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ON THE NUMBER OF POLYNOMIALS WITH COEFFICIENTS IN $[n]$

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Abstract

In this paper we introduce several natural sequences related to polynomials of degree s having coefficients in $\{1, 2, ..., n\}$ ($n \in \mathbb{N}$) which factor completely over the integers. These sequences can be seen as generalizations of A006218. We provide precise methods for calculating the terms and investigate the asymptotic behavior of these sequences for $s \in \{1, 2, 3\}$.

1 Introduction

For any two positive integers s and n, we denote by $\mathcal{A}_n^{(s)}$ the set of polynomials of degree s,

$$P(x) = c_s x^s + c_{s-1} x^{s-1} + \cdots + c_1 x + c_0, \tag{1}$$

having s integer roots, where the coefficients c_i belong to the set $[n] := \{1, 2, \cdots, n\}$. Let us denote by $A_n^{(s)}$ the cardinality of $\mathcal{A}_n^{(s)}$. We are mainly interested in these sequences $A_n^{(s)}$ and their asymptotic behavior.

It turns out that if $s = 1$, one ends up with a classical problem known as the Dirichlet divisor problem. Dirichlet (1849) showed that

$$A_n^{(1)} = n \ln n + (2\gamma - 1)n + O(\sqrt{n}),$$

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where γ is the Euler-Mascheroni number.

For $s = 2$, the sequence $A_n^{(2)}$ appeared in the following problem (see [5]) proposed by L. Panaitopol in the Romanian Mathematical Olympiad-Final Round 2004 : *For every $n \geq 4$, we have*

$$n < A_n^{(2)} < n^2. \tag{2}$$

The original solution of the author was the following. Since the polynomials $x^2 + (k+1)x + k$, $k = 1, \ldots, n-1$, and $2x^2 + 4x + 2$, $x^2 + 4x + 4$ are in $\mathcal{A}_n^{(2)}$ with $n \geq 4$, we obviously get that $n+1 \leq A_n^{(2)}$. Therefore, the first inequality in ([5]) must be true.

In order to show the second inequality in ([5]), we observe that if $P \in \mathcal{A}_n^{(2)}$, then $P(x) = a(x+x_1)(x+x_2)$, where $x_1, x_2 \in \mathbb{N}$, and a, $a(x_1+x_2)$, and ax_1x_2 are in $\{1, 2, \ldots, n\}$. We conclude that $x_2 \leq \frac{n}{ax_1}$, and so

$$A_n^{(2)} \leq \sum_{\substack{1 \leq x_1 \leq n \\ 1 \leq a \leq n}} \frac{n}{ax_1} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)^2.$$

It is easy to show (by induction for instance) that for every $n \geq 5$, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \sqrt{n}.$$

It is not difficult to check that $A_4^{(2)} = 5$. Combining the two inequalities above shows that $A_n^{(2)} < n^2$ for every $n \geq 4$.

These inequalities can be obtained from the exact formula (7) of $A_n^{(2)}$ which we derive in Section 3. In fact, this formula allows us to be a little more precise about the growth of the sequence $A_n^{(2)}$ in Section 5:

$$A_n^{(2)} = \frac{1}{4}n(\ln n)^2 + Cn(\ln n) + O(n),$$

with some constants $C \in [\gamma - 1, \gamma + \frac{1}{2}]$.

In Section 4, we provide an exact formula for $A_n^{(3)}$. All monic polynomials in $\mathcal{A}_{10}^{(3)}$ are included next:

$$\begin{gathered} p_1(x) = x^3 + 3x^2 + 3x + 1, \quad p_2(x) = x^3 + 4x^2 + 5x + 2, \\ p_3(x) = x^3 + 5x^2 + 7x + 3, \quad p_4(x) = x^3 + 5x^2 + 8x + 4, \\ p_5(x) = x^3 + 6x^2 + 9x + 4. \end{gathered}$$

Similarly, in $\mathcal{A}_{20}^{(4)}$ we have

$$q_1(x) = x^4 + 4x^3 + 6x^2 + 4x + 1, \quad q_2(x) = x^4 + 5x^3 + 9x^2 + 7x + 2,$$
$$q_3(x) = x^4 + 6x^3 + 12x^2 + 10x + 3, \quad q_4(x) = x^4 + 6x^3 + 13x^2 + 12x + 4,$$
$$q_5(x) = x^4 + 7x^3 + 15x^2 + 13x + 4, \quad q_6(x) = x^4 + 7x^3 + 17x^2 + 17x + 6$$
$$q_7(x) = x^4 + 8x^3 + 18x^2 + 16x + 5, \text{ and } q_8(x) = x^4 + 7x^3 + 18x^2 + 20x + 8.$$

