DOI: 10.2478/auom-2013-0020
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 21(2),2013, 5–19

Extremal orders of some functions connected to regular integers modulo n

Brăduţ Apostol

Abstract

Let $V(n)$ denote the number of positive regular integers (mod n) less than or equal to n. We give extremal orders of $\frac{V(n)\sigma(n)}{n^2}$, $\frac{V(n)\psi(n)}{n^2}$, $\frac{\sigma(n)}{V(n)}$, $\frac{\psi(n)}{V(n)}$, where $\sigma(n)$, $\psi(n)$ are the sum-of-divisors function and the Dedekind function, respectively. We also give extremal orders for $\frac{\sigma^*(n)}{V(n)}$ and $\frac{\phi^*(n)}{V(n)}$, where $\sigma^*(n)$ and $\phi^*(n)$ represent the sum of the unitary divisors of n and the unitary function corresponding to $\phi(n)$, the Euler's function. Finally, we study some extremal orders of compositions $f(g(n))$, involving the functions from above.

1 Introduction

Let $n > 1$ be a positive integer. An integer a is called regular (mod n) if there exists an integer x such that $a^2x \equiv a \pmod{n}$.

Properties of regular integers have been investigated by several authors. In a recent paper O.Alkam and E.A. Osba [1], using ring theoretic considerations, rediscovered some of the statements proved elementary by J.Morgado [3], [4]. It was proved in [3], [4] that $a > 1$ is regular (mod n) if and only if gcd (a, n) is a unitary divisor of n. In [11] L.Tóth gives direct proofs of some properties,

Key Words: Arithmetical function, composition, regular integers (mod n), extremal orders.
2010 Mathematics Subject Classification: 11A25, 11N37.
Received:February 2012
Revised:June 2012
Accepted:July 2012

because the proofs of [3], [4] are lenghty and those of [1] are ring theoretical. Let $\mathrm{Reg}_n = \{a : 1 \le a \le n$ and a is regular (mod n)$\}$, and $V(n) = \#\mathrm{Reg}_n$. The function V is multiplicative and $V(p^\alpha) = \phi(p^\alpha) + 1 = p^\alpha - p^{\alpha-1} + 1$, where ϕ is the Euler function. Consequently, $V(n) = \sum_{d \| n} \phi(d)$, for every $n \ge 1$, where $d \| n$ means unitary divisor (defined later). Also $\phi(n) < V(n) \le n$, for every $n > 1$, and $V(n) = n$ if and only if n is a squarefree, see [4], [11], [1]. L.Tóth [11] proved results concerning the minimal and maximal orders of the functions $V(n)$ and $V(n)/\phi(n)$. The minimal order of $V(n)$ was investigated by O.Alkam and E.A.Osba in [1]. J. Sándor and L. Tóth [7] studied the extremal orders of compositions of certain functions. In the present paper we investigate the extremal orders of the function $V(n)$ in connection with the functions $\sigma(n)$, $\psi(n)$, $\sigma^*(n)$, $\phi^*(n)$. We also study extremal orders of certain composite functions involving $V(n)$, $\phi(n)$, $\sigma(n)$, $\psi(n)$, $\phi^*(n)$, $\sigma^*(n)$ and pose some open problems.
For other arithmetic functions defined by regular integers modulo n we refer to the papers [2] and [10].
In what follows let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} > 1$ be a positive integer. We will use throughout the paper the following notation:

- $p_1, p_2, ...$ - the sequence of the primes;
- $d \| n$ - d is a unitary divisor of n, that is $d \mid n$ and $(d, \frac{n}{d}) = 1$;
- $\sigma(n)$ - the sum of the divisors of the natural number n;
- $\psi(n)$ - the Dedekind function, $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$;
- $\zeta(n)$ - the Riemann zeta function, $\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$, $s = \sigma + it \in \mathbb{C}$ and $\sigma > 1$;
- $\phi(n)$ - the Euler function, $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$;
- γ - the Euler constant, $\gamma = \lim_{n\to\infty} (1 + \frac{1}{2} + ... + \frac{1}{n} - \log n)$;
- $\phi^*(n)$ - the unitary function corresponding to $\phi(n)$, $\phi^*(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - 1)$;
- $\sigma^*(n)$ - the unitary function corresponding to $\sigma(n)$, $\sigma^*(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} + 1)$.

