = cb,\ ac = ca,\ a^{p^{\alpha}} = 1,\ b^{p^{\beta}} = 1,\ c^{p^{\gamma}} = 1\Big\rangle \quad (4)$$

where $\alpha \geq \beta \geq \gamma \geq 1$ and

$$\mathcal{P}_{G_2} = \Big\langle a,\ b\ ;\ ab = ba^{1+p^{\alpha-\gamma}},\ a^{p^{\alpha}} = 1,\ b^{p^{\beta}} = 1\Big\rangle \quad (5)$$

where $\alpha \geq 2\gamma$, $\beta \geq \gamma \geq 1$ and $\alpha + \beta > 3$. In [1], the authors showed that the groups defined by the presentations (4) and (5) have the orders

$$p^{\alpha+\beta+\gamma} \quad \text{and} \quad p^{\alpha+\beta}.$$

In this paper, our aim is to study on the efficieny of the groups G_1 and G_2 presented by (4) and (5), by using the works given [2, 3, 5, 8, 9, 10].

Therefore we can give the main results of this paper as follows.

Theorem 2.1. *For every prime number p and integers α, β and γ with $\alpha \geq \beta > \gamma \geq 1$, the group G_1 presented by (4) is efficient.*

Theorem 2.2. *For every prime number p and integers α, β and γ with $\alpha \geq 2\gamma$, $\beta > \gamma \geq 1$ and $\alpha + \beta > 3$, the group G_2 presented by (5) is efficient.*

3 Proof of the main results

3.1 Proof of Theorem 2.1

Consider the group G_1. Since we have the following relations $ab = bac, bc = cb,\ ac = ca,\ a^{p^{\alpha}} = 1,\ b^{p^{\beta}} = 1,\ c^{p^{\gamma}} = 1$, we have to think about the following

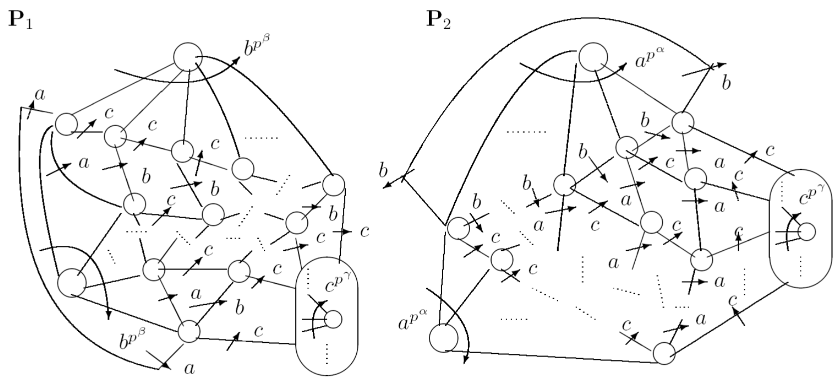

Figure 1

overlapping word pairs ab^{p^β}, $a^{p^\alpha}b$, ac^{p^γ}, bc^{p^γ}, $a^{p^\alpha}c$ and $b^{p^\beta}c$ for defining the elements of $\pi_2(\mathcal{P}_{G_1})$. It is known that spherical pictures which are obtained from the resolutions of these pairs give the elements of $\pi_2(\mathcal{P}_{G_1})$ by [5].

Now, let us consider the pairs ab^{p^β} and $a^{p^\alpha}b$. Then by using the relations of the group G_1, the resolutions for these pairs can be given as pictures $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively in Figure 1.

Now, let us also consider the discs in the pictures $\mathbf{P}_1$ and $\mathbf{P}_2$. To prove this theorem, we need to count the exponent sums of the discs in these pictures. So let us calculate the number of S_1-discs, S_2-discs, S_3-discs, S_4-discs, S_5-discs and S_6-discs in $\mathbf{P}_1$, $\mathbf{P}_2$ where $S_1 : b^{p^\beta} = 1$, $S_2 : c^{p^\gamma} = 1$, $S_3 : ab = bac$, $S_4 : a^{p^\alpha} = 1$, $S_5 : bc = cb$ and $S_6 : ac = ca$. At this point, it can be seen that