2 General observations and case $s = 1$

Let us observe that for a polynomial $P \in \mathcal{A}_n^{(s)}$ we can write $P(x) = c_s Q(x)$ where $Q(x) = (x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_s)$ for some integers α_i. It is clear that all the coefficients of Q must be positive integers. Hence c_s divides all c_i for all $i = 0, 1, ..., s$. This implies that $Q \in \mathcal{A}_k^{(s)}$ with $k = \frac{m}{c_s}$, where $m = \max\{c_0, c_1, ..., c_s\}$.

It is natural then to introduce the following related sequence and the subset of $\mathcal{A}_n^{(s)}$, say $\mathcal{B}_n^{(s)}$, of polynomials as in (1), where $c_s = 1$ and we have $\max\{c_0, c_1, ..., c_{s-1}\} = n$. We let then $B_n^{(s)}$ the cardinality of $\mathcal{B}_n^{(s)}$. Clearly, for fixed s, the sequence $\{A_n^{(s)}\}_n$ is non-decreasing, but one can easily check that $\{B_n^{(s)}\}_n$ is not a monotone sequence.

In what follows we are going to use $\lfloor x \rfloor$ for the integer part of a real number x. Let us first show the following relation between $A_n^{(s)}$ and $B_k^{(s)}$.

Theorem 2.1. *For any two positive integers s and n, we have*

$$A_n^{(s)} = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor B_k^{(s)}. \tag{3}$$

Proof. First, we have the simple inclusion of sets

$$\cup_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \mathcal{B}_k^{(s)} \subset \mathcal{A}_n^{(s)}, \tag{4}$$

where $\ell\mathcal{B}_k^{(s)}$ means all polynomials ℓP with $P \in \mathcal{B}_k^{(s)}$ and $\ell \in \mathbb{N}$. The sets $\{\ell\mathcal{B}_k^{(s)}\}_{\ell,k}$ are disjoint since if ℓP and $\ell' P'$ are identical polynomials, with $P \in \mathcal{B}_k^{(s)}$ and $P' \in \mathcal{B}_{k'}^{(s)}$, we must have $\ell = \ell'$ and then $P = P'$ implies $k = k'$.

We have seen earlier that every polynomial in $P \in \mathcal{A}_n^{(s)}$ is the result of the product of some number $c_s \in [n]$ and a polynomial $Q \in \mathcal{B}_k^{(s)}$ with $k = \frac{m}{c_s}$, and $m = \max\{c_0, c_1, ..., c_s\} \leq n$. In other words, every polynomial in $\mathcal{A}_n^{(s)}$ is in one of the sets $\ell\mathcal{B}_k^{(s)}$ with $\ell k \leq n$. This shows that (4) is an equality and then (3) follows. ■

Another simple observation is that if $Q \in \mathcal{B}_n^{(s)}$,

$$Q(x) = (x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_s)$$

and α_i are integers, then all α_i must be positive. This can be seen by observing that $Q(x) > 0$ for every $x \geq 0$ so there is no positive root of Q.

As a result, we have as a simple consequence of Theorem 2.1 the following corollary.

Corollary 2.1. *For $n \in \mathbb{N}$ we have*

$$A_n^{(1)} = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor . \tag{5}$$

Proof. For $s = 1$, there is clearly only one polynomial satisfying the definition of $\mathcal{B}_n^{(1)}$, namely $P(x) = x + n$. Hence, the equation (3) gives (5). ■

Actually, the sequence $A_n^{(1)}$, or A006218, is well known and one can read more about it in The On-Line Encyclopedia of Integer Sequences. It is important [1, pp.112-113], to observe that

$$A_n^{(1)} = \sum_{k=1}^{n} \tau(k), \tag{6}$$

where $\tau(m)$ counts the number of positive integer divisors of m.

