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Two-channel sampling in wavelet subspaces

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Abstract

We develop two-channel sampling theory in the wavelet subspace V_1 from the multi resolution analysis $\{V_j\}_{j\in\mathbb{Z}}$. Extending earlier results by G. G. Walter [11], W. Chen and S. Itoh [2] and Y. M. Hong et al [5] on the sampling theory in the wavelet or shift invariant spaces, we find a necessary and sufficient condition for two-channel sampling expansion formula to hold in V_1.

1 Indroduction

The classical Whittaker-Shannon-Kotel'nikov(WSK) sampling theorem [4] states that any signal $f(t)$ with finite energy and the bandwidth π can be completely reconstructed from its discrete values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

WSK sampling theorem has been extended in many directions (see [1], [2], [5], [6], [7], [8], [10], [11], [12] and references therein). G. G. Walter [11] developed a sampling theorem in wavelet subspaces, noting that the sampling function sinc$t := \sin\pi t/\pi t$ in the WSK sampling theorem is a scaling function of a multi-resolution analysis. A. J. E. M. Janssen [6] used the Zak transform to generalize Walter's work to regular shifted sampling. Later, W. Chen and S. Itoh [2] (see also [12]) extended Walter's result further by relaxing conditions

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on the scaling function $\phi(t)$. Recently in [5],[8] general sampling expansion are handled on shift invariant spaces([9]). In this work, we find a necessary and sufficient condition for two-channel sampling expansion to hold in the wavelet subspace V_1 of a multi resolution analysis $\{V_j\}_{j\in\mathbb{Z}}$.

2 Preliminaries

For a measurable function $f(t)$ on $\mathbb{R}$, we let

$$\|f(t)\|_0 := \sup_{|E|=0} \inf_{\mathbb{R}\setminus E} |f(t)| \quad \text{and} \quad \|f(t)\|_\infty := \inf_{|E|=0} \sup_{\mathbb{R}\setminus E} |f(t)|$$

be the *essential infimum* and the *essential supremum* of $|f(t)|$ on $\mathbb{R}$ respectively, where $|E|$ is the Lebesgue measure of E. For any $f(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we let

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt$$

be the Fourier transform of $f(t)$ so that $\frac{1}{\sqrt{2\pi}}\mathcal{F}(\cdot)$ becomes a unitary operator on $L^2(\mathbb{R})$. A sequence $\{\phi_n : n \in \mathbb{Z}\}$ in a Hilbert space H is called a Riesz sequence if $\{\phi_n : n \in \mathbb{Z}\}$ is a Riesz basis of the closed subspace $V := \overline{\text{span}}\{\phi_n : n \in \mathbb{Z}\}$ of H.

Definition 1. *A function $\phi(t) \in L^2(\mathbb{R})$ is called a scaling function of a multiresolution analysis (MRA in short) $\{V_j\}_{j\in\mathbb{Z}}$ if the closed subspaces V_j of $L^2(\mathbb{R})$,*

$$V_j := \overline{span}\Big\{\phi(2^j t - k) : k \in \mathbb{Z}\Big\}, \quad j \in \mathbb{Z}$$

satisfy the following properties;

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$;
2. $\overline{\bigcup V_j} = L^2(\mathbb{R})$;
3. $\bigcap V_j = \{0\}$;
4. $f(t) \in V_j$ *if and only if* $f(2t) \in V_{j+1}$;
5. $\{\phi(t-n) : n \in \mathbb{Z}\}$ *is a Riesz basis of* V_0.

The wavelet subspace W_j is the orthogonal complement of V_j in V_{j+1} so that $V_{j+1} = V_j \oplus W_j$. Then there is a wavelet $\psi(t) \in L^2(\mathbb{R})$ that induces a Riesz basis $\{\psi(2^j t - k) : k \in \mathbb{Z}\}$ of W_j. Moreover, $\{\phi(2^j t - k), \psi(2^j t - k) : k \in \mathbb{Z}\}$ forms a Riesz basis of V_{j+1}.

