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A viscosity projection method for class $\mathfrak{T}$ mappings

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Abstract

In this paper, we firstly introduce a viscosity projection method for the class $\mathfrak{T}$ mappings

$$x_{n+1} = \alpha_n P_{H(x_n,S_nx_n)} f(x_n) + (1-\alpha_n)S_nx_n,$$

where $S_n = (1-w)I + wT_n$, $w \in (0,1), T_n \in \mathfrak{T}$ and prove strong convergence theorems of the proposed method. It is verified that the viscosity projection method converges locally faster than the viscosity method. Furthermore, we present a viscosity projection method for a quasi-nonexpansive and nonexpansive mappings in Hilbert spaces. A numerical test provided in the paper shows that the viscosity projection method converges faster than the viscosity method.

1 Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in H$. The set of fixed points of T is $Fix(T) := \{x \in H : Tx = x\}$. A mapping $T : H \to H$ is said to be quasi-nonexpansive if $Fix(T)$ is nonempty and $\|Tx - p\| \le \|x - p\|$ for all $x \in H$ and $p \in$Fix(T). A mapping $f : H \to H$ is said to be a contraction with constant $\rho \in [0, 1)$ if

$$\|f(x) - f(y)\| \le \rho\|x - y\| \quad \forall x, y \in H.$$

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Given $x, y \in H$, let

$$H(x,y) := \{z \in H : \langle z - y, x - y\rangle \leq 0\},$$

be the half-space generated by (x, y). The boundary ∂H of H is

$$\partial H(x,y) = \{z \in H : \langle z - y, x - y\rangle = 0\}.$$

It is clear that $\partial H(x, y)$ is a closed and convex subset of H. A mapping $T : H \to H$ is said to be the class $\mathfrak{T}$ (or a cutter) if $T \in \mathfrak{T} = \{T : H \to H|\, dom(T) = H\, and\, Fix(T) \subset H(x, Tx),\, for\, all\, x \in H\}$

Remark 1.1. *The class $\mathfrak{T}$ is fundamental because it contains several types of operators commonly found in various areas of applied mathematics and in particular in approximation and optimization theory (see [1, 2] for details).*

Let C be a nonempty closed convex subset of a Hilbert space H. For a mapping $T : C \to C$, Moudafi [10] and many other researchers (eg.[7, 8, 11, 12, 13, 14]) studied the viscosity approximation method as follow: for given $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1}$$

where $\{\alpha_n\} \subset (0, 1)$ and $f : C \to C$ is a contraction. It was proved in [10] (also see Xu [13]) that the sequence $\{x_n\}$ generated by (1) converges strongly to the unique solution of the variational inequality problem $VI(I - f, Fix(T))$: find x^* in $Fix(T)$ such that

$$\forall v \in Fix(T), \quad \langle (I - f)x^*, v - x^*\rangle \geq 0.$$

A special case of (1) was considered by Halpern [5] who introduced following iterative process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where $u, x_0 \in C$ are arbitrary (but fixed) and $\{\alpha_n\} \subset (0, 1)$.

Recently, Maingé [9] studied following algorithm for a quasi-nonexpansive mapping T:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_w x_n, \tag{2}$$

where $\{\alpha_n\} \subset (0, 1)$, $T_w = (1 - w)I + wT$, $w \in (0, 1)$. He proposed a new analysis of the viscosity approximation method to prove the convergence of the algorithm (2).

Inspired by Maingé [9] and others (e.g. [1, 2, 3, 6]), in this paper we firstly discuss the following viscosity projection method for a sequence of class $\mathfrak{T}$ mappings $T_n : H \to H$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n)S_n x_n, \tag{3}$$

where $\{\alpha_n\} \subset (0,1)$, $S_n = (1-w)I + wT_n$, $w \in (0,1)$, I is the identity mapping on H and P_K denotes the metric projection from H onto a closed convex subset K of H (see below Lemma 1.3 for the definition). We prove that the sequence $\{x_n\}$ generated by (3) converges strongly to the unique solution of the variational inequality problem $VI(I-f, \bigcap_{n=1}^{\infty} Fix(T_n))$: find x^* in $\bigcap_{n=1}^{\infty} Fix(T_n)$ such that

$$\forall v \in \bigcap_{n=1}^{\infty} Fix(T_n), \quad \langle (I-f)x^*, v - x^* \rangle \geq 0. \tag{4}$$

We will use the following notations:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit of $\{x_n\}$.