2 Extremal orders concerning classical arithmetic functions

We know that $\phi(n) < n < \sigma(n)$ for every $n > 1$. It is easy to see that $\dfrac{6}{\pi^2} < \dfrac{\phi(n)\sigma(n)}{n^2} < 1$, $n > 1$, $\liminf\limits_{n\to\infty} \dfrac{\phi(n)\sigma(n)}{n^2} = \dfrac{6}{\pi^2}$ and $\limsup\limits_{n\to\infty} \dfrac{\phi(n)\sigma(n)}{n^2} = 1$.
In [5] it was proved that $\liminf\limits_{n\to\infty} \dfrac{\phi(n)\psi(n)}{n^2} = \dfrac{6}{\pi^2}$ and $\limsup\limits_{n\to\infty} \dfrac{\phi(n)\psi(n)}{n^2} = 1$.

We recall that an integer $n > 1$ is called powerful if it is divisible by the square of each of its prime factors. A powerful integer is also called a squarefull integer.

The investigation of the minimal and maximal order of $V(n)\sigma(n)$ led us to

Proposition 1.

$$\frac{V(n)\sigma(n)}{n^2} > 1, \tag{i}$$

for every $n > 1$.

$$\liminf_{n\to\infty} \frac{V(n)\sigma(n)}{n^2} = 1, \tag{ii}$$

$$\frac{V(n)\sigma(n)}{n^2} \le \frac{\zeta(2)}{\zeta(6)}, \tag{iii}$$

for every powerful number n.

$$\limsup_{\substack{n\to\infty \\ n \text{ powerful}}} \frac{V(n)\sigma(n)}{n^2} = \frac{\zeta(2)}{\zeta(6)}. \tag{iv}$$

Proof.

(i) Let $n > 1$ be an integer with the prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$.
Since $\left(1 - \dfrac{1}{p} + \dfrac{1}{p^\alpha}\right) \cdot \dfrac{p - \frac{1}{p^\alpha}}{p-1} > 1$, it follows that

$$\frac{V(n)\sigma(n)}{n^2} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i} + \frac{1}{p_i^{\alpha_i}}\right) \cdot \frac{p_i - \frac{1}{p_i^{\alpha_i}}}{p_i - 1} > 1.$$

(ii) Since $\lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \dfrac{V(p)\sigma(p)}{p^2} = \lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \dfrac{p^2+p}{p^2} = 1$, taking (i) into account, we obtain

$$\liminf_{n\to\infty} \frac{V(n)\sigma(n)}{n^2} = 1.$$

(iii) Let $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$, $q_1 < q_2 < ... < q_k$, $\alpha_i \geq 2$, $1 \leq i \leq k$ and p_1,...,p_k the first k primes. We have $\dfrac{q^\alpha - q^{\alpha-1}+1}{q^\alpha} \cdot \dfrac{q^{\alpha+1}-1}{q^\alpha(q-1)} \leq \dfrac{1}{1-\frac{1}{q^2}} \cdot \left(1 - \dfrac{1}{q^6}\right)$ for $\alpha \geq 2$ and q prime, so

$$\frac{V(n)\sigma(n)}{n^2} = \prod_{i=1}^{k} \frac{q_i^{\alpha_i} - q_i^{\alpha_i - 1} + 1}{q_i^{\alpha_i}} \cdot \frac{q_i^{\alpha_i+1}-1}{q_i^{\alpha_i}(q_i - 1)} \leq \prod_{i=1}^{k} \frac{1}{1-\frac{1}{q_i^2}} \cdot \left(1 - \frac{1}{q_i^6}\right).$$

Since $q_i \geq p_i$ for $1 \leq i \leq k$, it follows that
$\dfrac{1}{1-\frac{1}{q_i^2}} \cdot \left(1 - \dfrac{1}{q_i^6}\right) \leq \dfrac{1}{1-\frac{1}{p_i^2}} \cdot \left(1 - \dfrac{1}{p_i^6}\right)$ for $1 \leq i \leq k$, so

$$\frac{V(n)\sigma(n)}{n^2} \leq \prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i^2}} \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^6}\right).$$

Taking $k \to \infty$, we obtain

$$\frac{V(n)\sigma(n)}{n^2} \leq \frac{\zeta(2)}{\zeta(6)}.$$

(iv) Taking $n_k = p_1^2 \cdots p_k^2$ ($p_1, ..., p_k$ being the first k primes),

$$\frac{V(n_k)\sigma(n_k)}{n_k^2} = \prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i^2}} \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^6}\right),$$

so

$$\lim_{k\to\infty} \frac{V(n_k)\sigma(n_k)}{n_k^2} = \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^2}} \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p^6}\right) = \frac{\zeta(2)}{\zeta(6)}.$$

In view of *(iii)* , we obtain

$$\limsup_{\substack{n\to\infty \\ n \text{ powerful}}} \frac{V(n)\sigma(n)}{n^2} = \frac{\zeta(2)}{\zeta(6)}. \qquad \square$$

Corollary 1. *The minimal order of* $\dfrac{V(n)\sigma(n)}{n^2}$ *is* 1 *and the maximal order of* $\dfrac{V(n)\sigma(n)}{n^2}$ *for n powerful is* $\dfrac{\zeta(2)}{\zeta(6)}$.

