



# Common fixed point theorems for generalized contraction involving rational expressions in complex valued metric spaces

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## Abstract

The purpose of this paper is to study common fixed points in complex valued metric spaces and obtain sufficient conditions for the existence of common fixed points of a pair of mappings satisfying generalized contraction involving rational expressions.

## 1 Introduction

The Banach contraction principle [4] is a very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

**Theorem 1.1.** [4]. *Let  $(X, d)$  be a complete metric space and  $T$  be a mapping of  $X$  into itself satisfying:*

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X, \quad (1)$$

*where  $k$  is a constant in  $(0, 1)$ . Then,  $T$  has a unique fixed point  $x^* \in X$ .*

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There are in the literature a great number of generalizations of the Banach contraction principle (see [1, 2] and others). Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, D-metric spaces, and cone metric spaces (see [5]- [10]).

Recently, Azam et al. [3] introduced the notion of complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. In a continuation of Azam et al. [3], in this paper, we prove a common fixed point theorem for a pair of mappings satisfying a more general contraction involving rational expression in complex valued metric spaces.

## 2 Preliminaries

First of all, we introduce some notations and definitions that will be used later.

### 2.1 Notations and Definitions

The following definition was introduced by Azam et al. in [3].

Let  $\mathbb{C}$  be the set of complex numbers and  $z_1, z_2 \in \mathbb{C}$ . Define a partial order  $\preceq$  on  $\mathbb{C}$  as follows:

$$z_1 \preceq z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2), \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

It follows that  $z_1 \preceq z_2$  if one of the following conditions is satisfied:

- (i)  $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$ ,
- (ii)  $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$ ,
- (iii)  $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$ ,
- (iv)  $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$ .

In particular, we will write  $z_1 \prec z_2$  if  $z_1 \neq z_2$  and one of (i), (ii), and (iii) is satisfied and we will write  $z_1 \prec z_2$  if only (iii) is satisfied. Note that

$$0 \preceq z_1 \prec z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

**Definition 2.1.** Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow \mathbb{C}$ , satisfies:

1.  $0 \preceq d(x, y)$ , for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
3.  $d(x, y) \preceq d(x, z) + d(z, y)$ , for all  $x, y, z \in X$ .

Then  $d$  is called a complex valued metric on  $X$ , and  $(X, d)$  is called a complex valued metric space.

A point  $x \in X$  is called an *interior point* of a set  $A \subseteq X$  whenever there exists  $0 \prec r \in \mathbb{C}$  such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A point  $x \in X$  is called a *limit point* of  $A$  whenever for every  $0 \prec r \in \mathbb{C}$ ,

$$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset.$$

$A$  is called *open* whenever each element of  $A$  is an interior point of  $A$ . A subset  $B \subseteq X$  is called *closed* whenever each limit point of  $B$  belongs to  $B$ . The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}.$$

is a sub-basis for a Hausdorff topology  $\tau$  on  $X$ .

Let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . If for every  $c \in \mathbb{C}$ , with  $0 \prec c$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$ ,  $d(x_n, x) \prec c$ , then  $\{x_n\}$  is said to be *convergent*,  $\{x_n\}$  *converges* to  $x$  and  $x$  is the *limit point* of  $\{x_n\}$ . We denote this by  $\lim_n x_n = x$ , or  $x_n \rightarrow x$ , as  $n \rightarrow \infty$ . If for every  $c \in \mathbb{C}$  with  $0 \prec c$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$ ,  $d(x_n, x_{n+m}) \prec c$ , then  $\{x_n\}$  is called a *Cauchy sequence* in  $(X, d)$ . If every Cauchy sequence is convergent in  $(X, d)$ , then  $(X, d)$  is called a *complete complex valued metric space*.

**Lemma 2.2.** [3]. Let  $(X, d)$  be a complex valued metric space and let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  converges to  $x$  if and only if  $|d(x_n, x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.3.** [3]. Let  $(X, d)$  be a complex valued metric space and let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is a Cauchy sequence if and only if  $|d(x_n, x_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ .

### 3 Common fixed point results in complete complex valued metric space

**Theorem 3.1.** Let  $(X, d)$  be a complete complex valued metric space and let the mappings  $S, T : X \rightarrow X$  satisfy:

$$\begin{aligned} d(Sx, Ty) \preceq & \alpha d(x, y) + \frac{\beta[1 + d(x, Sx)]d(y, Ty)}{1 + d(x, y)} \\ & + \gamma[d(x, Sx) + d(y, Ty)] + \delta[d(x, Ty) + d(y, Sx)] \end{aligned}$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma, \delta$  are nonnegative reals with  $\alpha + \beta + 2\gamma + 2\delta < 1$ . Then  $S$  and  $T$  have a unique common fixed point.