We need some facts and tools in a real Hilbert space H which are listed below.

Definition 1.1. *Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to a point p. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge locally faster than $\{y_n\}_{n=1}^{\infty}$ if $x_n = y_n$ implies*

$$\|x_{n+1} - p\| \leq \|y_{n+1} - p\|$$

for any $n \in \mathbb{N}$.

Lemma 1.1. *Let H be a Hilbert space and I be the identity operator of H.*

(i) If dom $T = H$, then $2T - I$ is quasi-nonexpansive if and only if $T \in \mathfrak{T}$,

(ii) If $T \in \mathfrak{T}$, then $\lambda I + (1-\lambda)T \in \mathfrak{T}$, $\forall \lambda \in [0,1]$.

(iii) If $T \in \mathfrak{T}$, then T is quasi-nonexpansive.

(iv) If $T \in \mathfrak{T}$, then $\|x - Tx\|^2 \leq \langle x - Tx, x - u\rangle$ for all $x \in H$ and $u \in Fix(T)$.

(v) If $T \in \mathfrak{T}$ and $S = wI + (1-w)T$, $w \in (0,1)$, then $H(x, Tx) \subset H(x, Sx)$, $\forall x \in H$.

Proof. The proof of (i)-(iv) can be found in [1]. Here we just prove (v). For any $y \in H(x, Tx)$, we have

$$\langle y - Tx, x - Tx\rangle \leq 0.$$

So, we get

$$\langle y - Sx, x - Sx\rangle = (1-w)\langle y - Tx, x - Tx\rangle - (1-w)w\|x - Tx\|^2 \leq 0,$$

which implies $y \in H(x, Sx)$.

Remark 1.2. *Let $T \in \mathfrak{T}$ with $Fix(T) \neq \emptyset$ and set $T_w := (1-w)I + wT$ for $w \in (0,1)$. Then the following statements are reached:*

(a1) $Fix(T) = Fix(T_w)$ *if* $w \neq 0$;

(a2) $Fix(T)$ *is a closed convex subset of* H.

(a3) $\langle x - T_w x, x - q\rangle \geq w\|x - Tx\|^2$ *for all* $x \in H$, $q \in Fix(T)$.

From Lemma 1.1 (i) and (ii), it is an easy matter to show (a1)-(a3) by using Remarks 1.2 and 2.1 in [9].

Definition 1.2. *A sequence of mappings $\{T_n\}$ having common fixed points is said to satisfy the condition (Z) if every bounded sequence $\{x_n\}$ with $\|x_n - T_n x_n\| \to 0$ satisfies $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} Fix(T_n)$.*

Definition 1.3. *A mapping T is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $Tx_n \to y$.*

Next Lemma shows that nonexpansive mappings are demeiclosed at 0.

Lemma 1.2. *[4] Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 1.3. *[4] Let K be a closed convex subset of real Hilbert space H and let P_K be the (metric or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \textit{for all } y \in K.$$

Lemma 1.4. *[6] Let $C = \{z \in H : \langle x - u, z - u\rangle \leq 0\}$. Assume $x \neq u$ and $x_0 \notin C$. Then*

$$P_C x_0 = x_0 - \frac{\langle x - u, x_0 - u\rangle}{\|x - u\|^2}(x - u). \tag{5}$$

Lemma 1.5. *Let $F := I - P_{H(x,Tx)} f$, where $x \in H$ and f is the contraction with constant ρ. Then the operator F is $(1-\rho)$-strongly monotone, i.e.,*

$$\langle Fy - Fz, y - z\rangle \geq (1-\rho)\|y - z\|^2 \quad \textit{for all } x, y \in H.$$

Proof. Note that $P_{H(x,Tx)}$ is a metric projection, so it is firmly nonexpansive and thus is nonexpansive. It is easy to see that, for all $y, z \in H$,

$$\|P_{H(x,Tx)} f(y) - P_{H(x,Tx)} f(z)\| \leq \|f(y) - f(z)\| \leq \rho\|y - z\|. \tag{6}$$