We now prove an analogous result for $V(n)\psi(n)$:

Proposition 2.

$$\liminf_{\substack{n\to\infty \\ n \text{ squarefree}}} \frac{V(n)\psi(n)}{n^2} = 1, \tag{i}$$

$$\frac{V(n)\psi(n)}{n^2} \le \frac{\zeta(3)}{\zeta(6)}, \tag{ii}$$

for every powerful number n.

$$\limsup_{\substack{n\to\infty \\ n \text{ powerful}}} \frac{V(n)\psi(n)}{n^2} = \frac{\zeta(3)}{\zeta(6)}. \tag{iii}$$

Proof.

(i) Let $n = p_1 \cdots p_k$, where p_1,...,p_k are distinct prime numbers. We have $\frac{V(n)\psi(n)}{n^2} = \frac{(p_1+1)\cdots(p_k+1)}{p_1\cdots p_k} > 1$. Since $\lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \frac{V(p)\psi(p)}{p^2} = 1$, we obtain $\liminf\limits_{\substack{n\to\infty \\ n \text{ squarefree}}} \frac{V(n)\psi(n)}{n^2} = 1$.

(ii) If $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$, $\alpha_i \ge 2$, and $1 \le i \le k$, then we have

$$\frac{V(n)\psi(n)}{n^2} = \prod_{i=1}^{k} \frac{q_i^{\alpha_i+1} - q_i^{\alpha_i-1} + q_i + 1}{q_i^{\alpha_i+1}}.$$

It is immediate that $\frac{q^{\alpha+1} - q^{\alpha-1} + q + 1}{q^{\alpha+1}} \le \left(1 - \frac{1}{q^2}\right)\left(1 + \frac{1}{q^2 - q}\right) = 1 + \frac{1}{q^3}$ for $\alpha \ge 2$ and q prime, so

$$\frac{V(n)\psi(n)}{n^2} \le \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^2}\right)\left(1 + \frac{1}{q_i^2 - q_i}\right) = \prod_{i=1}^{k}\left(1 + \frac{1}{q_i^3}\right).$$

Let $p_1, ..., p_k$ the first k primes. Since $q_i \ge p_i$ for $1 \le i \le k$, we get

$1 + \frac{1}{q_i^3} \le 1 + \frac{1}{p_i^3}$ for $1 \le i \le k$, hence $\frac{V(n)\psi(n)}{n^2} \le \prod_{i=1}^{k}\left(1 + \frac{1}{p_i^3}\right)$. Since the right hand side tends increasingly to $\frac{\zeta(3)}{\zeta(6)}$ as $k \to \infty$, we get $\frac{V(n)\psi(n)}{n^2} \le \frac{\zeta(3)}{\zeta(6)}$, for every powerful number n.

(iii) Take $n_k = p_1^2 \cdots p_k^2$ ($p_1, ..., p_k$ being the first k primes). Then
$\dfrac{V(n_k)\psi(n_k)}{n_k^2} = \prod_{i=1}^{k}\left(1 - \dfrac{1}{p_i^2}\right) \cdot \prod_{i=1}^{k}\left(1 + \dfrac{1}{p_i^2 - p_i}\right) = \prod_{i=1}^{k}\left(1 + \dfrac{1}{p_i^3}\right) \to \dfrac{\zeta(3)}{\zeta(6)}$,
($k \to \infty$) so, if we take into account *(ii)*, we deduce that
$\limsup\limits_{\substack{n\to\infty \\ n \text{ powerful}}} \dfrac{V(n)\psi(n)}{n^2} = \dfrac{\zeta(3)}{\zeta(6)}$, implying that the maximal order of $\dfrac{V(n)\psi(n)}{n^2}$
for n powerful is $\dfrac{\zeta(3)}{\zeta(6)}$. □

In order to prove the properties below we apply the following result ([12], Corollary 1) :

Lemma 1. *If f is a nonnegative real-valued multiplicative arithmetic function such that for each prime p,*

$$(i) \qquad \rho(p) = \sup_{\alpha \geq 0}(f(p^\alpha)) \leq \left(1 - \frac{1}{p}\right)^{-1}, \text{ and}$$

$$(ii) \qquad \text{there is an exponent } e_p = p^{o(1)} \in \mathbb{N} \text{ satisfying } f(p^{e_p}) \geq 1 + \frac{1}{p},$$

$$\text{then} \quad \limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^\gamma \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)\rho(p).$$