$$\begin{aligned}
exp_{S_1}(\mathbf{P}_1) &= 1-1=0, \quad exp_{S_2}(\mathbf{P}_1) = p^{\beta-\gamma},\\
exp_{S_2}(\mathbf{P}_2) &= p^{\alpha-\gamma}, \quad exp_{S_3}(\mathbf{P}_1) = p^{\beta},\\
exp_{S_3}(\mathbf{P}_2) &= p^{\alpha}, \quad exp_{S_4}(\mathbf{P}_2) = 1-1=0,\\
exp_{S_5}(\mathbf{P}_1) &= 1+2+3+\cdots+(p^\beta-1) = \frac{(p^\beta-1)p^\beta}{2},\\
exp_{S_6}(\mathbf{P}_2) &= 1+2+3+\cdots+(p^\alpha-1) = \frac{(p^\alpha-1)p^\alpha}{2}
\end{aligned}$$

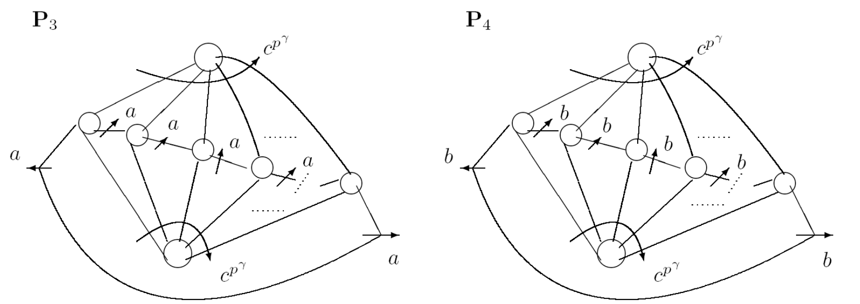

Figure 2

and to q-Cockcroft property be hold for some prime q, we need to have

$$
\begin{array}{lcl}
exp_{S_2}(\mathbf{P}_1) \equiv 0\,(mod\ q) & \Leftrightarrow & p^{\beta-\gamma} \equiv 0\,(mod\ q), \\
exp_{S_2}(\mathbf{P}_2) \equiv 0\,(mod\ q) & \Leftrightarrow & p^{\alpha-\gamma} \equiv 0\,(mod\ q), \\
exp_{S_3}(\mathbf{P}_1) \equiv 0\,(mod\ q) & \Leftrightarrow & p^{\beta} \equiv 0\,(mod\ q), \\
exp_{S_3}(\mathbf{P}_2) \equiv 0\,(mod\ q) & \Leftrightarrow & p^{\alpha} \equiv 0\,(mod\ q), \\
exp_{S_5}(\mathbf{P}_1) \equiv 0\,(mod\ q) & \Leftrightarrow & \dfrac{(p^{\beta}-1)p^{\beta}}{2} \equiv 0\,(mod\ q), \\
exp_{S_6}(\mathbf{P}_2) \equiv 0\,(mod\ q) & \Leftrightarrow & \dfrac{(p^{\alpha}-1)p^{\alpha}}{2} \equiv 0\,(mod\ q).
\end{array}
$$

Now, let us consider the pairs $ac^{p^{\gamma}}$ and $bc^{p^{\gamma}}$. Then by using the relations S_2, S_5 and S_6, the resolutions for these pairs can be given as pictures $\mathbf{P}_3$ and $\mathbf{P}_4$, respectively in Figure 2.

Similarly, as in the above, we need to count the exponent sums of the discs in these pictures. Therefore let us give the number of S_2-discs, S_5-discs and S_6-discs in $\mathbf{P}_3$, $\mathbf{P}_4$ as follows;

$$
\begin{array}{lcll}
exp_{S_2}(\mathbf{P}_3) & = & 1-1=0, & exp_{S_2}(\mathbf{P}_4) = 1-1=0, \\
exp_{S_5}(\mathbf{P}_4) & = & p^{\gamma}, & exp_{S_6}(\mathbf{P}_3) = p^{\gamma}.
\end{array}
$$

So in order to give q-Cockcroft property for some prime q, we need to have

$$exp_{S_5}(\mathbf{P}_4) = exp_{S_6}(\mathbf{P}_3) \equiv 0\,(mod\ q) \quad \Leftrightarrow \quad p^{\gamma} \equiv 0\,(mod\ q).$$

Similarly, let us consider the pairs $a^{p^{\alpha}}c$ and $b^{p^{\beta}}c$. Then by using the relations S_1, S_4, S_5 and S_6, the resolutions for these pairs can be given as pictures $\mathbf{P}_5$ and $\mathbf{P}_6$, respectively in Figure 3.