From (6), we have

$$\begin{aligned}\langle Fy - Fz, y - z\rangle &= \|y - z\|^2 - \langle P_{H(x,Tx)}f(y) - P_{H(x,Tx)}f(z), y - z\rangle \\ &\geq \|y - z\|^2 - \|P_{H(x,Tx)}f(y) - P_{H(x,Tx)}f(z)\|\|y - z\| \\ &\geq (1 - \rho)\|y - z\|^2.\end{aligned}$$

Lemma 1.6. *([9] (Lemma 2.1)). Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n | \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $lim_{n\to\infty}\tau(n) = \infty$ and, for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}.$$

2 Main results

Lemma 2.1. *Let $T_n \in \mathfrak{T}$ with $\mathcal{F} := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$, $\{\alpha_n\} \subset (0, 1)$ and $w \in (0, 1)$. Let f be a contraction with constant ρ. The sequence $\{x_n\}$ generated by (3) is bounded.*

Proof. By $T_n \in \mathfrak{T}$ and Lemma 1.1 (v), $Fix(T_n) \subset H(x, S_n x)$, for all $x \in H$, therefore, we have $P_{H(x,S_n x)}p = p$, for all $p \in \mathcal{F}$. So, using Lemma 1.1 (ii)-(iii) and (6), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n P_{H(x_n,S_n x_n)}f(x_n) + (1 - \alpha_n)S_n x_n - p\| \\ &\leq \alpha_n\|P_{H(x_n,S_n x_n)}f(x_n) - p\| + (1 - \alpha_n)\|S_n x_n - p\| \\ &\leq \alpha_n\|P_{H(x_n,S_n x_n)}f(x_n) - P_{H(x_n,S_n x_n)}f(p)\| \\ &\quad + \alpha_n\|P_{H(x_n,S_n x_n)}f(p) - P_{H(x_n,S_n x_n)}p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \alpha_n\|f(p) - p\| + [1 - \alpha_n(1 - \rho)]\|x_n - p\| \\ &= \alpha_n(1 - \rho)\frac{\|f(p) - p\|}{1 - \rho} + [1 - \alpha_n(1 - \rho)]\|x_n - p\|.\end{aligned}$$

Thus, by induction on n,

$$\|x_n - p\| \leq \max\left\{\frac{\|f(p) - p\|}{1 - \rho}, \|x_0 - p\|\right\},$$

for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ is bounded, and hence, $\{P_{H(x_n,S_n x_n)}f(x_n)\}$ is also bounded.

Lemma 2.2. *Assume a sequence of mappings $T_n \in \mathfrak{T} : H \to H$ satisfies the condition (Z). If x^* is the solution of (4) and $\{x_n\}$ is a bounded sequence such that $\|T_n x_n - x_n\| \to 0$, then*

$$\liminf_{n\to\infty} \langle (I - P_{H(x_n, T_n x_n)} f) x^*, x_n - x^* \rangle \geq 0. \tag{7}$$

Proof. Since the sequence $\{T_n\}$ satisfies the condition (Z) and $\{x_n\}$ is a bounded sequence, $\omega_w(x_n) \subset \mathcal{F}$. It is also a simple matter to see that there exists $\bar{x}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$ (hence $\bar{x} \in \mathcal{F}$) and such that

$$\liminf_{n\to\infty} \langle (I - f) x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle (I - f) x^*, x_{n_k} - x^* \rangle,$$

which by (4) obviously leads to

$$\liminf_{n\to\infty} \langle (I - f) x^*, x_n - x^* \rangle = \langle (I - f) x^*, \bar{x} - x^* \rangle \geq 0.$$

So,

$$\liminf_{n\to\infty} \langle (I - f) x^*, x_n - x^* \rangle \geq 0. \tag{8}$$

If $f(x^*) \in H(x_n, T_n x_n)$, then $P_{H(x_n, T_n x_n)} f(x^*) = f(x^*)$ and (8) implies (7). Otherwise, assume $f(x^*) \notin H(x_n, T_n x_n)$. Then, by definition of $H(x_n, T_n x_n)$, we have

$$\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle > 0. \tag{9}$$

By $x^* \in \mathcal{F} \subset H(x_n, T_n x_n)$, we get

$$\langle x_n - T_n x_n, x_n - x^* \rangle = \|x_n - T_n x_n\|^2 + \langle x_n - T_n x_n, T_n x_n - x^* \rangle > 0. \tag{10}$$

