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Reduced diophantine quadruples with the binary recurrence $G_n = AG_{n-1} - G_{n-2}$

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Abstract

Given a positive integer $A \neq 2$. In this paper, we show that there do not exist two positive integer pairs $\{a, b\} \neq \{c, d\}$ such that the values of $ac+1$, $ad+1$ and $bc+1$, $bd+1$ are the terms of the sequence $\{G_n\}_{n\geq 0}$ which satisfies the recurrence relation $G_n = AG_{n-1} - G_{n-2}$ with the initial values $G_0 = 0,\ G_1 = 1$.

1 Introduction

A diophantine m-tuple is a set $\{a_1, a_2, \ldots, a_m\}$ of positive integers such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. This problem and its variations have an extensive history starting with Diophantus. He found such a but rational set: $\{1/16, 33/16, 68/16, 105/16\}$. Fermat was the first who could give an integer quadruple, namely the set $\{1, 3, 8, 120\}$.

It is well-known that infinitely many integer diophantine quadruples exist (see, for instance [6]). A widely believed conjecture foreshadows that no quintuple exists. The famous theorem of Dujella [4] states that there are only finitely many quintuples.

A variant of the problem is obtained if one replaces the squares by the terms of a given binary recurrence. For details, see articles [5], [8], [9] and

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[1]. The first cited paper investigates a general case and provides sufficient and necessary conditions to be finitely many diophantine triples with terms of the binary recurrent sequence. But the arguments in [5] give no hint how to find the triples themselves. The other cited papers describe methods to determine all diophantine triples for Fibonacci, Lucas and balancing numbers, respectively.

This work follows the treatment of the above results, but there is an essential difference, the binary recurrence we investigate here contains a positive integer parameter A. That happened also in [7], where we needed to include a new, additional idea to show that there is no positive integer triple $\{a, b, c\}$ such that all of $ab+1$, $ac+1$ and $bc+1$ are in the sequence $\{G_n\}_{n\geq 0}$ satisfies the relation

$$G_n = AG_{n-1} - G_{n-2} \tag{1.1}$$

with the initial values $G_0 = 0$, $G_1 = 1$. Further, we investigated there the analogous question for the quadruples $\{a, b, c, d\}$ with $abc+1 = G_w$, $bcd+1 = G_x$, $cda + 1 = G_y$ and $dab + 1 = G_z$, and deduced the non-existence of such quadruples.

In this paper, the following question linked to reduced quadruples will be solved. Are there two integer pairs $\{a, b\}$ and $\{c, d\}$ such that $ac + 1$, $ad + 1$ and $bc + 1$, $bd + 1$ are in the sequence $\{G\}$? Note that this kind of reduced quadruples were examined for higher power of integers in [2].

The Binet formula

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

gives G_n explicitly, where

$$\alpha = \frac{A + \sqrt{A^2 - 4}}{2} \qquad \text{and} \qquad \beta = \frac{A - \sqrt{A^2 - 4}}{2}.$$

We define $\{H_n\}$ as the associated sequence of $\{G_n\}$ by the usual manner. The recurrence relation for $\{G_n\}$ and $\{H_n\}$ coincide, but the initial conditions in the second case are $H_0 = 2$ and $H_1 = A$. Obviously, both $\{G_n\}$ and $\{H_n\}$ are strictly monotone increasing sequences if $A \geq 3$.

The main result of this work is the following.

Theorem 1. *Suppose that $A \neq 2$ is a positive integer. Then there do not exist sets $\{a, b\} \neq \{c, d\}$ of positive integers such that*

$$\begin{aligned} ac + 1 &= G_w, \\ ad + 1 &= G_x, \\ bc + 1 &= G_y, \\ bd + 1 &= G_z \end{aligned} \tag{1.2}$$

hold for the positive integers w, x, y and z.

Observe that $A = 2$ would admit the sequence $\{G_n\}$ as the sequence of natural numbers and in this case, trivially, system (1.2) is satisfied by arbitrary a, b and c, d. Further the case $A = 1$ provides the periodic sequence $G_n = 0, 1, 1, 0, -1, -1, \ldots$. Hence (1.2) cannot be fulfilled with $A = 1$. Thus, in the sequel, we assume $A \geq 3$.