For any $\phi(t) \in L^2(\mathbb{R})$, $\{\phi(t-n) : n \in \mathbb{Z}\}$ is a Bessel sequence if there is a constant $B > 0$ such that

$$\sum_{n\in\mathbb{Z}} |\langle f(t), \phi(t-n)\rangle|^2 \le B\|f\|^2_{L^2(\mathbb{R})}, \quad f \in L^2(\mathbb{R})$$

or equivalently (see Theorem 7.2.3 in [3]) $G_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)|^2 \le B$ a.e. on $[0, 2\pi]$.

Lemma 1. *(Lemma 2.2 in [5] and Lemma 7.2.1 in [3]) Let $\phi(t) \in L^2(\mathbb{R})$ be such that $\{\phi(t-k) : k \in \mathbb{Z}\}$ is a Bessel sequence. Then, for any $\{c_k\}_{k\in\mathbb{Z}} \in l^2, \sum_{k\in\mathbb{Z}} c_k\phi(t-k)$ converges in $L^2(\mathbb{R})$ and*

$$\mathcal{F}\Big(\sum_{k\in\mathbb{Z}} c_k\phi(t-k)\Big) = \Big(\sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}\Big)\hat{\phi}(\xi).$$

For any $\mathbf{c} = \{c_k\}_{k\in\mathbb{Z}} \in l^2$, let $\widehat{\mathbf{c}}(\xi) := \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$. Then, $\widehat{\mathbf{c}}(\xi) \in L^2[0, 2\pi]$ or $C[0, 2\pi]$ if $\{c_k\}_{k\in\mathbb{Z}} \in l^2$ or l^1 respectively.

Lemma 2. *If $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}}$, $\mathbf{b} = \{b_k\}_{k\in\mathbb{Z}} \in l^2$ and $\widehat{\mathbf{a}}(\xi) \in L^\infty[0, 2\pi]$, then $\mathbf{a} * \mathbf{b} := \big\{\sum_{j\in\mathbb{Z}} a_j b_{k-j}\big\}_{k\in\mathbb{Z}} \in l^2$ and*

$$\widehat{\mathbf{a} * \mathbf{b}}(\xi) = \widehat{\mathbf{a}}(\xi)\widehat{\mathbf{b}}(\xi).$$

Proof. Since $\widehat{\mathbf{a}}(\xi) \in L^\infty[0, 2\pi]$ and $\widehat{\mathbf{b}}(\xi) \in L^2[0, 2\pi]$, $\widehat{\mathbf{a}}(\xi)\widehat{\mathbf{b}}(\xi) \in L^2[0, 2\pi]$. Hence we can expand $\widehat{\mathbf{a}}(\xi)\widehat{\mathbf{b}}(\xi)$ into its Fourier series $\sum_n c_n e^{-in\xi}$ in $L^2[0, 2\pi]$, where

$$\begin{aligned} c_n &= \frac{1}{2\pi}\Big\langle \widehat{\mathbf{a}}(\xi)\widehat{\mathbf{b}}(\xi), e^{-in\xi}\Big\rangle_{L^2[0,2\pi]} = \frac{1}{2\pi}\Big\langle \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}, \Big(\sum_{k\in\mathbb{Z}} \overline{b_k} e^{ik\xi}\Big) e^{-in\xi}\Big\rangle_{L^2[0,2\pi]} \\ &= \frac{1}{2\pi}\Big\langle \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}, \sum_{k\in\mathbb{Z}} \overline{b_{n-k}} e^{-ik\xi}\Big\rangle_{L^2[0,2\pi]} = \sum_{k\in\mathbb{Z}} a_k b_{n-k} \end{aligned}$$

by Parseval's identity. Hence the conclusion follows. □

For any $\phi(t) \in L^2(\mathbb{R})$, let

$$H_\phi(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)| \quad \text{and} \quad C_\phi(t) := \sum_{n\in\mathbb{Z}} |\phi(t+n)|^2.$$

3 Main Result and an example

In the following, let $\phi(t) \in L^2(\mathbb{R})$ be a scaling function of an MRA $\{V_j\}_{j\in\mathbb{Z}}$ and $\psi(t)$ the associated wavelet, of which we always assume that $H_\phi(\xi)$ and $H_\psi(\xi)$ are in $L^\infty[0, 2\pi]$. Then (cf. Proposition 2.4 in [8]) $\phi(t)$ and $\psi(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\sup\limits_{\mathbb{R}} C_\phi(t) < \infty$, $\sup\limits_{\mathbb{R}} C_\psi(t) < \infty$. Hence for any $\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in l^2$, $\big(\mathbf{c} * \phi\big)(t) := \sum\limits_{n\in\mathbb{Z}} c_n\phi(t-n)$ converges both in $L^2(\mathbb{R})$ and uniformly in $\mathbb{R}$ so that each $V_j \subset L^2(\mathbb{R}) \cap C(\mathbb{R}), j \in \mathbb{Z}$.

