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Non-commutative finite monoids of a given order $n \geq 4$

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Abstract

For a given integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $(k \geq 2)$, we give here a class of finitely presented finite monoids of order n. Indeed the monoids $Mon(\pi)$, where

$$\pi = \langle a_1, a_2, \dots, a_k | a_i^{p_i^{\alpha_i}} = a_i,\ (i = 1, 2, \dots, k),\ a_i a_{i+1} = a_i,\ (i = 1, 2, \dots, k-1)\rangle.$$

As a result of this study we are able to classify a wide family of the k-generated p-monoids (finite monoids of order a power of a prime p). An interesting difference between the center of finite p-groups and the center of finite p-monoids has been achieved as well. All of these monoids are regular and non-commutative.

1. Introduction

The study of finite monoids is of interest because of its applications in several branches of science, for instance, its uses and advantages in mathematics, computer science and finite machines are well-known. So identifying a finite monoid of a given positive integer n could be significant. In this paper we present a class of finite monoids for every integer n.

First of all we give a short history on the finitely presented semigroups and monoids. Let A be an alphabet. We denote by A^+ the *free semigroup on A*

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consisting of all non-empty words over A, and by A^* the *free monoid* $A^+ \cup \{1\}$, where 1 denotes the empty word. A *semigroup (or monoid) presentation* is an ordered pair $\langle A|R\rangle$, where $R \subseteq A^+ \times A^+$ (or $R \subseteq A^* \times A^*$). A semigroup (or monoid) presentation S is said to be *defined* by the semigroup (or monoid) presentation $\langle A|R\rangle$ if $S \cong A^+/\rho$ (or $S \cong A^*/\rho$), where ρ is the congruence on A^+ (or A^*) generated by R. Note that the semigroup presentations for a semigroup S are precisely monoid presentations (without the trivial relation 1=1) for the monoid S^1 obtained from S by adjoining an identity, whether or not S already has one.

Here, our notation is standard and we follow [5, 11, 12]. one may consult [10] for more information on the presentation of groups. To distinguish between the semigroup, the monoid and the group defined by a presentation $\pi = \langle A|R\rangle$, we shall denote them by $Sg(\pi)$, $Mon(\pi)$ and $Gp(\pi)$, respectively.

On comparing the semigroup, monoids and groups defined by a presentation the articles [1, 2, 3, 9] studied certain interesting classes of such algebraic structure. The references [6, 7] study two special and outstanding classes of semigroups. For the recently obtained results on the study of subsemigroups and the efficiency of semigroups one may see the interesting results [4, 8].

Considering the presentation

$$\pi = \langle a_1, a_2, \ldots, a_k | a_i^{p_i^{\alpha_i}} = a_i,\ (i = 1, \ldots, k),\ a_i a_{i+1} = a_i,\ (i = 1, \ldots, k-1)\rangle,$$

it is clear that $Gp(\pi)$ is a cyclic group of order $p_1^{\alpha_1} - 1$. Our results on the $Sg(\pi)$ and $Mon(\pi)$ are the following:

Theorem A. *For every integer $n \geq 4$, the monoid $Mon(\pi)$ is of order n and the semigroup $Sg(\pi)$ is order $n-1$. Moreover, $Mon(\pi)$ is non-commutative and regular monoid but is not an inverse monoid, for every n.*

Corollary B. *For $p_1 = p_2 = \cdots = p_k = p$, if $m = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ is a partition of the integer $m \geq 2$, $Mon(\pi)$ is a p-monoid (monoids of order p^m). Moreover, all the different monoids are non-isomorphic for all partitions of m.*

2. The proofs

In this section, we prove Theorem A and Corollary B.