From (5), it follows

$$P_{H(x_n, T_n x_n)} f(x^*) = f(x^*) - \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} (x_n - T_n x_n). \tag{11}$$

Combining (9), (10) and (11), we obtain

$$\begin{aligned} \langle (I - P_{H(x_n, T_n x_n)} f) x^*, x_n - x^* \rangle &= \langle (I - f) x^*, x_n - x^* \rangle \\ &\quad + \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} \langle x_n - T_n x_n, x_n - x^* \rangle \\ &> \langle (I - f) x^*, x_n - x^* \rangle, \end{aligned} \tag{12}$$

which together with (8) implies

$$\liminf_{n\to\infty} \langle (I - P_{H(x_n, T_n x_n)} f) x^*, x_n - x^* \rangle \geq \liminf_{n\to\infty} \langle (I - f) x^*, x_n - x^* \rangle \geq 0.$$

Therefore, we obtain the desired result.

Theorem 2.1. *Suppose that a sequence* $\{T_n\} \subset \mathfrak{T}$ *satisfies* $\mathcal{F} := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ *and the condition (Z). Let* f *be a contraction with constant* $\rho \in [0,1)$*. Assume* $w \in (0,1)$*, and* $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Then,* $\{x_n\}$ *generated by (3) converges strongly to* $x^* \in \mathcal{F}$ *verifying*

$$x^* = (P_{\mathcal{F}} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \mathcal{F}, \quad \textit{and} \quad (\forall v \in \mathcal{F}), \quad \langle (I-f)x^*, v - x^* \rangle \geq 0. \tag{13}$$

Proof. Let x^* be the solution of (13). From (3) we obviously have

$$x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n,S_nx_n)}f(x_n)) = (1-\alpha_n)(S_nx_n - x_n), \tag{14}$$

hence

$$\langle x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n,S_nx_n)}f(x_n)), x_n - x^* \rangle = -(1-\alpha_n)\langle x_n - S_nx_n, x_n - x^* \rangle. \tag{15}$$

Moreover, by $x^* \in \mathcal{F}$, and using Remark 1.2 (a3), we have

$$\langle x_n - S_nx_n, x_n - x^* \rangle \geq w\|x_n - T_nx_n\|^2,$$

which together with (15) entails

$$\langle x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n,S_nx_n)}f(x_n)), x_n - x^* \rangle \leq -w(1-\alpha_n)\|x_n - T_nx_n\|^2,$$

or equivalently

$$\begin{aligned} -\langle x_n - x_{n+1}, x_n - x^* \rangle \leq & -\alpha_n \langle x_n - P_{H(x_n,S_nx_n)}f(x_n), x_n - x^* \rangle \\ & - w(1-\alpha_n)\|x_n - T_nx_n\|^2. \end{aligned} \tag{16}$$

Setting $\Gamma_n := \frac{1}{2}\|x_n - x^*\|^2$, we have

$$\langle x_n - x_{n+1}, x_n - x^* \rangle = -\Gamma_{n+1} + \Gamma_n + \frac{1}{2}\|x_n - x_{n+1}\|^2.$$

So that (16) can be equivalently rewritten as

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n - \frac{1}{2}\|x_n - x_{n+1}\|^2 \leq & -\alpha_n \langle x_n - P_{H(x_n,S_nx_n)}f(x_n), x_n - x^* \rangle \\ & - w(1-\alpha_n)\|x_n - T_nx_n\|^2. \end{aligned} \tag{17}$$

Now using (14) again, we have

$$\|x_{n+1} - x_n\|^2 = \|\alpha_n(P_{H(x_n,S_nx_n)}f(x_n) - x_n) + (1-\alpha_n)(S_nx_n - x_n)\|^2.$$

Hence it is a classical matter to see that

$$\|x_{n+1} - x_n\|^2 \leq 2\alpha_n^2 \|P_{H(x_n,S_nx_n)} f(x_n) - x_n\|^2 + 2(1-\alpha_n)^2 \|S_n x_n - x_n\|^2,$$

which by $\|S_nx_n - x_n\| = w\|T_nx_n - x_n\|$ and $(1-\alpha_n)^2 \leq (1-\alpha_n)$ yields

$$\frac{1}{2}\|x_{n+1}-x_n\|^2 \leq \alpha_n^2 \|P_{H(x_n,S_nx_n)} f(x_n)-x_n\|^2 + w^2(1-\alpha_n)\|T_nx_n - x_n\|^2. \tag{18}$$