Let $\mathcal{L}_j[\cdot]$ be the LTI (linear time invariant) systems with frequency responses $M_j(\xi) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for $j = 1, 2$. Then

$$\mathcal{L}_j[f](t) = \mathcal{F}^{-1}(\hat{f}M_j)(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R}), f \in L^2(\mathbb{R}) \text{ and } j = 1, 2$$

and $\lim\limits_{|t|\to\infty} \mathcal{L}_j[f](t) = 0$ by the Riemann-Lebesgue Lemma since $\hat{f}(\xi)M_j(\xi) \in L^1(\mathbb{R})$.

Moreover by the Poisson summation formula (cf. Lemma 5.1 in [8]), for any fixed $t \in \mathbb{R}$ and $j = 1, 2$,

$$\sum_{n\in\mathbb{Z}} \mathcal{L}_j[\phi](t+n)e^{-in\xi} = \sum_{n\in\mathbb{Z}} \hat{\phi}(\xi + 2n\pi)M_j(\xi + 2n\pi)e^{it(\xi+2n\pi)}$$

and

$$\sum_{n\in\mathbb{Z}} \mathcal{L}_j[\psi](t+n)e^{-in\xi} = \sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi)M_j(\xi + 2n\pi)e^{it(\xi+2n\pi)}$$

are in $L^\infty[0, 2\pi]$ as functions in ξ since $H_\phi(\xi), H_\psi(\xi) \in L^\infty[0, 2\pi]$.

In particular, $A_{i,j}(\xi) = A_{i,j}(\xi + 2\pi) \in L^\infty[0, 2\pi]$ for $i, j = 1, 2$, where

$$A_{1,j}(\xi) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j[\phi](n)e^{-in\xi}, \quad A_{2,j}(\xi) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j[\psi](n)e^{-in\xi}.$$

Let

$$A(\xi) = [A_{i,j}(\xi)]_{i,j=1}^2.$$

Then for any $f(t) = \sum\limits_{k\in\mathbb{Z}} c_{1,k}\phi(t-k) + \sum\limits_{k\in\mathbb{Z}} c_{2,k}\psi(t-k) \in V_1$, where $\{c_{1,k}\}_{k\in\mathbb{Z}}$ and $\{c_{2,k}\}_{k\in\mathbb{Z}}$ in l^2, we have for $i = 1, 2$ and $n \in \mathbb{Z}$

$$\mathcal{L}_i(f)(t) = \sum_{k\in\mathbb{Z}} c_{1,k}\mathcal{L}_i(\phi)(t-k) + \sum_{k\in\mathbb{Z}} c_{2,k}\mathcal{L}_i(\psi)(t-k),$$

which converges both in $L^2(\mathbb{R})$ and absolutely on $\mathbb{R}$. In particular

$$\mathcal{L}_i(f)(n) = \sum_{k\in\mathbb{Z}} c_{1,k}\mathcal{L}_i(\phi)(n-k) + \sum_{k\in\mathbb{Z}} c_{2,k}\mathcal{L}_i(\psi)(n-k), \ n \in \mathbb{Z} \text{ and } i = 1, 2. \tag{1}$$

Lemma 3. *Assume that* $\det A(\xi) \neq 0$ *a.e. in* $[0, 2\pi]$. *Let* $\lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi)$ *be eigenvalues of the Hermitian matrix* $B(\xi) = (A(\xi)^* A(\xi))^{-1}$. *If* $\| \det A(\xi)\|_0 > 0$, *then*

$$0 < \|\lambda_{1,B}(\xi)\|_0 \le \|\lambda_{2,B}(\xi)\|_\infty < \infty.$$

Proof. Since $B(\xi)$ is a nonsingular positive semi-definite Hermitian matrix a.e. in $[0, 2\pi]$,

$$0 < \lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi) \quad \text{a.e. in } [0, 2\pi].$$