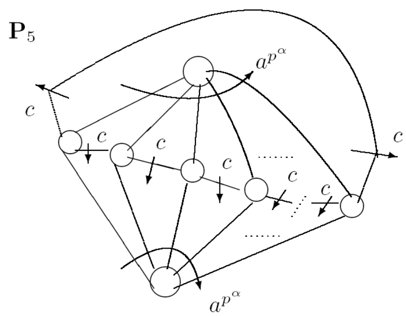

Figure 3

Here one can give the exponent sums of the discs in these pictures as follows;

$$\begin{array}{rcl} exp_{S_1}(\mathbf{P}_6) & = & 1-1=0, \quad exp_{S_4}(\mathbf{P}_5)=1-1=0, \\ exp_{S_5}(\mathbf{P}_6) & = & p^\beta, \qquad\qquad exp_{S_6}(\mathbf{P}_5)=p^\alpha. \end{array}$$

Thus in order to give q-Cockcroft property for some prime q, we have

$$\begin{array}{rcl} exp_{S_5}(\mathbf{P}_6) \equiv 0\,(mod\ q) & \Leftrightarrow & p^\beta \equiv 0\,(mod\ q), \\ exp_{S_6}(\mathbf{P}_5) \equiv 0\,(mod\ q) & \Leftrightarrow & p^\alpha \equiv 0\,(mod\ q). \end{array}$$

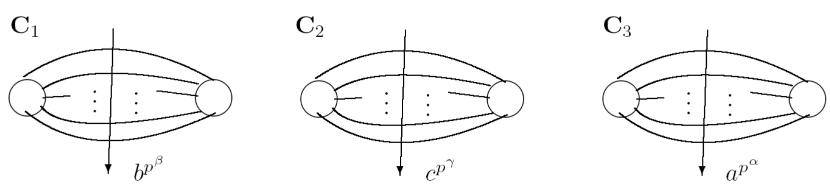

Figure 4

Also let us consider the pictures in Figure 4. Here we have

$$exp_{S_1}(\mathbf{C}_1) \;=\; 1-1=0, \;\; exp_{S_2}(\mathbf{C}_2)=1-1=0, \;\; exp_{S_4}(\mathbf{C}_3)=1-1=0.$$

Finally, we can see that $\pi_2(\mathcal{P}_{G_1})$ consists of the pictures $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{P}_6$, $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$. Thus in order to get q-Cockcroft property, we must calculate the exponent sums of the discs in these pictures. Then, by using the above arguments, for getting q-Cockcroft property for some prime q, we must have

$$\begin{aligned} p^{\beta-\gamma} &\equiv 0\,(mod\;q), \;\; p^{\alpha-\gamma} \equiv 0\,(mod\;q), \\ p^{\beta} &\equiv 0\,(mod\;q), \;\; p^{\alpha} \equiv 0\,(mod\;q), \;\; p^{\gamma} \equiv 0\,(mod\;q) \\ \frac{(p^{\beta}-1)p^{\beta}}{2} &\equiv 0\,(mod\;q), \;\; \frac{(p^{\alpha}-1)p^{\alpha}}{2} \equiv 0\,(mod\;q). \end{aligned}$$

Then by Theorem 1.1 we may say that the group G_1 is efficient if and only if it is q-Cockcroft for some prime q. At this point, since we have $\alpha \geq \beta > \gamma \geq 1$, then we choose $p = q$. This gives that the group G_1 presented by (4) is q-Cockcroft. This says that G_1 is efficient.