3 Exact formula for $A_n^{(2)}$

We recall that, in particular, $A_n^{(2)}$ is the number of quadratic polynomials $P(x) = c_2x^2 + c_1x + c_0$, where $c_0, c_1, c_2 \in [n]$, having only integer roots. For the calculation of $A_n^{(2)}$ we employ the divisor function again.

Theorem 3.1. *The sequence $A_n^{(2)}$ is given by*

$$A_n^{(2)} = \sum_{k=2}^{n} \left\lfloor \frac{n}{k} \right\rfloor \left\lfloor \frac{\tau(k)+1}{2} \right\rfloor . \tag{7}$$

Proof. From (3), it suffices to show that $B_k^{(2)} = \left\lfloor \frac{\tau(k)+1}{2} \right\rfloor$ if $k \geq 2$ and $B_1^{(2)} = 0$. It is clear that $B_1^{(2)} = 0$. If $k \geq 2$ and $P \in \mathcal{B}_k^{(2)}$ with $P(x) = (x+\alpha)(x+\beta)$

where α, β are positive integers, we may assume without loss of generality that $\alpha \le \beta$. If $\alpha = 1$ then we must have $\beta = k-1$. If $\beta \ge \alpha \ge 2$, we observe that $(\alpha-1)(\beta-1) \ge 1$, which in turn implies $k \ge \alpha\beta \ge \alpha+\beta$. Hence, we also get $\alpha\beta = k$. Let us write the set of divisors of k as

$$1 = d_1 < d_2 < \cdots < d_{\tau(k)} = k.$$

Summarizing, $B_k^{(2)}$ is the cardinality of the set

$$\mathcal{P} := \{(1, k-1)\} \cup \{(d_i, d_{\tau(M)-i+1}) | i \ge 1 \ \text{ with } 2 \le d_i \le d_{\tau(M)-i+1}\}.$$

In general, there is an even number of divisors of k, unless k is a perfect square. So, if k is not a perfect square, the cardinality of $\mathcal{P}$ is $\frac{\tau(k)}{2}$ and if k is a perfect square, the cardinality of $\mathcal{P}$ is $\frac{\tau(k)+1}{2}$. An unified formula for the number of elements of $\mathcal{P}$ is then $\left\lfloor \frac{1}{2}(\tau(k)+1) \right\rfloor$. This shows that $B_k^{(2)} = \left\lfloor \frac{\tau(k)+1}{2} \right\rfloor$ and the theorem follows from (3). ■

In order to derive the second inequality in ([5]), mentioned in the introduction, let us observe that from (7) we obtain

$$A_n^{(2)} \le n \sum_{k=2}^{n} \frac{\tau(k)+1}{2k} \le n \sum_{k=2}^{n} \frac{2\sqrt{k}+1}{2k} = n \sum_{k=2}^{n} \frac{1}{\sqrt{k}} + \frac{n}{2} \sum_{k=2}^{n} \frac{1}{k}$$

$$\le n(2\sqrt{n+1}-1) + \frac{n}{2}(\ln(n+1)+\gamma-1) < n^2, \quad n \ge 4.$$

We have used the classical inequality $\tau(k) \le 2\sqrt{k}$, which can be established by observing that if $1 = d_1 < d_2 < \cdots < d_s$ are the divisors of k not exceeding $\sqrt{k}$, we must have $s \le \sqrt{k}$ and the other divisors are simply $n = n/d_1$, $n/d_2, \cdots$, in number of $s-1$ or s depending upon k is a perfect square or not.

It does not appear that the sequence $A_n^{(2)}$ is a known sequence. Its first twenty terms are: 0, 1, 2, 5, 6, 10, 11, 16, 19, 23, 24, 33, 34, 38, 42, 50, 51, 60, 61, and 70.

4 Exact description for $A_n^{(3)}$

As before we need to calculate $B_k^{(3)}$ for all $k \ge 1$, and so we let $P \in \mathcal{B}_k^{(3)}$ and write

$$P(x) = (x+u)(x+v)(x+w),$$

where u, v and w are in $[k]$ with $\max(u,v,w) = k$. So, $B_k^{(3)}$ is the number of ordered triples (u,v,w) of integers, $1 \le u \le v \le w$ such that

$$\begin{cases} u+v+w \le k & (8.1) \\ uv+vw+wu \le k & (8.2) \\ uvw \le k & (8.3) \end{cases} \tag{8}$$

with at least one equality sign in (8). Let us assume that (u, v, w) is such a triple. Since $uv+vw+wu \ge u+v+w$ we may assume that the equality takes place either in the second or in the last inequality of (8).