Since $A_{i,j}(\xi) \in L^\infty[0, 2\pi]$ and $\| \det A(\xi)\|_0 > 0$, all entries of $B(\xi)$ are also in $L^\infty[0, 2\pi]$ so that the characteristic equation of $B(\xi)$ is of the form

$$\lambda(\xi)^2 + f(\xi)\lambda(\xi) + g(\xi) = 0$$

where $f(\xi)$ and $g(\xi)$ are real-valued functions in $L^\infty[0, 2\pi]$. Hence $0 < \|\lambda_{2,B}(\xi)\|_\infty < \infty$. Since $\lambda_{1,B}(\xi)\lambda_{2,B}(\xi) = \det B(\xi) = |\det A(\xi)|^{-2}$,

$$\| \det A(\xi)\|_\infty^{-2} \le \lambda_{1,B}(\xi)\lambda_{2,B}(\xi) \le \| \det A(\xi)\|_0^{-2} \quad \text{a.e. in } [0, 2\pi]$$

so that $0 < \| \det A(\xi)\|_\infty^{-2} \|\lambda_{2,B}(\xi)\|_\infty^{-1} \le \lambda_{1,B}(\xi)$ a.e. in $[0, 2\pi]$. □

Definition 2. *For any* $f_1(t)$ *and* $f_2(t)$ *in* $L^2(\mathbb{R})$, *let* $F(\xi) := [F_{i,j}(\xi)]_{i,j=1}^2$ *be the Gramian of* $\{f_1, f_2\}$, *where* $F_{i,j}(\xi) := \sum_{k\in\mathbb{Z}} \hat{f}_i(\xi + 2k\pi)\overline{\hat{f}_j(\xi + 2k\pi)}$.

Then as a Hermitian matrix, $F(\xi)$ has real eigenvalues.

Proposition 1. *([9]) Let* $\lambda_{1,F}(\xi) \le \lambda_{2,F}(\xi)$ *be eigenvalues of the Gramian* $F(\xi)$ *of* $\{f_1, f_2\}$. *Then* $\{f_1(t-n), f_2(t-n) : n \in \mathbb{Z}\}$ *is a Riesz sequence if and only if*

$$0 < \|\lambda_{1,F}(\xi)\|_0 \le \|\lambda_{2,F}(\xi)\|_\infty < \infty.$$

Lemma 4. *Assume* $\| \det A(\xi)\|_0 > 0$. *Let* $\begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix} := A(\xi)^{-1} \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix}$ *and* $S_i(t) := \mathcal{F}^{-1}(\hat{S}_i)(t)$ *for* $i = 1, 2$. *Then* $S_i(t) \in V_1$ *for* $i = 1, 2$ *and* $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ *is a Riesz sequence.*

Proof. Let $A(\xi)^{-1} = C(\xi) = [C_{i,j}(\xi)]_{i,j=1}^2$. Since $C_{i,j}(\xi) \in L^\infty(\mathbb{R})$, $\hat{S}_i(\xi) = C_{i,1}(\xi)\hat{\phi}(\xi) + C_{i,2}(\xi)\hat{\psi}(\xi) \in L^2(\mathbb{R})$ for $i = 1, 2$. Since $C_{i,j}(\xi) = C_{i,j}(\xi + 2\pi) \in L^\infty[0, 2\pi]$, we may expand $C_{i,j}(\xi)$ into its Fourier series $C_{i,j}(\xi) = \sum_{k\in\mathbb{Z}} c_{i,j,k}e^{-ik\xi}$ where $\{c_{i,j,k}\}_{k\in\mathbb{Z}} \in l^2$. Then by Lemma 1,

$$\hat{S}_i(\xi) = \sum_{k\in\mathbb{Z}} \left(c_{i,1,k}e^{-ik\xi}\hat{\phi}(\xi) + c_{i,2,k}e^{-ik\xi}\hat{\psi}(\xi) \right)$$

so that

$$S_i(t) := \mathcal{F}^{-1}(\hat{S}_i)(t) = \sum_{k\in\mathbb{Z}} \Big(c_{i,1,k}\phi(t-k) + c_{i,2,k}\psi(t-k)\Big) \in V_1.$$