Remark 3.1. *We realised that we choose $\alpha \geq \beta > \gamma \geq 1$. If we choose $\alpha \geq \beta \geq \gamma \geq 1$, then we may have $\beta = \gamma$ or $\alpha = \gamma$. This gives that $p^{\beta-\gamma} = p^0 = 1$ is not equivalent to 0 by the modulo q or $p^{\alpha-\gamma} = p^0 = 1$ is not equivalent to 0 by the modulo q, for some prime q. Also, for $p = 2$, if we choose $\alpha \geq \beta \geq \gamma \geq 1$, then we may have $\beta = 1$ or $\alpha = 1$. This says that $\frac{(p^{\beta}-1)p^{\beta}}{2} = 1$ is not equivalent to 0 by the modulo q or $\frac{(p^{\alpha}-1)p^{\alpha}}{2} = 1$ is not equivalent to 0 by the modulo q, for some prime q.*

Remark 3.2. *In [3, 8], it was shown that for a finitely presented group G with non-negative deficiency we have $def_S(G) = def_G(G)$. This says that a group G with non-negative deficiency is efficient as a group if and only if G is efficient as a semigroup. Therefore, since the group G_1 presented by (4) has non-negative deficiency and it is efficient as a group, then it is also efficient as a semigroup. Hence we get that the semigroup related to the certain group presentation (2) is also efficient.*

3.2 Proof of Theorem 2.2

Let us consider the group G_2. Here we have the following relations $a^{p^{\alpha}} = 1$, $b^{p^{\beta}} = 1$ and $ab = ba^{1+p^{\alpha-\gamma}}$. Thus we cocern about the following overlapping word pairs $ab^{p^{\beta}}$ and $a^{p^{\alpha}}b$ for defining the elements of $\pi_2(\mathcal{P}_{G_2})$.

Now, let us consider the pairs $ab^{p^{\beta}}$ and $a^{p^{\alpha}}b$. Then by using the relations of the group G_2, the resolutions for these pairs can be given as pictures $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively in Figure 5.

Now, let us also think the discs in the pictures $\mathbf{K}_1$ and $\mathbf{K}_2$. To prove this theorem, we need to count the exponent sums of the discs in these pictures. So let us calculate the number of R_1-discs, R_2-discs and R_3-discs in $\mathbf{K}_1$, $\mathbf{K}_2$

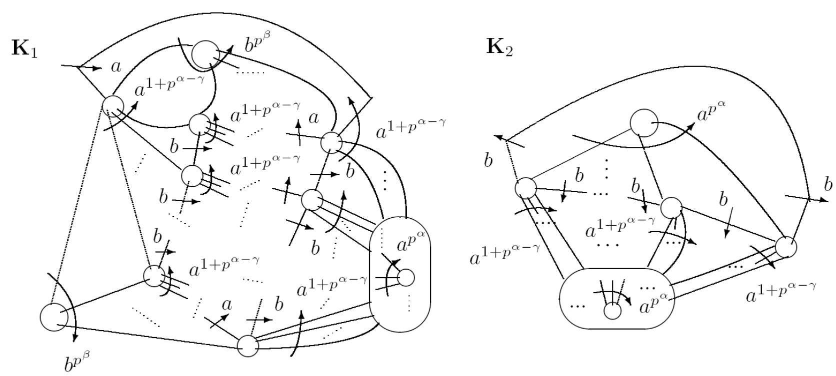

Figure 5

where $R_1 : a^{p^\alpha} = 1$, $R_2 : b^{p^\beta} = 1$ and $R_3 : ab = ba^{1+p^{\alpha-\gamma}}$. Here, it is seen that

$$\begin{aligned}
exp_{R_1}(\mathbf{K}_1) &= \frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^\alpha},\\
exp_{R_1}(\mathbf{K}_2) &= \frac{p^\alpha(1+p^{\alpha-\gamma})}{p^\alpha}-1 = p^{\alpha-\gamma},\\
exp_{R_2}(\mathbf{K}_1) &= 1-1=0,\\
exp_{R_3}(\mathbf{K}_1) &= 1+(1+p^{\alpha-\gamma})+(1+p^{\alpha-\gamma})^2+\cdots+(1+p^{\alpha-\gamma})^{p^\beta-1} = \frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^{\alpha-\gamma}},\\
exp_{R_3}(\mathbf{K}_2) &= p^\alpha
\end{aligned}$$