However, for $u \ge 3$, because $u(uv+vw+wu) \le 3uvw$, we get

$$uvw \ge \frac{u}{3}(uv+vw+wu) \ge (uv+vw+wu)$$

which shows that we must have equality in (8.3) in this case. Hence, we need to study separately what happens if equality happens in (8.2) and not in (8.3) (Cases **(I)** and **(II)** in what follows).

(I) If $u = 1$, the system (8) reduces to only one equation, namely, $vw + u + v = k$ or $(v+1)(w+1) = k+1$. As we have seen in Section 3, there are $\left\lfloor \frac{\tau(k+1)-1}{2} \right\rfloor$ polynomials of this form in $\mathcal{B}_k^{(3)}$ (we needed to exclude the pair for which $v+1 = 1$). We observe that for $k+1$ prime, the contribution for this type of polynomials is equal to zero.

(II) If $u = 2$ and equality is attained only in (8.2), we have equivalently $2v+2w+vw = k$ and $2vw < k$, or $(v+2)(w+2) = k+4$ and $(v-2)(w-2) < 4$. Hence, $(v-2)^2 < 4$ or $v < 4$.

If $v = 2$ then $w+2 = (k+4)/4$ which attracts $k \ge 12$ with k a multiple of 4 ($k = 4\ell$, $\ell \ge 3$). We will simply write $(2, 2, \ell-1) \in \mathcal{B}_{4\ell}^{(3)}$ and observe that equality is taking place only in (8.2).

If $v = 3$ then $w < 6$ which forces $k = (v+2)(w+2) - 4$ to be in the set $\{21, 26, 31\}$.

Let us define the following step (counting) function that is going to be the contribution in $B_k^{(3)}$ for all the above situations in case **(II)**:

$$f(k) = \begin{cases} 1, & \text{if } k = 4\ell \text{ with } \ell \ge 3; \\ 1, & \text{if } k \in \{21, 26, 31\}; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

(III) This case is characterized by (8) and the fact that equality is attained in (8.3), i.e. $uvw = k$. In terms of u, this is equivalent to

$$u \ge 3 \text{ or } (u = 2, 2vw = k, \text{ and } 2v+2w+vw \le k). \tag{10}$$

If $u = 2$, this means that $k = 2\ell$ and the system above becomes $vw = \ell$ and $(v-2)(w-2) \ge 4$. It is convenient to take the negation of the last inequality:

if $v = 2$ we have basically no restriction on w so $k = 4\ell$ with $\ell \geq 1$, and if $v = 3$ then $k \in \{18, 24, 30\}$.

We need the following lemma which is a simple combinatorial result that may be known but we do not have a good reference to it and so we include it for completion.

Lemma 4.1. *Given a natural number k whose prime (powers) factorization is $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, then the number of positive integers u, v, and w such that $uvw = k$ with $u \leq v \leq w$ is given by*

$$od_3(k) := \frac{1}{6}\left[2\delta_3(k) + \frac{1}{2^n}\prod_{j=1}^{n}(\alpha_j+1)(\alpha_j+2) + 3\prod_{j=1}^{n}\left(\left\lfloor\frac{\alpha_j}{2}\right\rfloor + 1\right)\right], \qquad (11)$$

where $\delta_3(k)$ is 1 if k is a perfect cube and 0 otherwise.

Proof. Clearly every divisor d_β of k is of the form $d_\beta := p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ with $\beta_i \in \{0, 1, 2, \cdots, \alpha_i\}$. So, for every index i we can take $\beta_i^{(1)}$, $\beta_i^{(2)}$ and $\beta_i^{(3)}$ such that $\beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)} = \alpha_i$ and form the divisors $d_{\beta^{(1)}}$, $d_{\beta^{(2)}}$ and $d_{\beta^{(3)}}$ which satisfy $d_{\beta^{(1)}} d_{\beta^{(2)}} d_{\beta^{(3)}} = k$. It is known (see [4], page 25), that the number of solutions $(x_1, x_2, ..., x_\ell)$ to $x_1 + x_2 + \cdots + x_\ell = m$ in nonnegative integers is equal to $\binom{m+\ell-1}{\ell-1}$. Hence, we get $\binom{\alpha_i+2}{2} = (\alpha_i+1)(\alpha_i+2)/2$ possible ways to choose $(\beta_i^{(1)}, \beta_i^{(2)}, \beta_i^{(3)})$. Thus we have $N_1 := \frac{1}{2^n}\prod_{j=1}^{n}(\alpha_j+1)(\alpha_j+2)$ possible ways to get ordered triples (d_1, d_2, d_3) which satisfy $d_1 d_2 d_3 = k$. Every single such triple can be ordered and we will denote the non-decreasing triple by $[u, v, w]$. If all of the divisors d_1, d_2 and d_3 are distinct, then all six possible permutations (which are counted in N_1) give the same $[u, v, w]$. If two of the divisors are equal, then only three possible permutations appear in the counting N_1: (u, u, w), (u, w, u) and (w, u, u).