Let $G(\xi)$ and $S(\xi)$ be the Gramians of $\{\phi,\psi\}$ and $\{S_1,S_2\}$ respectively and $\lambda_{1,G}(\xi) \le \lambda_{2,G}(\xi)$ and $\lambda_{1,S}(\xi) \le \lambda_{2,S}(\xi)$ the eigenvalues of $G(\xi)$ and $S(\xi)$ respectively. Then $S(\xi) = C(\xi)G(\xi)C(\xi)^*$. Let $U_S(\xi)$ and $U_G(\xi)$ be unitary matrices, which diagonalize $S(\xi)$ and $G(\xi)$ respectively, i.e.,

$$S(\xi) = U_S(\xi)\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi)\end{bmatrix} U_S(\xi)^*$$

and

$$G(\xi) = U_G(\xi)\begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi)\end{bmatrix} U_G(\xi)^*.$$

Then

$$\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi)\end{bmatrix} = R(\xi)\begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi)\end{bmatrix} R(\xi)^*$$

where

$$R(\xi) = U_S(\xi)^*C(\xi)U_G(\xi) := \begin{bmatrix} R_{1,1}(\xi) & R_{1,2}(\xi) \\ R_{2,1}(\xi) & R_{2,2}(\xi)\end{bmatrix}$$

so that

$$\lambda_{1,S}(\xi) = \lambda_{1,G}(\xi)|R_{1,1}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{1,2}(\xi)|^2; \qquad (2)$$

$$\lambda_{2,S}(\xi) = \lambda_{1,G}(\xi)|R_{2,1}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{2,2}(\xi)|^2. \qquad (3)$$

On the other hand,

$$\begin{aligned} R(\xi)R(\xi)^* &= U_S(\xi)^*C(\xi)C(\xi)^*U_S(\xi) = U_S(\xi)^*B(\xi)U_S(\xi) \qquad (4)\\ &= U_S(\xi)^*U_B(\xi)\begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi)\end{bmatrix} U_B(\xi)^*U_S(\xi), \end{aligned}$$

where $U_B(\xi)$ is the unitary matrix such that

$$B(\xi) = U_B(\xi)\begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi)\end{bmatrix} U_B(\xi)^*$$

with $\lambda_{1,B}(\xi) \le \lambda_{2,B}(\xi)$. Set $U_S(\xi)^*U_B(\xi) = \big[D_{i,j}(\xi)\big]_{i,j=1}^2$, which is also a unitary matrix. Then we have from diagonal entries of both sides of (4),

$$|R_{1,1}(\xi)|^2 + |R_{1,2}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{1,1}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{1,2}(\xi)|^2; \qquad (5)$$

$$|R_{2,1}(\xi)|^2 + |R_{2,2}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{2,1}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{2,2}(\xi)|^2. \qquad (6)$$

Then we have a.e. in $[0, 2\pi]$ (from (2), (3), (5), and (6))

$$\lambda_{1,S}(\xi) \geq \lambda_{1,G}(\xi)\Big(|R_{1,1}(\xi)|^2 + |R_{1,2}(\xi)|^2\Big) \geq \lambda_{1,G}(\xi)\lambda_{1,B}(\xi);$$

$$\lambda_{2,S}(\xi) \leq \lambda_{2,G}(\xi)\Big(|R_{,21}(\xi)|^2 + |R_{2,2}(\xi)|^2\Big) \leq \lambda_{2,G}(\xi)\lambda_{2,B}(\xi)$$

since $|D_{1,1}(\xi)|^2 + |D_{1,2}(\xi)|^2 = |D_{2,1}(\xi)|^2 + |D_{2,2}(\xi)|^2 = 1$ a.e. in $[0, 2\pi]$. Hence

$$\begin{aligned} 0 &< \|\lambda_{1,G}(\xi)\|_0 \|\lambda_{1,B}(\xi)\|_0 \leq \|\lambda_{1,S}(\xi)\|_0 \\ &\leq \|\lambda_{2,S}(\xi)\|_\infty \leq \|\lambda_{2,G}(\xi)\|_\infty \|\lambda_{2,B}(\xi)\|_\infty < \infty \end{aligned}$$

by Lemma 3 and Proposition 4 so that $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz sequence by Proposition 4. □

Now we are ready to give the main result of this work.