Prescribing the more strict conditions $1 \leq a < c < b < d$ and changing the order of the equations in system (1.2) we arrive at the so called cyclic variant with length 4 of the problem:

$$\begin{aligned} ac + 1 &= G_w, \\ cb + 1 &= G_y, \\ bd + 1 &= G_z, \\ da + 1 &= G_x. \end{aligned} \tag{1.3}$$

Clearly, Theorem 1 immediately implies

Corollary 1. *There is no solution to system (1.3) in positive integers $a < c < b < d$.*

Note that the auxiliary results we use in the proof of Theorem 1 are located in the last section.

2 Proof of Theorem 1

Recall that $A \geq 3$ is a positive integer, and suppose that $1 \leq a < b$ and $1 \leq c < d$ satisfy (1.2) with some positive integers w, x, y and z. Consequently, w is the smallest, z is the largest subscript in (1.2). Then, $1 \cdot 1 + 1 \leq ac + 1 = G_w$ implies $w \geq 2$. Subsequently, $x \geq 3$.

Moreover, there is no restriction in assuming $b < d$. Indeed, if $b = d$ holds then we have $a \neq c$, $b \neq c$ and system (1.2) contains $ab + 1$, $ac + 1$ and $bc + 1$ generated by the triple $\{a, b, c\}$, and according to Theorem 1 of [7] it is not possible. Now we split the proof into two parts.

Case 1. $z \leq 138$

In this case, we show that there is an upper bound for the coefficient A of the sequence $\{G_n\}$.

Lemma 1. *If there exist a solution to system (1.2) then $A \leq A_0$ with a suitable $A_0 \in \mathbb{N}^+$.*

Proof. Clearly, the terms of the sequence $\{G_n\}$ are monic polynomials in A with $\deg(G_n(A)) = n-1$ ($n \geq 1$), the first few terms are $G_0(A) = 0,\ G_1(A) = 1$ and

$$G_2(A) = A, \quad G_3(A) = A^2 - 1, \quad G_4(A) = A^3 - 2A, \quad \ldots .$$

Then, by (1.2)

$$ac = \sqrt{\frac{(G_w(A) - 1)\,(G_x(A) - 1)\,(G_y(A) - 1)}{G_z(A) - 1}} \tag{2.1}$$

must be integer for some A. If we divide the numerator in (2.1) by $G_z(A) - 1$, then we obtain polynomials $q(A) \in \mathbb{Z}[A]$ and $r(A) \in \mathbb{Z}[A]$ such that

$$(G_w(A) - 1)\,(G_x(A) - 1)\,(G_y(A) - 1) = q\,(A) \cdot (G_z\,(A) - 1) + r\,(A)\,,$$

where $\deg\,(r\,(A)) < \deg\,(G_z\,(A))$.

Checking the eligible possibilities for w, x, y and z ($2 \leq w < x, y < z \leq 138$) by computer, we observe that $r(A)$ is never the constant zero polynomial, further $r\,(A) \neq 0$ for $A \geq 3$. Hence

$$\frac{(G_w(A) - 1)\,(G_x(A) - 1)\,(G_y(A) - 1)}{(G_z\,(A) - 1)} = q\,(A) + \frac{r\,(A)}{G_z\,(A) - 1} \tag{2.2}$$

follows, with non-vanishing fraction on the right hand side. If for some A the left hand side of the equation (2.2) is an integer, then by $q\,(A) \in \mathbb{N}$ we deduce that

$$\frac{r\,(A)}{G_z\,(A) - 1}$$

is so. But $\deg\,(r\,(A)) < \deg\,(G_z\,(A))$, so A cannot be large since

$$\lim_{A \to \infty} \frac{r\,(A)}{G_z\,(A) - 1} = 0.$$

Consequently, $|r\,(A)\,| \geq G_z\,(A) - 1$ must hold since $r\,(A) \neq 0$ and $\frac{r(A)}{G_z(A)-1}$ is an integer, which proves $A \leq A_0$ with some positive integer A_0. To obtain the exact upper bound, we run a computer search with the conditions $2 \leq w < x, y < z \leq 138$, and we found that $A_0 = 2$. □

Then, by Lemma 1 we obtain immediately that there is no solution to system (1.2) in the first case.