Let us determine the number of all triples with two equal divisors, say $d_{\beta^{(1)}} = d_{\beta^{(2)}}$. Then the equation $2\beta_i^{(1)} + \beta_i^{(3)} = \alpha_i$ is uniquely determined by $\beta_i^{(1)}$, and we have a solution as long as $\beta_i^{(1)} \in \{0, 1, ..., \lfloor\frac{\alpha_i}{2}\rfloor\}$. So, the number of such solutions is $\lfloor\frac{\alpha_i}{2}\rfloor + 1$. Hence the number of lists $[d_1, d_1, d_3]$ such that $d_1^2 d_2 = k$ is $N_2 := \prod_{j=1}^{n}(\lfloor\frac{\alpha_j}{2}\rfloor + 1)$. Assuming that $d_1 \neq d_3$, in N_1 we count such a triple three times.

Let us first assume that k is not a perfect cube. Then $d_1 \neq d_3$ is automatic and the formula (11) follows. If k is a perfect cube, then $k = d^3$. The triple (d, d, d) is counted only one time in N_1 and 3 times in the term $3N_2$. This explains the term of $2\delta_3(k)$ in (11). ■

The sequence $od_3(k)$ is known as A034836 or the number of boxes with integer edge lengths and volume k.

Now we can give an expression for the last case.

Lemma 4.2. *Given a natural number k then the number of non-decreasing integer triples (u, v, w) such that (10) are satisfied, is given by*

$$od_3(k) - \left\lfloor \frac{\tau(k)+1}{2} \right\rfloor - g(k), \tag{12}$$

where $od_3(k)$ is defined by (11) and g by

$$g(k) = \begin{cases} 1, & \text{if } k = 4\ell,\ \ell \geq 1; \\ 1, & \text{if } k \in \{18, 24, 30\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Obviously, by Lemma 4.1, we have to exclude from the counting in $od_3(k)$ all the solutions with $u = 1$ and those for which $u = 2$ and $\{v, w\}$ do not satisfy (10). If $u = 1$, then $vw = k$ implies as before $\left\lfloor \frac{\tau(k)+1}{2} \right\rfloor$ such solutions with $v \leq w$. If $u = 2$ and $(v-2)(w-2) < 4$, then we have seen that means exactly the definition of g. ■

We can put all these cases together at this point.

Theorem 4.1. *We have for every $k \geq 1$ and $n \geq 1$*

$$B_k^{(3)} = \left\lfloor \frac{\tau(k+1)-1}{2} \right\rfloor + od_3(k) - \left\lfloor \frac{\tau(k)+1}{2} \right\rfloor + E(k), \text{ and} \tag{13}$$

$$A_n^{(3)} = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor B_k^{(3)},$$

where

$$E(k) = \begin{cases} -1, & \text{if } k \in \{8, 18, 24, 30\}; \\ 1, & \text{if } k \in \{21, 26, 31\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.2 and the analysis of cases above, we observe that the contribution for $k = 4\ell$ from f, cancel with the one from g if $\ell \geq 3$ and then (13) follows. ■

Numerical calculations show that the first fifty terms of $[k, B_k^{(3)}]$ are: [1, 0], [2, 0], [3, 1], [4, 0], [5, 1], [6, 0], [7, 1], [8, 1], [9, 1], [10, 0], [11, 2], [12, 1], [13, 1], [14, 1], [15, 2], [16, 1], [17, 2], [18, 0], [19, 2], [20, 2], [21, 2], [22, 0], [23, 3], [24, 2], [25, 1], [26, 2], [27, 3], [28, 1], [29, 3], [30, 0], [31, 3], [32, 3], [33, 1], [34, 1], [35, 4], [36, 3], [37, 1], [38, 1], [39, 3], [40, 2], [41, 3], [42, 1], [43, 2], [44, 3], [45, 2], [46, 0], [47, 4], [48, 5], [49, 2], and [50, 2].