Theorem 1. *There exist $S_i(t) \in V_1$ $(i = 1, 2)$ such that $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz basis of V_1 for which two-channel sampling formula*

$$f(t) = \sum_{n\in\mathbb{Z}} \mathcal{L}_1(f)(n)S_1(t-n) + \sum_{n\in\mathbb{Z}} \mathcal{L}_2(f)(n)S_2(t-n) \tag{7}$$

holds for $f \in V_1$ if and only if $\|\det A(\xi)\|_0 > 0$. In this case

$$S_i(t) = \mathcal{F}^{-1}\big(C_{i,1}(\xi)\hat{\phi}(\xi) + C_{i,2}(\xi)\hat{\psi}(\xi)\big)(t) \quad \text{for } i = 1, 2. \tag{8}$$

Proof. Assume $\|\det A(\xi)\|_0 > 0$ and define $S_i(t)$ by (8). Then $S_i(t) \in V_1$ $(i = 1, 2)$ and $\{S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz sequence by Lemma 5. For any $f(t) \in V_1$

$$f(t) = \sum_{k\in\mathbb{Z}} c_{1,k}\phi(t-k) + \sum_{k\in\mathbb{Z}} c_{2,k}\psi(t-k)$$

where $\{c_{i,k}\}_{k\in\mathbb{Z}} \in l^2$ for $i = 1, 2$, we have by Lemma 1,

$$\hat{f}(\xi) = \Big(\sum_{k\in\mathbb{Z}} c_{1,k}e^{-ik\xi}\Big)\hat{\phi}(\xi) + \Big(\sum_{k\in\mathbb{Z}} c_{2,k}e^{-ik\xi}\Big)\hat{\psi}(\xi). \tag{9}$$

Since $\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix} = A(\xi) \begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix}$, we have by (1), (9) and Lemma 1

$$\begin{aligned} \hat{f}(\xi) &= \sum_{j=1}^{2} \Big[\Big(\sum_{k\in\mathbb{Z}} c_{1,k}e^{-ik\xi}\Big)A_{1,j}(\xi) + \Big(\sum_{k\in\mathbb{Z}} c_{2,k}e^{-ik\xi}\Big)A_{2,j}(\xi)\Big]\hat{S}_j(\xi) \qquad (10) \\ &= \sum_{n\in\mathbb{Z}} \Big[\mathcal{L}_1[f](n)e^{-in\xi}\hat{S}_1(\xi)\Big] + \sum_{n\in\mathbb{Z}} \Big[\mathcal{L}_2[f](n)e^{-in\xi}\hat{S}_2(\xi)\Big]. \end{aligned}$$

Taking the inverse Fourier transform on (10) gives (7), which implies $V_1 = \overline{\text{span}}\,\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ so that $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz basis of V_1. Conversely assume that there exist $S_i(t) \in V_1$ $(i = 1,2)$ such that $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ is a Riesz basis of V_1 and (7) holds. In particular,

$$\begin{aligned} \phi(t) &= \sum_{n\in\mathbb{Z}} \mathcal{L}_1[\phi](n)S_1(t-n) + \sum_{n\in\mathbb{Z}} \mathcal{L}_2[\phi](n)S_2(t-n); \\ \psi(t) &= \sum_{n\in\mathbb{Z}} \mathcal{L}_1[\psi](n)S_1(t-n) + \sum_{n\in\mathbb{Z}} \mathcal{L}_2[\psi](n)S_2(t-n). \end{aligned}$$

By taking Fourier transform and using Lemma 1, we have

$$\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix} = A(\xi) \begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix}. \tag{11}$$