and for the q-Cockcroft property to be held for some q, we need to have

$$\begin{aligned}
exp_{R_1}(\mathbf{K}_1) \equiv 0\,(mod\ q) &\Leftrightarrow \frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^\alpha} \equiv 0\,(mod\ q),\\
exp_{R_1}(\mathbf{K}_2) \equiv 0\,(mod\ q) &\Leftrightarrow p^{\alpha-\gamma} \equiv 0\,(mod\ q),\\
exp_{R_3}(\mathbf{K}_1) \equiv 0\,(mod\ q) &\Leftrightarrow \frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^{\alpha-\gamma}} \equiv 0\,(mod\ q),\\
exp_{R_3}(\mathbf{K}_2) \equiv 0\,(mod\ p) &\Leftrightarrow p^\alpha \equiv 0\,(mod\ q).
\end{aligned}$$

Here let us denote $\frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^\alpha}$ by A and $\frac{(1+p^{\alpha-\gamma})^{p^\beta}-1}{p^{\alpha-\gamma}}$ by B.
Therefore, since we have

$$\begin{aligned}
(1+p^{\alpha-\gamma})^{p^\beta}-1 = p^\beta p^{\alpha-\gamma} &+ \frac{1}{2}p^\beta(p^\beta-1)p^{2(\alpha-\gamma)} + \frac{1}{6}p^\beta(p^\beta-1)(p^\beta-2)p^{3(\alpha-\gamma)}\\
&+ \cdots + p^{p^\beta(\alpha-\gamma)}
\end{aligned}$$

then we get that

$$\begin{aligned} A = p^{\beta-\gamma} &+ \frac{1}{2}p^{\beta}(p^{\beta}-1)p^{(\alpha-2\gamma)} + \frac{1}{6}p^{\beta}(p^{\beta}-1)(p^{\beta}-2)p^{(2\alpha-3\gamma)} \\ &+ \cdots + p^{p^{\beta}(\alpha-\gamma)-\alpha} \end{aligned}$$

and

$$\begin{aligned} B = p^{\beta} &+ \frac{1}{2}p^{\beta}(p^{\beta}-1)p^{(\alpha-\gamma)} + \frac{1}{6}p^{\beta}(p^{\beta}-1)(p^{\beta}-2)p^{(2\alpha-2\gamma)} \\ &+ \cdots + p^{p^{\beta}(\alpha-\gamma)-\alpha+\gamma}. \end{aligned}$$

Finally, we can see that $\pi_2(\mathcal{P}_{G_2})$ consists of the pictures $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{C}_1$ and $\mathbf{C}_3$. Thus in order to get q-Cockcroft property, we must calculate the exponent sums of the discs in these pictures. Then, by using the above arguments, in order to get q-Cockcroft property for some prime q, we must have

$$\begin{aligned} A &\equiv 0\,(mod\; q), \quad p^{\alpha-\gamma} \equiv 0\,(mod\; q), \\ B &\equiv 0\,(mod\; q), \quad p^{\alpha} \equiv 0\,(mod\; q). \end{aligned}$$

Then by Theorem 1.1 we can say that the group G_2 is efficient if and only if it is q-Cockcroft for some prime q. Here since we have $\alpha \geq 2\gamma$ and $\beta > \gamma \geq 1$, then we choose $p = q$. So we get that the group G_2 presented by (5) is q-Cockcroft. This says that G_2 is efficient.

Remark 3.3. *We realised that we take $\beta > \gamma \geq 1$. If we take $\beta \geq \gamma \geq 1$, then we may have $\beta = \gamma$. This says that A is not equivalent to 0 by the modulo q for some prime q.*

Remark 3.4. *By using smilar argumets as in Remark 3.2, since the group G_2 presented by (5) has non-negative deficiency and it is efficient as a group, then it is also efficient as a semigroup. So we deduce that the semigroup related to the certain group presentation (3) is also efficient.*

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