For the quotient $\dfrac{\sigma(n)}{V(n)}$, we notice that $\dfrac{\sigma(n)}{V(n)} \geq 1$ for every $n \geq 1$.
Since $\lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \dfrac{\sigma(p)}{V(p)} = 1$, we get $\liminf\limits_{n\to\infty} \dfrac{\sigma(n)}{V(n)} = 1$, hence the minimal order of $\dfrac{\sigma(n)}{V(n)}$ is 1. Proposition 3 shows that the maximal order of $\dfrac{\sigma(n)}{V(n)}$ is $e^{2\gamma}(\log\log n)^2$:

Proposition 3.
$$\limsup_{n\to\infty} \frac{\sigma(n)}{V(n)(\log\log n)^2} = e^{2\gamma}.$$

Proof. Take $f(n) = \sqrt{\dfrac{\sigma(n)}{V(n)}}$. Then

$$f(p^\alpha) = \sqrt{\frac{p^{\alpha+1} - 1}{(p-1)(p^\alpha - p^{\alpha-1} + 1)}} \leq \left(1 - \frac{1}{p}\right)^{-1} = \rho(p),$$

and

$$f(p^2) = \sqrt{\frac{p^2+p+1}{p^2-p+1}} \geq 1 + \frac{1}{p}$$

for every prime p, so (ii) in the above Lemma is satisfied. We obtain

$$\limsup_{n\to\infty} \frac{\sqrt{\sigma(n)}}{\sqrt{V(n)}\log\log n} = e^{\gamma},$$

so

$$\limsup_{n\to\infty} \frac{\sigma(n)}{V(n)(\log\log n)^2} = e^{2\gamma}. \qquad \square$$

Consider now the quotient $\dfrac{\psi(n)}{V(n)}$. Since $\dfrac{\psi(n)}{V(n)} \geq 1$ for every $n \geq 1$ and $\dfrac{\psi(p)}{V(p)} = \dfrac{p+1}{p}$ for every prime p, it is immediate that $\liminf\limits_{n\to\infty} \dfrac{\psi(n)}{V(n)} = 1$. Thus, the minimal order of $\dfrac{\psi(n)}{V(n)}$ is 1.

Proposition 4.

$$\limsup_{n\to\infty} \frac{\psi(n)}{V(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}.$$

Proof. Let $f(n) = \sqrt{\dfrac{\psi(n)}{V(n)}}$ in Lemma 1. Here

$$f(p^\alpha) = \sqrt{\frac{p^\alpha + p^{\alpha-1}}{p^\alpha - p^{\alpha-1}+1}} \leq \sqrt{\frac{p+1}{p-1}} = \rho(p) < \left(1-\frac{1}{p}\right)^{-1},$$

and

$$f(p^4) = \sqrt{\frac{p^4+p^3}{p^4-p^3+1}} \geq 1 + \frac{1}{p},$$

so (ii) is fulfilled in the cited Lemma, for every prime p. We obtain

$$\limsup_{n\to\infty} \frac{\sqrt{\psi(n)}}{\sqrt{V(n)}\log\log n} = e^{\gamma} \prod_{p \text{ prime}} \sqrt{1-\frac{1}{p^2}} = e^{\gamma}\sqrt{\frac{6}{\pi^2}},$$

so

$$\limsup_{n\to\infty} \frac{\psi(n)}{V(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}. \qquad \square$$

3 Extremal orders concerning unitary analogues

In what follows we consider the functions $\sigma^*(n)$ and $\phi^*(n)$, representing the sum of the unitary divisors of n and the unitary Euler function, respectively. The functions $\sigma^*(n)$ and $\phi^*(n)$ are multiplicative. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorisation of $n > 1$, then

$$\phi^*(n) = (p_1^{\alpha_1} - 1) \cdots (p_k^{\alpha_k} - 1), \qquad \sigma^*(n) = (p_1^{\alpha_1} + 1) \cdots (p_k^{\alpha_k} + 1)$$

Note that $\sigma^*(n) = \sigma(n)$, $\phi^*(n) = \phi(n)$ for all squarefree n, and for every $n \geq 1$