*Proof.* Let  $x_0$  be an arbitrary point in  $X$  and define

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = Tx_{2k+1}, \quad k = 0, 1, 2, \dots$$

Then,

$$\begin{aligned} & d(x_{2k+1}, x_{2k+2}) \\ = & d(Sx_{2k}, Tx_{2k+1}) \\ \lesssim & \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta[1 + d(x_{2k}, Sx_{2k})]d(x_{2k+1}, Tx_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \\ & + \gamma[d(x_{2k}, Sx_{2k}) + d(x_{2k+1}, Tx_{2k+1})] + \delta[d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})] \\ \lesssim & \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta[1 + d(x_{2k}, x_{2k+1})]d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1})} \\ & + \gamma[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] + \delta[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})] \\ \lesssim & \alpha d(x_{2k}, x_{2k+1}) + \beta d(x_{2k+1}, x_{2k+2}) + \gamma[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\ & + \delta[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\ \lesssim & \frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta} d(x_{2k}, x_{2k+1}). \end{aligned}$$

Similarly,

$$\begin{aligned} & d(x_{2k+2}, x_{2k+3}) \\ = & d(Sx_{2k+1}, Tx_{2k+2}) \\ \lesssim & \alpha d(x_{2k+2}, x_{2k+1}) + \frac{\beta[1 + d(x_{2k+1}, Sx_{2k+1})]d(x_{2k+2}, Tx_{2k+2})}{1 + d(x_{2k+1}, x_{2k+2})} \\ & + \gamma[d(x_{2k+1}, Sx_{2k+1}) + d(x_{2k+2}, Tx_{2k+2})] \\ & + \delta[d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})] \\ \lesssim & \alpha d(x_{2k+1}, x_{2k+2}) + \frac{\beta[1 + d(x_{2k}, x_{2k+2})]d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k}, x_{2k+2})} \\ & + \gamma[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] + \delta[d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})] \\ \lesssim & \alpha d(x_{2k+1}, x_{2k+2}) + \beta d(x_{2k+2}, x_{2k+3}) + \gamma[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\ & + \delta[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\ \lesssim & \frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta} d(x_{2k+1}, x_{2k+2}). \end{aligned}$$

Putting

$$h = \frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta},$$

we have

$$d(x_{n+1}, x_{n+2}) \lesssim h d(x_n, x_{n+1}) \lesssim \dots \lesssim h^{n+1} d(x_0, x_1).$$

Hence, for any  $m > n$ ,

$$\begin{aligned} d(x_n, x_m) &\lesssim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\lesssim [h^n + h^{n+1} + \cdots + h^{m-1}]d(x_0, x_1) \\ &\lesssim \frac{h^n}{1-h}d(x_0, x_1) \end{aligned}$$

and so

$$|d(x_m, x_n)| \leq \frac{h^n}{1-h} |d(x_0, x_1)| \rightarrow 0, \text{ as } m, n \rightarrow \infty.$$

This implies that  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete, there exists  $u \in X$  such that  $x_n \rightarrow u$ . It follows that  $u = Su$ , otherwise  $d(u, Su) = z > 0$  and we would then have

$$\begin{aligned} z &\lesssim d(u, x_{2k+2}) + d(x_{2k+2}, Su) \\ &\lesssim d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\ &\lesssim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta[1 + d(u, Su)]d(x_{2k+1}, Tx_{2k+1})}{1 + d(u, x_{2k+1})} \\ &\quad + \gamma[d(u, Su) + d(x_{2k+1}, Tx_{2k+1})] \\ &\quad + \delta[d(u, Tx_{2k+1}) + d(x_{2k+1}, Su)] \\ &\lesssim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta[1 + d(u, Su)]d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})} \\ &\quad + \gamma[d(u, Su) + d(x_{2k+1}, x_{2k+2})] + \delta[d(u, x_{2k+2}) + d(x_{2k+1}, Su)]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &\leq |d(u, x_{2k+2})| + \alpha |d(x_{2k+1}, u)| + \frac{\beta|1 + z||d(x_{2k+1}, x_{2k+2})|}{|1 + d(u, x_{2k+1})|} \\ &\quad + \gamma[|z| + |d(x_{2k+1}, x_{2k+2})|] + \delta|d(u, x_{2k+2}) + d(x_{2k+1}, Su)|. \end{aligned}$$

Letting  $n \rightarrow \infty$ , it follows that

$$|z| \leq (\gamma + \delta)|z| \leq (\alpha + \beta + 2\gamma + 2\delta)|z| < |z|,$$

a contradiction and so  $|z| = 0$ , that is,  $u = Su$ .