We then have as in the proof of Lemma 5, $G(\xi) = A(\xi)S(\xi)A(\xi)^*$, where $G(\xi)$ and $S(\xi)$ are Gramians of $\{\phi, \psi\}$ and $\{S_1, S_2\}$ respectively. Hence $\det G(\xi) = \det S(\xi)|\det A(\xi)|^2$ so that

$$|\det A(\xi)|^2 = \frac{\det G(\xi)}{\det S(\xi)} = \frac{\lambda_{1,G}(\xi)\lambda_{2,G}(\xi)}{\lambda_{1,S}(\xi)\lambda_{2,S}(\xi)} \geq \frac{\lambda_{1,G}(\xi)^2}{\lambda_{2,S}(\xi)^2} \quad \text{a.e. in } [0, 2\pi],$$

where $\lambda_{1,G}(\xi) \leq \lambda_{2,G}(\xi)$ and $\lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi)$ are eigenvalues of $G(\xi)$ and $S(\xi)$ respectively. Therefore,

$$|\det A(\xi)| \geq \frac{\lambda_{1,G}(\xi)}{\lambda_{2,S}(\xi)} \geq \frac{\|\lambda_{1,G}(\xi)\|_0}{\|\lambda_{2,S}(\xi)\|_\infty} \quad \text{a.e. in } [0, 2\pi]$$

so that $\|\det A(\xi)\|_0 > 0$ since both $\{\phi(t-n), \psi(t-n) : n \in \mathbb{Z}\}$ and $\{S_i(t-n) : i = 1,2 \text{ and } n \in \mathbb{Z}\}$ are Riesz sequences. Finally (8) comes from (11) immediately. □

Note that if $\{\mathcal{L}_i(\phi)(n)\}_{n\in\mathbb{Z}}$ and $\{\mathcal{L}_i(\psi)(n)\}_{n\in\mathbb{Z}} \in l^1$, then $A_{i,j}(\xi) \in C[0, 2\pi]$ for i, $j = 1,2$ so that $A_{i,j}(\xi) \in L^\infty[0, 2\pi]$ and $\|\det A(\xi)\|_0 > 0$ is equivalent to $\det A(\xi) \neq 0$ on $[0, 2\pi]$.

Example. *(2-channel sampling in Paley-Wiener space)*
Let $\phi(t) = \text{sinc}t$ so that $V_0 = \overline{\text{span}}\{\phi(t-n) : n \in \mathbb{Z}\} = PW_\pi$ and $V_1 = \overline{\text{span}}\{\phi(2t-n) : n \in \mathbb{Z}\} = PW_{2\pi}$. Then $V_1 = V_0 \oplus W_0$ where $W_0 = \overline{\text{span}}\{\psi(t-n) : n \in \mathbb{Z}\}$ and $\psi(t) = (\cos\frac{3}{2}\pi t)(\text{sinc}\frac{1}{2}t)$.

Note that $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\xi)$ and $\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(\xi)$, where $\chi_E(\cdot)$ is the characteristic function of a set E in $\mathbb{R}$. We have by (12)

$$\sum_{n\in\mathbb{Z}} \mathcal{L}_i[\phi](n)e^{-in\xi} = \begin{cases} M_i(\xi) & , \ \xi \in [0,\pi) \\ M_i(\xi-2\pi) & , \ \xi \in [\pi, 2\pi] \end{cases}$$

and

$$\sum_{n\in\mathbb{Z}} \mathcal{L}_i[\psi](n)e^{-in\xi} = \begin{cases} M_i(\xi-2\pi) & , \ \xi \in [0,\pi) \\ M_i(\xi) & , \ \xi \in [\pi, 2\pi] \end{cases}.$$

Hence

$$A(\xi) = \begin{cases} \begin{bmatrix} M_1(\xi) & M_2(\xi) \\ M_1(\xi-2\pi) & M_2(\xi-2\pi) \end{bmatrix} & \text{on } [0,\pi) \\ \\ \begin{bmatrix} M_1(\xi-2\pi) & M_2(\xi-2\pi) \\ M_1(\xi) & M_2(\xi) \end{bmatrix} & \text{on } [\pi,2\pi] \end{cases}$$

so that the determinant condition $\|\text{det}A(\xi)\|_0 > 0$ is equivalent to $\|\text{det}M(\xi)\|_0 > 0$ where

$$M(\xi) = \begin{bmatrix} M_1(\xi) & M_1(\xi-2\pi) \\ M_2(\xi) & M_2(\xi-2\pi) \end{bmatrix}.$$