$$\phi(n) \leq \phi^*(n) \leq n \leq \sigma^*(n) \leq \sigma(n).$$

We give extremal orders for the quotients $\dfrac{\sigma^*(n)}{V(n)}$ and $\dfrac{\phi^*(n)}{V(n)}$, the minimal order of $\dfrac{\phi^*(n)}{V(n)}$ being studied for powerful numbers. Since $\dfrac{\sigma^*(n)}{V(n)} \geq 1$ and for prime numbers p, $\lim\limits_{p\to\infty} \dfrac{\sigma^*(p)}{V(p)} = \lim\limits_{p\to\infty} \dfrac{p+1}{p} = 1$, it follows that $\liminf\limits_{n\to\infty} \dfrac{\sigma^*(n)}{V(n)} = 1$. If n is powerful, it is easy to see that $\dfrac{\phi^*(n)}{V(n)} \geq 1$, taking into account that $\dfrac{\phi^*(p^\alpha)}{V(p^\alpha)} \geq 1$ for $\alpha \geq 2$. For prime numbers p, we notice that $\lim\limits_{p\to\infty} \dfrac{\phi^*(p^2)}{V(p^2)} = \lim\limits_{p\to\infty} \dfrac{p^2-1}{p^2-p+1} = 1$, which implies that $\liminf\limits_{n\to\infty} \dfrac{\phi^*(n)}{V(n)} = 1$, so the minimal order of $\dfrac{\phi^*(n)}{V(n)}$ is 1. For the maximal orders of these quotients we give:

Proposition 5.

$$\limsup_{n\to\infty} \frac{\sigma^*(n)}{V(n)\log\log n} = e^\gamma, \qquad (i)$$

$$\limsup_{n\to\infty} \frac{\phi^*(n)}{V(n)\log\log n} = e^\gamma. \qquad (ii)$$

Proof.

(i) Take $f(n) = \dfrac{\sigma^*(n)}{V(n)}$, which is a nonnegative real-valued multiplicative arithmetic function. We have $f(p^\alpha) = \dfrac{p^\alpha + 1}{p^\alpha - p^{\alpha-1} + 1} \leq \left(1 - \dfrac{1}{p}\right)^{-1} = \rho(p)$, and

$f(p) = 1 + \frac{1}{p} \geq 1 + \frac{1}{p}$ for every prime p. Applying Lemma 1, we get

$$\limsup_{n\to\infty} \frac{\sigma^*(n)}{V(n)\log\log n} = e^\gamma.$$

(ii) Now let $f(n) = \frac{\phi^*(n)}{V(n)}$. Here
$f(p^\alpha) = \frac{p^\alpha - 1}{p^\alpha - p^{\alpha-1} + 1} \leq \left(1 - \frac{1}{p}\right)^{-1} = \rho(p)$, and
$f(p^4) = \frac{p^4 - 1}{p^4 - p^3 + 1} \geq 1 + \frac{1}{p}$, for every prime p. According to Lemma 1,

$$\limsup_{n\to\infty} \frac{\phi^*(n)}{V(n)\log\log n} = e^\gamma. \qquad \square$$

Corollary 2. *The maximal order of both* $\frac{\sigma^*(n)}{V(n)}$ *and* $\frac{\phi^*(n)}{V(n)}$ *is* $e^\gamma \log\log n$.

4 Extremal orders regarding compositions of functions

We now move to the study of extremal orders of some composite arithmetic functions. We start with $V(V(n))$ and $\phi(V(n))$.
We know that $V(n) \leq n$ for every $n \geq 1$, so $\frac{V(V(n))}{n} \leq \frac{V(n)}{n} \leq 1$ and $\lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \frac{V(V(p))}{p} = \lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \frac{V(p)}{p} = 1$, so the maximal order of $V(V(n))$ is n.
Since $\phi(n) \leq n$ and $V(n) \leq n$ for any $n \geq 1$, we have $\frac{\phi(V(n))}{n} \leq \frac{V(n)}{n} \leq 1$.
But $\lim\limits_{\substack{p\to\infty \\ p \text{ prime}}} \frac{\phi(V(p))}{p} = \lim\limits_{p\to\infty} \frac{p-1}{p} = 1$, so the maximal order of $\phi(V(n))$ is n.
In [7] was investigated the maximal order of $\phi^*(\phi(n))$. Using the general idea of that proof, we show:

Proposition 6. *The maximal order of* $V(\phi(n))$ *is* n.

Proof. We will use Linnik's theorem which states that if $(k, \ell) = 1$, then there exists a prime p such that $p \equiv \ell \pmod{k}$ and $p \ll k^c$, where c is a constant (one can take $c \leq 11$).
Let $A = \prod\limits_{\substack{p \leq x \\ p \text{ prime}}} p$. Since $(A^2, A+1) = 1$, by Linnik's theorem there is a prime number q such that $q \equiv A + 1 \pmod{A^2}$ and $q \ll (A^2)^c = A^{2c}$, where c

satisfies $c \le 11$. Let q be the least prime satisfying the above condition. So, $q - A - 1 = kA^2$, for some k. We have
$\phi(q) = q - 1 = A + kA^2 = A(1 + kA) = AB$, where $B = 1 + kA$. Thus $(A, B) = 1$, so B is free of prime factors $\le x$. We have $q - 1 = AB$, so $q = AB + 1$.
Since $V(n)$ is multiplicative, we have

$$\frac{V(\phi(q))}{q} = \frac{V(AB)}{AB+1} = \frac{V(A)}{A} \cdot \frac{V(B)}{B} \cdot \frac{AB}{AB+1}. \tag{1}$$