It follows similarly that  $u = Tu$ .

We now show that  $S$  and  $T$  have unique common fixed point. For this, assume that  $u^*$  in  $X$  is a second common fixed point of  $S$  and  $T$ . Then

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\lesssim \alpha d(u, u^*) + \frac{\beta[1 + d(u, Su)]d(u^*, Tu^*)}{1 + d(u, u^*)} + \gamma[d(u, Su) + d(u^*, Tu^*)] \\ &\quad + \delta[d(u, Tu^*) + d(u^*, Su)] \\ &\lesssim (\alpha + 2\delta)d(u, u^*) \end{aligned}$$

and so  $d(u, u^*) = 0$ , since  $(\alpha + 2\delta) < 1$ . This implies that  $u^* = u$ , completing the proof of the theorem.  $\square$

Putting  $S = T$ , we have

**Corollary 3.2.** *Let  $(X, d)$  be a complete complex valued metric space and let the mappings  $T : X \rightarrow X$  satisfy:*

$$\begin{aligned} d(Tx, Ty) &\lesssim \alpha d(x, y) + \frac{\beta[1 + d(x, Tx)]d(y, Ty)}{1 + d(x, y)} + \gamma[d(x, Tx) + d(y, Ty)] \\ &\quad + \delta[d(x, Ty) + d(y, Tx)] \end{aligned}$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma, \delta$  are nonnegative reals with  $\alpha + \beta + 2\gamma + 2\delta < 1$ . Then  $T$  has a unique fixed point.

**Corollary 3.3.** *Let  $(X, d)$  be a complete complex valued metric space and let the mappings  $T : X \rightarrow X$  satisfy:*

$$\begin{aligned} d(T^n x, T^n y) &\lesssim \alpha d(x, y) + \frac{\beta[1 + d(x, T^n x)]d(y, T^n y)}{1 + d(x, y)} + \gamma[d(x, T^n x) + d(y, T^n y)] \\ &\quad + \delta[d(x, T^n y) + d(y, T^n x)] \end{aligned}$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma, \delta$  are nonnegative reals with  $\alpha + \beta + 2\gamma + 2\delta < 1$ . Then  $T$  has a unique fixed point.

*Proof.* By Corollary 3.2 there exists  $v \in X$  such that  $T^n v = v$ . Then

$$\begin{aligned} d(Tv, v) &= d(TT^n v, T^n v) = d(T^n Tv, T^n v) \\ &\lesssim \alpha d(Tv, v) + \frac{\beta[1 + d(Tv, T^n Tv)]d(v, T^n v)}{1 + d(Tv, v)} \\ &\quad + \gamma[d(Tv, T^n Tv) + d(v, T^n v)] + \delta[d(Tv, T^n v) + d(v, T^n Tv)] \\ &\lesssim \alpha d(Tv, v) + \frac{\beta[1 + d(Tv, TT^n v)]d(v, v)}{1 + d(Tv, v)} \\ &\quad + \gamma d(Tv, TT^n v) + \delta[d(Tv, v) + d(v, TT^n v)] \\ &= (\alpha + 2\delta)d(Tv, v) \end{aligned}$$

and so  $d(Tv, v) = 0$ .  $\square$

## References

- [1] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Series: Topological Fixed Point Theorems and its Applications, 6, Springer, New York, 2009.

- [2] M. A. Alghamdi, V. Berinde, N. Shahzad, *Fixed points of non-self almost contractions*, Carpathian Journal of Mathematics, **30**(1)(2014), 7-14.
- [3] A. Azam, B. Fisher, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numerical Functional Analysis and Optimization, **3**(3) (2011), 243-253.
- [4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3**(1922), 133-181.
- [5] O. Hadzic, E. Pap, *Fixed Point Theory in PM-Spaces*, Kluwer Academic, Dordrecht, The Netherlands(2001).
- [6] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468-1476.
- [7] J.L. Kelley, *General Topology*, Van Nostrand Reinhold, New York(1955).
- [8] H.P.A. Kunzi, *A note on sequentially compact quasi-pseudo-metric spaces*, Monatshefte Math., **95**(1983), 219-220.
- [9] M. Telci, B. Fisher, *On a fixed point theorem for fuzzy mappings in quasi-metric spaces*, Thai J. Math., **2**(2003), 1-8.
- [10] J.S. Ume, *Fixed point theorems in generalizing spaces of quasi metric family and applications*, Indian J. Pure App. Math., **33**(2002), 1041-1051.

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