Then from (17) and (18) we obtain

$$\begin{aligned}&\Gamma_{n+1} - \Gamma_n + (1-w)w(1-\alpha_n)\|x_n - T_nx_n\|^2 \\ &\quad \leq \alpha_n(\alpha_n \|P_{H(x_n,S_nx_n)} f(x_n) - x_n\|^2 - \langle x_n - P_{H(x_n,S_nx_n)} f(x_n)), x_n - x^*\rangle).\end{aligned} \tag{19}$$

The rest of the proof will be divided into two parts:

Case 1. Suppose that there exists n_0 such that $\{\Gamma_n\}_{n \geq n_0}$ is nonincreasing. In this situation, $\{\Gamma_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below), so that $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$, hence, in light of (19) together with $\alpha_n \to 0$, and the boundedness of $\{x_n\}$ (hence, thanks Lemma 2.1, $\{P_{H(x_n,S_nx_n)} f(x_n)\}$ is also bounded), we obtain

$$\lim_{n\to\infty} \|x_n - T_nx_n\| = 0,$$

which together with $S_n = (1-w)I + wT_n$, $w \in (0,1)$, implies

$$\lim_{n\to\infty} \|x_n - S_nx_n\| = 0. \tag{20}$$

From (19) again, we have

$$\alpha_n(-\alpha_n \|(P_{H(x_n,S_nx_n)} f(x_n)) - x_n\|^2 + \langle x_n - P_{H(x_n,S_nx_n)} f(x_n)), x_n - x^*\rangle) \leq \Gamma_n - \Gamma_{n+1}.$$

Then, by $\sum_n \alpha_n = \infty$, we obviously deduce that

$$\liminf_{n\to\infty}(-\alpha_n \|P_{H(x_n,S_nx_n)} f(x_n) - x_n\|^2 + \langle x_n - P_{H(x_n,S_nx_n)} f(x_n), x_n - x^*\rangle) \leq 0,$$

or equivalently (as $\alpha_n \|P_{H(x_n,S_nx_n)} f(x_n))x_n\|^2 \to 0$)

$$\liminf_{n\to\infty} \langle x_n - P_{H(x_n,S_nx_n)} f(x_n)), x_n - x^*\rangle \leq 0. \tag{21}$$

Moreover, by Lemma 1.5, we have

$$2(1-\rho)\Gamma_n + \langle x^* - P_{H(x_n,S_nx_n)} f(x^*), x_n - x^*\rangle \leq \langle x_n - P_{H(x_n,S_nx_n)} f(x_n), x_n - x^*\rangle, \tag{22}$$

which by (21) entails

$$\liminf_{n\to\infty}(2(1-\rho)\Gamma_n + \langle x^* - P_{H(x_n,S_nx_n)}f(x^*)), x_n - x^*\rangle) \leq 0.$$

Hence, recalling that $\lim_{n\to\infty}\Gamma_n$ exists, we equivalently obtain

$$2(1-\rho)\lim_{n\to\infty}\Gamma_n + \liminf_{n\to\infty}\langle x^* - P_{H(x_n,S_nx_n)}f(x^*), x_n - x^*\rangle \leq 0,$$

namely,

$$2(1-\rho)\lim_{n\to\infty}\Gamma_n \leq -\liminf_{n\to\infty}\langle x^* - P_{H(x_n,S_nx_n)}f(x^*), x_n - x^*\rangle. \tag{23}$$

From (20) and invoking Lemma 2.2, we have

$$\liminf_{n\to\infty}\langle x^* - P_{H(x_n,S_nx_n)}f(x^*), x_n - x^*\rangle \geq 0,$$

which by (23) yields $\lim_{n\to\infty}\Gamma_n = 0$, so that $\{x_n\}$ converges strongly to x^*.