Here $\dfrac{AB}{AB+1} \to 1$ as $x \to \infty$, so it is sufficient to study $\dfrac{V(A)}{A}$ and $\dfrac{V(B)}{B}$.
Clearly,

$$\frac{V(A)}{A} = \frac{V\big(\prod\limits_{p \le x} p\big)}{\prod\limits_{p \le x} p} = \frac{\prod\limits_{p \le x} V(p)}{\prod\limits_{p \le x} p} = 1. \tag{2}$$

It is well-known that $A = \prod\limits_{p \le x} p = e^{O(x)}$. Since $q \ll A^{2c}$ and $A = e^{O(x)}$, from $B \ll A^{10}$ we have $B \ll \left(e^{O(x)}\right)^{10} = e^{O(x)}$, so

$$\log B \ll x. \tag{3}$$

If $B = \prod\limits_{i=1}^{k} q_i^{b_i}$ is the prime factorization of B, we obtain
$\log B = \sum\limits_{i=1}^{k} b_i \log q_i > (\log x) \sum\limits_{i=1}^{k} b_i$, as $q_i > x$ for all $i \in \{1, 2, ..., k\}$. But $\sum\limits_{i=1}^{k} b_i \ge k$, so $\log B > k \log x$, implying that $k < \dfrac{\log B}{\log x} \ll \dfrac{x}{\log x}$ (by(3)). We get:

$$\frac{V(B)}{B} = \frac{V\left(\prod\limits_{i=1}^{k} q_i^{b_i}\right)}{\prod\limits_{i=1}^{k} q_i^{b_i}} = \frac{\prod\limits_{i=1}^{k} (q_i^{b_i} - q_i^{b_i - 1} + 1)}{\prod\limits_{i=1}^{k} q_i^{b_i}} > \frac{\prod\limits_{i=1}^{k} (q_i^{b_i} - q_i^{b_i - 1})}{\prod\limits_{i=1}^{k} q_i^{b_i}} =$$

$$= \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right) > \left(1 - \frac{1}{x}\right)^k \ge \left(1 - \frac{1}{x}\right)^{O(\frac{x}{\log x})} > 1 + O\left(\frac{1}{\log x}\right),$$

because $1 - \frac{1}{q_i} > 1 - \frac{1}{x}$. So,

$$\frac{V(B)}{B} > 1 + O\left(\frac{1}{\log x}\right). \tag{4}$$

By (1), (2), (4) and $\frac{AB}{AB+1} \to 1$ as $x \to \infty$, we obtain

$$\frac{V(\phi(q))}{q} > 1 + O\left(\frac{1}{\log x}\right). \tag{5}$$

By relation (5), and since $\frac{V(\phi(n))}{n} \leq \frac{\phi(n)}{n} \leq 1$, it follows that

$$\limsup_{n\to\infty} \frac{V(\phi(n))}{n} = 1. \qquad \square$$

Proposition 7. *The maximal order of $V(\phi^*(n))$ is n.*

Proof. We apply the following result:
If a is an integer, $a > 1$, p is a prime number and $f(n)$ is an arithmetical function satisfying $\phi(n) \leq f(n) \leq \sigma(n)$, one has

$$\lim_{p\to\infty} \frac{f(N(a,p))}{N(a,p)} = 1, \tag{6}$$

where $N(a,p) = \frac{a^p - 1}{a - 1}$ (see e.g. D.Suryanarayana [9]).
Since $\phi^*(n) \leq n$, it follows that $V(\phi^*(n)) \leq \phi^*(n) \leq n$, so

$$\frac{V(\phi^*(n))}{n} \leq 1. \tag{7}$$

Let $n = 2^p$, p prime number. Then we have

$$\frac{V(\phi^*(2^p))}{2^p} = \frac{V(2^p - 1)}{2^p - 1} \cdot \frac{2^p - 1}{2^p}. \tag{8}$$

Since $\phi(n) \leq V(n) \leq \sigma(n)$ and $N(2,p) = 2^p - 1$, it follows that

$$\lim_{p\to\infty} \frac{V(2^p - 1)}{2^p - 1} = 1,$$

taking into account (6). By (8), taking $p \to \infty$, we obtain

$$\lim_{p\to\infty} \frac{V(\phi^*(2^p))}{2^p} = 1. \tag{9}$$

Now (7) and (9) imply $\limsup_{n\to\infty} \frac{V(\phi^*(n))}{n} = 1$. $\square$