Case 2. $z \geq 139$

Put $P = \gcd(G_z - 1, G_x - 1)$. Obviously, $d \leq P$. By Lemma 2 (3) and (1), we have

$$\begin{aligned} P &\leq \gcd(G_{z-1}G_{z+1}, G_{x-1}G_{x+1}) \\ &\leq \prod_{i,j \in \{\pm 1\}} \gcd(G_{z-i}, G_{x-j}) = \prod_{i,j \in \{\pm 1\}} G_{\gcd(z-i, x-j)}. \end{aligned} \tag{2.3}$$

Let say that $\gcd(z-i, x-j) = (z-i)/k_{ij}$ is valid for some positive integer k_{ij}.

Firstly, assume that $k_{ij} \geq 8$, hold for all $i, j \in \{\pm 1\}$. Then Lemma (3) implies that

$$\alpha^{\frac{z-1}{2}} < \sqrt{G_z} < d \leq P \leq G^2_{\frac{z-1}{8}} G^2_{\frac{z+1}{8}} < \alpha^{4\left(\frac{z+1}{8} - 0.83\right)}. \tag{2.4}$$

If we compare the exponents of α in (2.4), we arrive at a contradiction.

In what follows, assume that $k_{ij} \leq 7$ appears for some i and j. Let k denote this k_{ij}. Further suppose that

$$\frac{z-i}{k} = \frac{x-j}{\ell}$$

holds for a suitable positive integer ℓ such that $\gcd(k, \ell) = 1$.

Now, we investigate three separated cases related to k and ℓ. Firstly, assume that $\ell > k$. Then $z - i < x - j$ implies $z = x + 1$ via $x < z$. Thus

$$\begin{aligned} \alpha^{\frac{z-1}{2}} &< P = \gcd(G_x - 1, G_{x+1} - 1) \\ &\leq \gcd(G_{x+1}G_{x-1}, G_x G_{x+2}) \\ &= \gcd(G_{x-1}, G_{x+2}) \leq G_3 < \alpha^{2.17}, \end{aligned}$$

lead to the contradiction $z < 5.4$.

Now assume $\ell = k$. Since k and ℓ are coprimes, $k = \ell = 1$ follows. By $z - i = x - j$, we obtain $z = x + 2$. According to Lemma 5,

$$\alpha^{\frac{z-1}{2}} < P = \gcd(G_{x+2} - 1, G_x - 1) < 2(A^2 - 2)$$

hold, which together with Lemma 4 is a contradiction again by $z < 7.1$.

Lastly, assume that $\ell < k$. We distinguish two cases. Analyse first when the condition $2 \leq \frac{k}{\ell}$ fulfils. Here

$$z = \frac{k}{\ell}(x - j) + i \geq 2(x - 1) - 1 = 2x - 3,$$

which, together with Lemma (8) implies one of the following three possibilities: $z = 2x-3$, $z = 2x-2$ and $z = 2x-1$.

If $z = 2x-3$ then, according to Lemma (6),

$$\alpha^{\frac{z-1}{2}} < P = \gcd\left(G_x - 1, G_{2x-3} - 1\right) < \alpha^{5.7}$$

is valid. But, this is not possible.

Now, assume that $z = 2x-2$. Then by Lemma 7, it follows that

$$\alpha^{\frac{z-1}{2}} < P = \gcd\left(G_x - 1, G_{2x-2} - 1\right) < \alpha^{6.4}$$

which is impossible since $z \geq 139$.

When $z = 2x-1$ holds, then we get

$$\alpha^{x-1.17} = \frac{\alpha^{2x-2}}{\alpha^{x-0.83}} < \frac{G_{2x-1}}{G_x} = \frac{bd+1}{ad+1} < \frac{b}{a},$$

and

$$a^2\alpha^{x-1.17} < ab = G_w - 1 < G_w < \alpha^{w-0.83}$$

follow. Thus

$$a^2 < \alpha^{w-x+0.34} \leq \alpha^{-0.66}$$

mean again a contradiction.