Proof of Theorem A. First we show that the relation $a_i^l a_j^m = a_i^l$ holds for every positive integers l and m, where $1 \leq i < j \leq k$. Since $a_i a_{i+1} = a_i$ then we get $a_i^l a_{i+1}^m = a_i^l$. Therefore,

$$\begin{aligned}
a_i^l a_j^m &= a_i^l a_{i+1} a_j^m = a_i^l a_{i+1} a_{i+2} a_j^m = \cdots = a_i^l a_{i+1} a_{i+2} \cdots a_{j-2} a_{j-1} a_j^m \\
&= a_i^l a_{i+1} a_{i+2} \cdots a_{j-2} a_{j-1} \\
&= a_i^l a_{i+1} a_{i+2} \cdots a_{j-2} \\
&= \cdots \\
&= a_i^l a_{i+1} a_{i+2} \\
&= a_i^l a_{i+1} \\
&= a_i^l.
\end{aligned}$$

This implies that every elements of $Mon(\pi)$ may be uniquely presented in the form $a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_1^{t_1}$, where $0 \leq t_i \leq p_i^{\alpha_i} - 1$. Thus $Mon(\pi)$ is of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = n$.

To prove the regularity of the monoid $Mon(\pi)$, let $w = a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i}$ be an arbitrary element of $Mon(\pi)$, where i is the least positive integer such that $t_i \neq 0$. Two cases occur:

Case 1. If $t_i + 1 < p_i^{\alpha_i}$ then we set $w' = a_i^{p_i^{\alpha_i} - t_i - 1}$. So,

$$\begin{aligned}
ww'w &= (a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i}) a_i^{p_i^{\alpha_i} - t_i - 1} (a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i}) \\
&= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{p_i^{\alpha_i} - 1} a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} \\
&= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{p_i^{\alpha_i} - 1} a_i^{t_i} \\
&= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{p_i^{\alpha_i} + t_i - 1} \\
&= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} \\
&= w.
\end{aligned}$$

Case 2. If $t_i + 1 = p_i^{\alpha_i}$ we may set $w' = a_i^{p_i^{\alpha_i}-1}$. Then,

$$\begin{aligned} ww'w &= (a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i}) a_i^{p_i^{\alpha_i}-1} (a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i}) \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{p_i^{\alpha_i}+t_i-1} a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} a_i^{t_i} \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i+1} a_i^{t_i-1} \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{p_i^{\alpha_i}} a_i^{t_i-1} \\ &= a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_i^{t_i} \\ &= w. \end{aligned}$$

Every inverse monoid is indeed a regular monoid however, the converse is not the case in general. Here we show that the monoid $Mon(\pi)$ is not an inverse monoid, to do this we take into consideration three cases:
Case 1. There exists i, $(1 \leq i \leq k-1)$, such that $p_i^{\alpha_i} > 2$. Then the element $a_{i+1}a_i$ has the inverses $a_{i+1}^t a_i^{p_i^{\alpha_i}-2}$, where $0 \leq t \leq p_{i+1}^{\alpha_{i+1}} - 1$.
Case 2. For every i, $(1 \leq i \leq k-1)$, $p_i^{\alpha_i} = 2$ and $p_k^{\alpha_k} > 2$. Then the element $a_{i+1}a_i$ has the inverses $a_{i+1}^t a_i$, where $0 \leq t \leq p_{i+1}^{\alpha_{i+1}} - 1$.
Case 3. For every i, $(1 \leq i \leq k)$, $p_i^{\alpha_i} = 2$. Then the element a_1 has the inverses a_1 and a_2a_1. This shows that there are different inverses for some elements of $Mon(\pi)$. □