For the maximal orders of $\frac{\sigma(\phi^*(n))}{V(\phi^*(n))}, \frac{\psi(\phi^*(n))}{V(\phi^*(n))}$ we give

Proposition 8.

$$(i)\ \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2} = e^{2\gamma},$$

$$(ii)\ \limsup_{n\to\infty} \frac{\psi(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2} = \frac{6}{\pi^2}e^{\gamma}.$$

Proof.
(i) Let

$$l_1 := \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2}$$

and

$$l_2 := \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2}.$$

Since $\phi^*(n) \le n$ for every $n \ge 1$,
$l_1 = \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} \le l_2 = \limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log \phi^*(n))^2} \le$
$\limsup_{m\to\infty} \frac{\sigma(m)}{V(m)(\log\log m)^2} = e^{2\gamma}$, by Proposition 3.
Since $(n,1) = 1$, by Linnik's theorem, there exists a prime number p such that $p \equiv 1 \pmod n$ and $p \ll n^c$. Let p_n be the least prime such that $p_n \equiv 1 \pmod n$, for every n. Then $n \mid p_n - 1$ and $p_n \ll n^c$, so $\log\log p_n \sim \log\log n$.
Observe that $a \mid b$ implies $\frac{\sigma(a)}{V(a)} \le \frac{\sigma(b)}{V(b)}$. If $p^\beta \mid p^\alpha$ $(\beta \le \alpha)$, it is easy to see that $\frac{\sigma(p^\beta)}{V(p^\beta)} \le \frac{\sigma(p^\alpha)}{V(p^\alpha)}$. The general case follows, taking into account that $\frac{\sigma(n)}{V(n)}$ is multiplicative. So,
$\frac{\sigma(\phi^*(p_n))}{V(\phi^*(p_n))(\log\log p_n)^2} =$
$\frac{\sigma(p_n-1)}{V(p_n-1)(\log\log p_n)^2} \sim \frac{\sigma(p_n-1)}{V(p_n-1)(\log\log n)^2} \ge \frac{\sigma(n)}{V(n)(\log\log n)^2}.$

But
$\limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{V(\phi^*(n))(\log\log n)^2} \ge \limsup_{n\to\infty} \frac{\sigma(\phi^*(p_n))}{V(\phi^*(p_n))(\log\log p_n)^2} \ge$

$$\limsup_{n\to\infty} \frac{\sigma(n)}{V(n)(\log\log n)^2} = e^{2\gamma}.$$

We obtain $e^{2\gamma} \le l_1 \le l_2 \le e^{2\gamma}$, hence $l_1 = l_2 = e^{2\gamma}$.

(ii) The proof is similar to the proof of (i), taking into account that $a \mid b$ impies $\dfrac{\psi(a)}{V(a)} \le \dfrac{\psi(b)}{V(b)}$ and $\limsup\limits_{n\to\infty} \dfrac{\psi(n)}{V(n)(\log\log n)^2} = \dfrac{6}{\pi^2}e^{2\gamma}$, by Proposition 4. □

So, the maximal orders of $\dfrac{\sigma(\phi^*(n))}{V(\phi^*(n))}$, $\dfrac{\psi(\phi^*(n))}{V(\phi^*(n))}$ are $e^{2\gamma}(\log\log n)^2$ and $\dfrac{6}{\pi^2}e^{2\gamma}(\log\log n)^2$, respectively. In a similar manner, since $\limsup\limits_{n\to\infty} \dfrac{\sigma^*(n)}{V(n)\log\log n} = \limsup\limits_{n\to\infty} \dfrac{\phi^*(n)}{V(n)\log\log n} = e^{\gamma}$ (Proposition 5), $a \mid b$ implies $\dfrac{\sigma^*(a)}{V(a)} \le \dfrac{\sigma^*(b)}{V(b)}$ and $\dfrac{\phi^*(a)}{V(a)} \le \dfrac{\phi^*(b)}{V(b)}$, respectively, it can be shown that

$$\limsup_{n\to\infty} \frac{\sigma^*(\phi^*(n))}{V(\phi^*(n))\log\log n} = \limsup_{n\to\infty} \frac{\sigma^*(\phi^*(n))}{V(\phi^*(n))\log\log \phi^*(n)} = e^{\gamma} \text{ and}$$

$$\limsup_{n\to\infty} \frac{\phi^*(\phi^*(n))}{V(\phi^*(n))\log\log n} = \limsup_{n\to\infty} \frac{\phi^*(\phi^*(n))}{V(\phi^*(n))\log\log \phi^*(n)} = e^{\gamma}.$$

5 Open Problems

Problem 1. Note that

$$\limsup_{n\to\infty} \frac{V(n)\sigma(n)}{n^2} = \limsup_{n\to\infty} \frac{V(n)\psi(n)}{n^2} = \infty,$$

since for $n_k = p_1 \cdots p_k$ (the product of the first k primes),

$$\frac{V(n_k)\sigma(n_k)}{n_k^2} = \frac{(p_1+1)\cdots(p_k+1)}{p_1\cdots p_k} = \prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right) \to \infty,\ k \to \infty;$$

the other relation follows in a similar manner. What are the maximal orders for $\dfrac{V(n)\sigma(n)}{n^2}$ and $\dfrac{V(n)\psi(n)}{n^2}$?