Finally assume that $\frac{k}{\ell} < 2$. Note that it implies $k \geq 3$. Taking any pair $(i_0, j_0) \neq (i, j)$, we have

$$z - i_0 = \frac{k}{\ell}\left(x - j\right) + i - i_0.$$

Now the goal is to calculate the best upper bound for $P_0 = \gcd\left(z - i_0, v - j_0\right)$. Starting with

$$\begin{aligned} P_0 &= \gcd\left(\frac{k}{\ell}\left(x-j\right) + i - i_0, x - j_0\right) \\ &\leq \gcd\left(k\left(x-j\right) + \ell(i - i_0), k(x - j_0)\right) = |k(j_0 - j) + \ell(i - i_0)|, \end{aligned}$$

we need to consider the last expression. The three cases $j \neq j_0$, $i \neq i_0$ and $j \neq j_0$, $i = i_0$ and $j = j_0$, $i \neq i_0$ give $P_0 \leq 2(k+\ell)$, $2k$, 2ℓ, respectively. Then using inequality (2.3), it yields

$$\begin{aligned} \alpha^{\frac{z-1}{2}} \leq P &= \gcd\left(G_x - 1, G_z - 1\right) < \prod_{i,j \in \{\pm 1\}} G_{\gcd(z-i, x-j)} \\ &\leq \alpha^{\frac{z+1}{k} + 2(k+\ell) + 2k + 2\ell - 4\cdot 0.83}. \end{aligned}$$

Going through the eligible pairs

$$(k, \ell) = (3,2),\ (4,3),\ (5,3),\ (5,4),\ (6,5),\ (7,4),\ (7,5),\ (7,6),$$

the previous argument provides the upper bounds

$$z \ < \ 105.1,\ \ 101.8,\ \ 98,\ \ 111.3,\ \ 124.1,\ \ 115.8,\ \ 127,\ \ 138.2,$$

respectively. The assertion of the second part of the proof contradicts any of these upper bounds. Thus the proof of Theorem 2 is complete.

3 Lemmata

In the proof of Theorem 1, we needed some lemmata. Apart from the last lemma of the list, and in part from Lemma 2, the proofs of them can be found in [7]. In Lemma 2, the first two identities are known from [3]. Further, paper [11] contains (3), the remaining two properties are also in [7].

Lemma 2. *Assume that $n, m \in \mathbb{N}$. Then the following identities hold.*

1. $\gcd(G_n, G_m) = G_{\gcd(n,m)}$,
2. $\gcd(G_n, H_m) = 1$ *or* 2 *or* $H_{\gcd(n,m)}$, *especially* $\gcd(G_n, H_n) = 1$ *or* 2,
3. $(G_n - 1)(G_n + 1) = G_{n-1}G_{n+1}$,
4. $G_{2n+1} - 1 = G_n H_{n+1}$,
5. $2G_{n+m} = G_n H_m + H_n G_m$.

Lemma 3. *Suppose that $A \geq 3$. Then for all integers $n \geq 3$, the inequalities*

$$\alpha^{n-1} < G_n < \alpha^{n-0.83} \tag{3.1}$$

and

$$\alpha^{n} < H_n < \alpha^{n+0.004} \tag{3.2}$$

hold.

Lemma 4. *Suppose $A \geq 3$. Then* $\log_\alpha(2(A^2-2)) < 3.1$.

Lemma 5. *Assume that $n \geq 3$ and $A \geq 3$ are integers. Then*

$$\gcd\left(G_n - 1, G_{n-2} - 1\right) \leq 2(A^2 - 2).$$

Lemma 6. *Any integer $n \geq 2$ satisfies*

$$\gcd\left(G_{2n-3} - 1, G_n - 1\right) < \alpha^{5.7}$$

Lemma 7. *Any integer $n \geq 2$ satisfies*

$$\gcd\left(G_{2n-2} - 1, G_n - 1\right) < \alpha^{6.4}$$

Lemma 8. *All positive solutions to the system (1.2) satisfy $z \leq 2x - 1$.*

Proof. Considering the second and fourth equations of the system (1.2) we have

$$d \mid \gcd\left(G_x - 1, G_z - 1\right).$$

Moreover $G_z = bd + 1 < d^2$, therefore $\sqrt{G_z} < d$ holds. By (3.1), we obtain

$$\sqrt{\alpha^{z-1}} < \sqrt{G_z} < d < G_x < \alpha^{x-0.83}, \tag{3.3}$$

which implies $z - 1 < 2(x - 0.83)$, and then $z \leq 2x - 1$. □

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