Case 2. Suppose there exists a subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{\tau(n)\}$ as defined in Lemma 1.6. It follows that $\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0$, which by (19) amounts to

$$\begin{aligned}(1-w)w(1-\alpha_{\tau(n)})&\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\|^2 \\ &< \alpha_{\tau(n)}(\alpha_{\tau(n)}\|P_{H(x_{\tau(n)},S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \\ &\quad - \langle x_{\tau(n)} - P_{H(x_{\tau(n)},S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}), x_{\tau(n)} - x^*\rangle).\end{aligned} \tag{24}$$

Hence, by the boundedness of $\{x_n\}$ and $\{P_{H(x_n,S_nx_n)}f(x_n)\}$, and $\alpha_n \to 0$, we immediately obtain

$$\lim_{n\to\infty}\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\| = 0, \tag{25}$$

which together with $S_{\tau(n)} = (1-w)I + wT_{\tau(n)}$, $w \in (0,1)$, implies

$$\lim_{n\to\infty}\|x_{\tau(n)} - S_{\tau(n)}x_{\tau(n)}\| = 0. \tag{26}$$

Using (3), we have

$$\begin{aligned}\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq\ &\alpha_{\tau(n)}\|P_{H(x_{\tau(n)},S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}) - x_{\tau(n)}\| \\ &+ (1-\alpha_{\tau(n)})\|x_{\tau(n)} - S_{\tau(n)}x_{\tau(n)}\|,\end{aligned}$$

which together with (26) and $\alpha_n \to 0$ yields

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{27}$$

Now by (24), we clearly have

$$\begin{aligned}\langle x_{\tau(n)} - &P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x_{\tau(n)}), x_{\tau(n)} - x^*\rangle \\ &\leq \alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x_{\tau(n)}) - x_{\tau(n)}\|^2,\end{aligned}$$

which in the light of (22) yields

$$\begin{aligned}2(1-\rho)\Gamma_{\tau(n)} + &\langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x^*), x_{\tau(n)} - x^*\rangle \\ &\leq \alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x_{\tau(n)}) - x_{\tau(n)}\|^2.\end{aligned}$$

Hence (as $\alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \to 0$) it follows that

$$2(1-\rho) \limsup_{n\to\infty} \Gamma_{\tau(n)} \leq - \liminf_{n\to\infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x^*), x_{\tau(n)} - x^*\rangle. \tag{28}$$

From (26) and invoking Lemma 2.2, we have

$$\liminf_{n\to\infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})} f(x^*), x_{\tau(n)} - x^*\rangle \geq 0,$$

which by (28) yields $\limsup_{n\to\infty} \Gamma_{\tau(n)} = 0$, so that $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. Applying (27), we have $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Then, recalling that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ (by Lemma 1.6), we get $\lim_{n\to\infty} \Gamma_n = 0$, so that $x_n \to x^*$ strongly.

Remark 2.1. *Assume that $f(x_n) \notin H(x_n, S_n x_n)$. From Lemma 1.4, we have*

$$P_{H(x_n, S_n x_n)} f(x_n) = f(x_n) - \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n\rangle}{\|x_n - S_n x_n\|^2}(x_n - S_n x_n). \tag{29}$$

So, the algorithm (3) can be rewritten as the form:

$$x_{n+1} = \begin{cases} \alpha_n f(x_n) + (1-\alpha_n) S_n x_n, & \text{if} \quad f(x_n) \in H(x_n, S_n x_n) \\ \alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1-\alpha_n) S_n x_n, & \text{if} \quad f(x_n) \notin H(x_n, S_n x_n) \end{cases} \tag{30}$$

where $P_{H(x_n, S_n x_n)} f(x_n)$ is given by (29). From (30), we know the algorithm (3) can be easily realized although there is a metric projection.

From (2), the classical viscosity method for class $\mathfrak{T}$ mappings $\{T_n\}$ is

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n) S_n y_n, \tag{31}$$

where $S_n = (1-w)I + wT_n$.

Next, we will compare the convergence rate of the viscosity projection method with the viscosity method.

Theorem 2.2. *Suppose that a sequence* $\{T_n\} \subset \mathfrak{T}$ *satisfies* $\mathcal{F} := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. *Take the same parameters* $\{\alpha_n\}$ *and* w *in (3) and (31). Let* $y_n = x_n$ *and* $p \in \mathcal{F}$. *Then it holds*

$$\|x_{n+1} - p\| \leq \|y_{n+1} - p\|. \tag{32}$$

Proof. From $T_n \in \mathfrak{T}$ and Lemma 1.1 (v), it follows $\mathcal{F} \in H(x_n, S_n x_n)$. If $f(x_n) \in H(x_n, S_n x_n)$ and then $P_{H(x_n,S_n x_n)} f(x_n) = f(x_n)$, then, it is obvious that $y_{n+1} = x_{n+1}$ and (32) follows.