Problem 2. Note that

$$\liminf_{n\to\infty} \frac{V(\phi(n))}{n} = \liminf_{n\to\infty} \frac{V(\phi^*(n))}{n} = \liminf_{n\to\infty} \frac{\phi^*(V(n))}{n} = 0.$$

For $n_k = p_1 \cdots p_k$ (the product of the first k primes),

$$\frac{V(\phi(n_k))}{n_k} = \frac{V((p_1-1)\cdots(p_k-1))}{p_1\cdots p_k} \leq \frac{(p_1-1)\cdots(p_k-1)}{p_1\cdots p_k}$$
$$= \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right),$$

so

$$\lim_{k\to\infty}\frac{V(\phi(n_k))}{n_k} = \lim_{k\to\infty}\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right) = 0,$$

similarly the other relations. What are the minimal orders for the $V(\phi(n))$, $V(\phi^*(n))$, $\phi^*(V(n))$?

Problem 3. Taking $n_k = p_1 \cdots p_k$ (the product of the first k primes),

$$\frac{\sigma^*(V(n_k))}{n_k} = \frac{\sigma^*(p_1\cdots p_k)}{p_1\cdots p_k} = \frac{(p_1+1)\cdots(p_k+1)}{p_1\cdots p_k}$$
$$= \left(1+\frac{1}{p_1}\right)\cdots\left(1+\frac{1}{p_k}\right) \to \infty$$

as $k \to \infty$, so $\limsup_{n\to\infty} \frac{\sigma^*(V(n))}{n} = \infty$. What is the maximal order for $\sigma^*(V(n))$?

ACKNOWLEDGEMENT. The author would like to thank Professor L.Tóth and Professor G. Mincu for the valuable suggestions which improved the presentation.

The author also thanks the referees for helpful suggestions.

References

[1] O. Alkam, E. Osba, On the regular elements in $\mathbb{Z}_n$, *Turk.J.Math.*, 32(2008).

[2] P. Haukkanen, L. Tóth, An analogue of Ramanujan's sum with respect to regular integers (mod r), *Ramanujan J.*, 27(2012) 71-88.

[3] J. Morgado, A property of the Euler ϕ-function concerning the integers which are regular modulo n, *Portugal. Math.*, 33 (1974), 185–191.

[4] J. Morgado, Inteiros regulares módulo n, *Gazeta de Matematica (Lisboa)*, 33(1972), No. 125–128, 1–5.

[5] J. Sándor, Geometric Theorems, Diophantine Equations, And Arithmetic Functions, *American Research Press Rehoboth*, 2002.

[6] J. Sándor, B. Crstici, Handbook of number theory II, *Kluwer Academic Publishers*, Dordrecht, 2004.

[7] J. Sándor, L. Tóth, Extremal orders of compositions of certain arithmetical functions, *Integers: Electronic Journal of Combinatorial Number Theory*, 8 (2008), # A34.

[8] R. Sivaramakrishnan, Classical theory of arithmetic functions, *Monographs and Textbooks in Pure and Applied Mathematics*, 126 . Marcel Dekker, Inc., New York, 1989.

[9] D. Suryanarayana, On a class of sequences of integers, *Amer. Math. Monthly*, 84(1977), 728–730.

[10] L. Tóth, A gcd-sum function over regular integers modulo n, *J. Integer Seq.*, 12(2009), no.2, Article 09.2.5, 8 pp.

[11] L. Tóth, Regular integers (mod n), *Annales Univ. Sci. Budapest., Sect Comp.*, 29(2008), 264–275, see http://front.math.ucdavis.edu/0710.1936.

[12] L. Tóth, E. Wirsing, The maximal order of a class of multiplicative arithmetical functions, *Annales Univ. Sci. Budapest., Sect Comp.*, 22(2003), 353–364.

Brăduţ APOSTOL,
"Spiru Haret" Pedagogical High School,
1 Timotei Cipariu St. , RO - 620004 Focşani, ROMANIA,
Email:apo_brad@yahoo.com