We assume that similar formulae exist for $s \geq 4$ but certainly, one would expect to get pretty complicated expressions because of the combinatorial complications that appear between the case $\alpha_1\alpha_2 \cdots \alpha_s = k$ and the other situations.

5 Asymptotic formulae for $A_n^{(1)}$ and $A_n^{(2)}$

Using the inequalities $x - 1 < \lfloor x \rfloor \leq x$, from (5) we obtain

$$\sum_{k=1}^{n} \frac{n}{k} - n < A_n^{(1)} < \sum_{k=1}^{n} \frac{n}{k}. \tag{14}$$

According to well-known asymptotic results (see T. Apostol [2, pp.70])

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O(\frac{1}{n}), \tag{15}$$

and

$$\sum_{k=2}^{n} \frac{\tau(k)}{k} = \frac{1}{2} \ln^2 n + 2\gamma \ln n + O(1). \tag{16}$$

We see that the inequalities (14) imply that $A_n^{(1)} = n \ln n + Cn + O(1)$ for some constants C such that $C \in [\gamma - 1, \gamma]$.
Some more recent progress has been made (see [4]) into showing the stronger fact:

$$A_n^{(1)} = n \ln n + (2\gamma - 1)n + O(n^{1/3} \ln n).$$

It seems like estimates better than (16) can be derived such as

$$\sum_{k=2}^{n} \frac{\tau(k)}{k} = \frac{1}{2} \ln^2 n + 2\gamma \ln n + \gamma^2 - 2\gamma_1 + O(1/\sqrt{n}),$$

where $\gamma_1 \approx -0.07281584548$ is one of the Stieltjes constants defined by

$$\gamma_1 := \lim_{n \to \infty} \left(-\frac{(\ln n)^2}{n+1} + \sum_{k=1}^{n} \frac{\ln k}{k} \right).$$

Theorem 5.1. *We have the following asymptotic inequalities*

$$A_n^{(2)} \leq \frac{n}{4} (\ln n)^2 + (\gamma + \frac{1}{2}) n \ln n + O(n), \text{ and} \tag{17}$$

$$A_n^{(2)} \geq \frac{n}{4} (\ln n)^2 + (\gamma - 1) n \ln n + O(n). \tag{18}$$

Proof. From Theorem 3.1 we have

$$A_n^{(2)} \leq \frac{n}{2}\sum_{k=2}^{n}\frac{\tau(k)}{k} + \frac{n}{2}\sum_{k=2}^{n}\frac{1}{k},$$

which in conjunction with (15) and (16) gives (17). To show (18), we observe that

$$A_n^{(2)} > \sum_{k=2}^{n}(\frac{n}{k}-1)(\frac{\tau(k)-1}{2}) = \frac{n-1}{2} + \frac{n}{2}\sum_{k=2}^{n}\frac{\tau(k)}{k} - \frac{n}{2}\sum_{k=2}^{n}\frac{1}{k} - \frac{1}{2}\sum_{k=2}^{n}\tau(k).$$

Since we know that $\sum_{k=1}^{n}\tau(k) = \sum_{k=1}^{n}\frac{n}{k}$ we can arrive at $\sum_{k=2}^{n}\tau(k) = (n-1) + \sum_{k=2}^{n}\frac{n}{k}$. This shows that

$$A_n^{(2)} > \frac{n}{2}\sum_{k=2}^{n}\frac{\tau(k)}{k} - n\sum_{k=2}^{n}\frac{1}{k},$$

which in turn implies (18). ■

From Theorem 5.1 we can easily see that

$$\lim_{n\to\infty}\frac{A_n^{(2)}}{n\ln^2 n} = \frac{1}{4}. \tag{19}$$

For the general situation we conjecture that

$$\lim_{n\to\infty}\frac{A_n^{(s)}}{n\ln^s n} = \frac{1}{(s!)^2}. \tag{20}$$

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