Next, assume $f(x_n) \notin H(x_n, S_n x_n)$, then it is easy to verify $P_{H(x_n,S_n x_n)} f(x_n) \in \partial H(x_n, S_n x_n)$. Actually, from (29), it follows

$$\begin{aligned}
&\langle P_{H(x_n,S_n x_n)} f(x_n) - S_n x_n, x_n - S_n x_n\rangle \\
&= \langle f(x_n) - S_n x_n - \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n\rangle}{\|x_n - S_n x_n\|^2}(x_n - S_n x_n), x_n - S_n x_n\rangle \\
&= \langle f(x_n) - S_n x_n, x_n - S_n x_n\rangle - \\
&\quad \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n\rangle}{\|x_n - S_n x_n\|^2}\langle x_n - S_n x_n, x_n - S_n x_n\rangle \\
&= 0,
\end{aligned}$$

which yields

$$\begin{aligned}
&\langle P_{H(x_n,S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n,S_n x_n)} f(x_n)\rangle \\
&= \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n\rangle}{\|x_n - S_n x_n\|^2}\langle x_n - S_n x_n, P_{H(x_n,S_n x_n)} f(x_n) - S_n x_n\rangle \\
&= 0.
\end{aligned} \tag{33}$$

On the other hand, since $p \in \mathcal{F} \subset H(x_n, S_n x_n)$, using Lemma 1.3, we get

$$\langle P_{H(x_n,S_n x_n)} f(x_n) - f(x_n), P_{H(x_n,S_n x_n)} f(x_n) - p\rangle \leq 0. \tag{34}$$

Applying (33), (34) and $x_n = y_n$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n P_{H(x_n,S_n x_n)} f(x_n) + (1-\alpha_n) S_n x_n - p\|^2 \\
&= \|\alpha_n (P_{H(x_n,S_n x_n)} f(x_n) - f(y_n)) + (y_{n+1} - p)\|^2 \\
&\leq \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n,S_n x_n)} f(x_n) - f(x_n), x_{n+1} - p\rangle \\
&= \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n,S_n x_n)} f(x_n) - f(x_n), P_{H(x_n,S_n x_n)} f(x_n) - p\rangle \\
&\quad + 2\alpha_n (1-\alpha_n)\langle P_{H(x_n,S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n,S_n x_n)} f(x_n)\rangle \\
&\leq \|y_{n+1} - p\|^2,
\end{aligned}$$

which implies $\|x_{n+1} - p\| \leq \|y_{n+1} - p\|$.

Remark 2.2. *From the Definition 1.1 and Theorem 2.2, it follows that the viscosity projection method converges locally faster than viscosity method.*

Remark 2.3. *In [3], Dong et al proved the strong convergence theorem of the shrinking projection methods under the assumption that a sequence of class $\mathfrak{T}$ mappings $\{T_n\}$ is coherent (see definition 1.1 in [3]). In Theorem 2.1, the condition (Z) is needed for a sequence of class $\mathfrak{T}$ mappings $\{T_n\}$. Comparing the definition of coherent and condition (Z), it is obvious that a sequence $\{T_n\}$ satisfies condition (Z) if it is coherent. So, in order to obtain strong convergence results, in this paper we just need a weaker condition than that in [3].*

3 Deduced results

In this section, using Theorem 2.1, we obtain some strong convergence results for a class $\mathfrak{T}$ mapping, a quasi-nonexpansive mapping and a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Assume $T \in \mathfrak{T}$ with $Fix(T) \neq \emptyset$ satisfies that $I - T$ is demiclosed at 0. Let f be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:*

$$x_{n+1} = \alpha_n P_{H(x_n, Sx_n)} f(x_n) + (1 - \alpha_n) Sx_n, \tag{35}$$

where $S = (1-w)I + wT$, $w \in (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in Fix(T)$ verifying

$$x^* = (P_{Fix(T)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in Fix(T), \quad \text{and} \quad (\forall v \in Fix(T)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

Proof. Let $T_n = T$ in (3) for all $n \in \mathbb{N}$. From Lemma 2.1, it follows that $\{x_n\}$ is bounded. Using the definition of demiclosed, we get that T satisfies condition (Z). From Theorem 2.1, the desired result follows.