Proof of Corollary B. The first part is a straightforward result of the proof of Theorem A. To prove the second part, let $m = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and $m = \beta_1 + \beta_2 + \cdots + \beta_l$ be different partitions of the integer $m \geq 2$, where $k \geq 2$ and $l \geq 2$. Also, let

$$\pi_1 = \langle a_1, a_2, \ldots, a_k | a_i^{p^{\alpha_i}} = a_i,\ (1 \leq i \leq k),\ a_i a_{i+1} = a_i,\ (1 \leq i \leq k-1) \rangle,$$

and

$$\pi_2 = \langle b_1, b_2, \ldots, b_l | b_j^{p^{\beta_j}} = b_j,\ (1 \leq j \leq l),\ b_j b_{j+1} = b_j,\ (1 \leq j \leq l-1) \rangle,$$

be the related presentations for the partitions $(\alpha_i)_{i=1}^{i=k}$ and $(\beta_j)_{j=1}^{j=l}$, respectively. If $k \neq l$, then the number of generators in $Mon(\pi_1)$ and $Mon(\pi_2)$ is not equal and hence the monoids are non-isomorphic. Now suppose $k = l$. Since these partitions are different and have the same length there exists an integer r, $(1 \leq r \leq k)$, such that $\alpha_r \neq \beta_j$, for every j, $(1 \leq j \leq k)$. Now if $f : Mon(\pi_1) \longrightarrow Mon(\pi_2)$ is a monoid isomorphism and $f(a_r) = b_k^{t_k} b_{k-1}^{t_{k-1}} \cdots b_1^{t_1}$, where $0 \leq t_j \leq p^{\beta_j} - 1$, then the period$(a_r)$ is equal to the period$(f(a_r))$. This implies $\alpha_r = \beta_s$, for an s, $(1 \leq s \leq k)$, which is a contradiction. $\square$

3. Remarks

Remark 1. *Let $w = a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_{i+1}^{t_{i+1}} a_i^{t_i}$ be an arbitrary non-identity element of $Mon(\pi)$, where i is the least positive integer such that $t_i \neq 0$ and j is the greatest positive integer such that $t_j \neq 0$, $(1 \leq i \leq j \leq k)$. Then,*

$$C_{Mon(\pi)}(w) = \{a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_{i+1}^{t_{i+1}} a_i^{t'_i} | 1 \leq t'_i \leq p_i^{\alpha_i} - 1\} \cup \{1\}.$$

Proof. Let $w' = a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_r^{t'_r}$ be a non-identity element of $Mon(\pi)$, where r is the least positive integer such that $t'_r \neq 0$ and s is the greatest positive integer such that $t'_s \neq 0$, $(1 \leq r \leq s \leq k)$. If $ww' = w'w$, then

$$(a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_i^{t_i})(a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_r^{t'_r}) = (a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_r^{t'_r})(a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_i^{t_i}).$$

Three cases occur:
Case 1. $i > s$. Then,

$$(a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_i^{t_i})(a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_r^{t'_r}) = a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_r^{t'_r}.$$

Since every elements of $Mon(\pi)$ have unique presentation as $a_k^{t_k} a_{k-1}^{t_{k-1}} \cdots a_1^{t_1}$, it is necessary that $t_j = t_{j-1} = \cdots = t_i = 0$, which is a contradiction with $w \neq 1$.
Case 2. There exists l, $(r < l \leq s)$ such that $i = l$. Then,

$$a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_i^{t_i + t'_i} a_{l-1}^{t'_{l-1}} \cdots a_r^{t'_r} = a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_i^{t'_i} a_{l-1}^{t'_{l-1}} \cdots a_r^{t'_r}.$$

Therefore, $t_i = 0$, which is a contradiction.
Case 3. $i = r$. Then,

$$a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_i^{t_i + t'_i} = a_s^{t'_s} a_{s-1}^{t'_{s-1}} \cdots a_i^{t'_i + t_i}.$$

Hence, $j = s$ and $t_l = t'_l$, for every l, $(j \leq l \leq i - 1)$. Consequently, $w' = a_j^{t_j} a_{j-1}^{t_{j-1}} \cdots a_{i+1}^{t_{i+1}} a_i^{t'_i}$. This completes the proof. $\square$

As an important result of Remark 1 we get:

Remark 2. *$Mon(\pi)$ is centerless. So, there exist finite p-monoids which have the trivial center. (In spite of the fact that finite p-groups have non-trivial center.)*

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