Take $M_1(\xi) = 1$ and $M_2(\xi) = -i\text{sgn}\xi$ so that $\mathcal{L}_1[f](t) = f(t)$ and $\mathcal{L}_2[f](t) = \tilde{f}(t)$ where $\tilde{f}(t)$ is the Hilbert transform of $f(t)$. Then

$$M(\xi) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

so that $\|\text{det}M(\xi)\|_0 = 2$. As a consequence, the sampling formula holds on $V_1 = PW_{2\pi}$. In fact, we have from (11),

$$\begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix} = \begin{cases} \frac{1}{2\sqrt{2\pi}} \begin{bmatrix} 1 \\ -i \end{bmatrix} & \text{on } [-2\pi, 0) \\ \\ \frac{1}{2\sqrt{2\pi}} \begin{bmatrix} 1 \\ i \end{bmatrix} & \text{on } [0, 2\pi] \\ \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{otherwise} \end{cases}$$

so that

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}2(t-n) - \sum_{n\in\mathbb{Z}} \tilde{f}(n)\sin\pi(t-n)\text{sinc}(t-n), \ f \in PW_{2\pi}.$$

As another example, take $M_1(\xi) = 1$ and $M_2(\xi) = i\xi$ so that $\mathcal{L}_1[f](t) = f(t)$ and $\mathcal{L}_2[f](t) = f'(t)$. Then

$$M(\xi) = \begin{bmatrix} 1 & 1 \\ i\xi & i(\xi - 2\pi) \end{bmatrix}$$

so that $\|\mathrm{det} M(\xi)\|_0 = 2\pi$. By the similar procedure as above, we obtain a sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}^2 2(t-n) + \frac{1}{\pi}\sum_{n\in\mathbb{Z}} f'(n)\sin\pi(t-n)\mathrm{sinc}(t-n), \;\; f \in PW_{2\pi}.$$

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References

[1] A. Aldroubi and K. Gröchenig, *Nonuniform sampling and reconstruction in shift-invariant spaces*, SIAM Review, vol. **43**, no. 4, pp. 585-620, 2001.

[2] W. Chen and S. Itoh, *A sampling theorem for shift-invariant subspace*, IEEE Trans. Signal Processing, vol. **46**, pp. 2822-2824, 1998.

[3] O. Christensen, *An Introduction to frames and Riesz Bases*, Birkhäuser, 2002.

[4] J. R. Higgins, *Sampling theory in Fourier and signal analysis*, Oxford University Press, 1996.

[5] Y. M. Hong, J. M. Kim, K. H. Kwon and E. H. Lee, *Channeled sampling in shift invariant spaces*, Int. J. Wavelets Multiresolut. Inf. Process., vol. **5**, pp. 753-767, 2007.

[6] A. J. E. M. Janssen, *The Zak-transform and sampling theorem for wavelet subspaces*, IEEE Trans. Signal Processing, vol. **41**, pp. 3360-3364, 1993.

[7] A. J. Jerri, *The Shannon sampling theorem-Its various extentions and applications: A tutorial review*, Proc. IEEE, vol. **65**, pp. 1565-1596, 1977.

[8] J. M. Kim and K. H. Kwon, *Sampling expansion in shift invariant spaces*, Int. J. Wavelets Multiresolut. Inf. Process., vol. **6**, no. 2, pp. 223-248, 2008.

[9] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L^2(\mathbb{R})$, Canadian J. Math., vol. **47**, pp.1051-1094, 1995.

[10] M. Unser, *Sampling-50 years after Shannon*, Proc. IEEE, vol. **88**, pp. 567-587, 2000.

[11] G. G. Walter, *A sampling theorem for wavelet subspaces*, IEEE Trans. Inform. Theory, vol. **38**, pp. 881-884, 1992.

[12] P. Zhao, G. Liu and S. Itoh, *Frames and sampling theorem for trnaslation-invariant subspaces*, IEEE Signal Processing Letters, vol. **11**, pp. 8-11, 2004.

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