Theorem 3.2. *Assume $U : H \to H$ is a quasi-nonexpansive mapping with $Fix(U) \neq \emptyset$ and satisfies that $I - U$ is demiclosed at 0. Let f be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:*

$$x_{n+1} = \alpha_n P_{H(x_n, Vx_n)} f(x_n) + (1 - \alpha_n) Vx_n,$$

where $V = (1-\gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then,* $\{x_n\}$ *converges strongly to* $x^* \in Fix(U)$ *verifying*

$$x^* = (P_{Fix(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in Fix(U), \quad \text{and} \quad (\forall v \in Fix(U)), \quad \langle (I-f)x^*, v - x^* \rangle \geq 0.$$

Proof. By Lemma 1.1 (i), $\frac{U+I}{2} \in \mathfrak{T}$. Substitute T in (35) by $\frac{U+I}{2}$. Then,

$$\begin{aligned} S &= (1-w)I + wT = (1-w)I + w\frac{U+I}{2} \\ &= (1 - \frac{w}{2})I + \frac{w}{2}U. \end{aligned}$$

Set $\gamma = \frac{w}{2} \in (0, \frac{1}{2})$ and $V = S = (1-\gamma)I + \gamma U$. Since $I - U$ is demiclosed at 0, $I - \frac{U+I}{2} = \frac{I-U}{2}$ is demiclosed at 0. So we can obtain the result by using Theorem 3.1.

Since a nonexpansive mapping is quasi-nonexpansive and demiclosed (see Lemma 1.2), using Theorem 3.2, we have following theorem.

Theorem 3.3. *Let* $U : H \to H$ *be a nonexpansive mapping with* $Fix(U) \neq \emptyset$ *and* f *be a contraction with constant* $\rho \in [0,1)$. *Define a sequence* $\{x_n\}$ *as follow:*

$$x_{n+1} = \alpha_n P_{H(x_n, Vx_n)} f(x_n) + (1-\alpha_n)Vx_n,$$

where $V = (1-\gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, *and* $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then,* $\{x_n\}$ *converges strongly to* $x^* \in Fix(U)$ *verifying*

$$x^* = (P_{Fix(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in Fix(U), \quad \text{and} \quad (\forall v \in Fix(U)), \quad \langle (I-f)x^*, v - x^* \rangle \geq 0.$$

4 Numerical tests

For comparing the convergent rate of viscosity projection with viscosity method, we compute two simple examples. Let $w = \frac{1}{3}$, $\alpha_n = \frac{1}{n}$, and $x_0 = y_0 = -0.3$. Consider two cases:
Case 1. $T_1(x) = \sin(x)$ and $f_1(x) = \cos(\frac{x}{2})$ with constant $\frac{1}{2}$;
Case 2. $T_2(x) = \cos(x)$ and $f_2(x) = \sin(\frac{x}{2})$ with constant $\frac{1}{2}$.

It is obvious T_1 and T_2 are two nonexpansive mappings on $\mathbb{R}$. From Figure 1, It illustrates that viscosity projection methods converges faster than viscosity methods for the given examples.

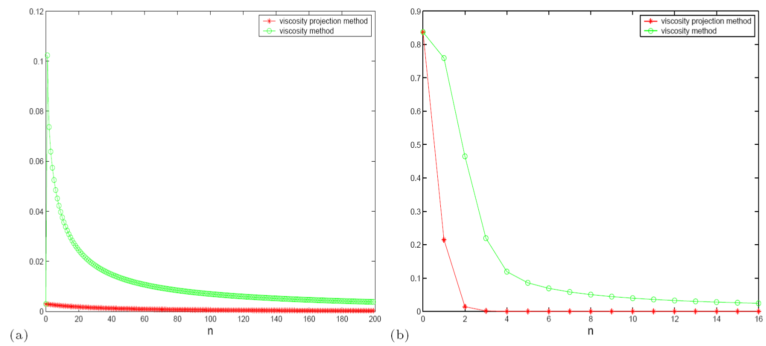

Figure 1: (a) Case 1 $\|x_n - Tx_n\|$; (b) Case 2 $\|x_n - Tx_n\|$.

Remark 4.1. *We just prove that viscosity projection method converges locally faster than viscosity in Theorem 2.2, and don't know if viscosity projection method converges faster than viscosity. It is an open problem.*

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