DOI: 10.2478/auom-2013-0022
An. Şt. Univ. Ovidius Constanţa **Vol. 21(2),2013, 35–52**

On the Growth of Solutions of Some Second Order Linear Differential Equations With Entire Coefficients

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Abstract

In this paper, we investigate the order and the hyper-order of growth of solutions of the linear differential equation

$$f'' + Q\left(e^{-z}\right) f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)^n f = 0,$$

where $n \geqslant 2$ is an integer, $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ are entire functions with $\max\{\sigma(A_j) : j = 1, 2\} < 1$, $Q(z) = q_m z^m + \cdots + q_1 z + q_0$ is a nonconstant polynomial and a_1, a_2 are complex numbers. Under some conditions, we prove that every solution $f(z) \not\equiv 0$ of the above equation is of infinite order and hyper-order 1.

1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [8], [13]). Let $\sigma(f)$ denote the order of growth of an entire function f and the hyper-order $\sigma_2(f)$ of f is defined by (see [9], [13])

$$\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log\log T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log\log\log M(r, f)}{\log r},$$

Key Words: Linear differential equations, Entire solutions, Order of growth, Hyper-order, Fixed points.
2010 Mathematics Subject Classification: Primary 34M10; Secondary 30D35.
Received: February 2013
Accepted: June 2013

where $T(r,f)$ is the Nevanlinna characteristic function of f and $M(r,f) = \max_{|z|=r} |f(z)|$.

In order to give some estimates of fixed points, we recall the following definition.

Definition 1.1 ([3], [10]) Let f be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \overline{\lambda}(f-z) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$. We also define

$$\overline{\lambda}(f-\varphi) = \limsup_{r\to+\infty} \frac{\log \overline{N}(r, \frac{1}{f-\varphi})}{\log r}$$

for any meromorphic function $\varphi(z)$.

In [11], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem A ([11]) *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leqslant |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f \not\equiv 0$ of the equation*

$$f'' + e^{-z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0$$

has infinite order and $\sigma_2(f) = 1$.

The main purpose of this paper is to extend and improve the results of Theorem A to some second order linear differential equations. In fact we will prove the following results.

Theorem 1.1 *Let $n \geqslant 2$ be an integer, $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\max\{\sigma(A_j) : j = 1, 2\} < 1$, $Q(z) = q_m z^m + \cdots + q_1 z + q_0$ be nonconstant polynomial and a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$. If (1) $\arg a_1 \neq \pi$ and $\arg a_1 \neq \arg a_2$ or (2) $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ and*

$|a_2| > n\,|a_1|$ *or* (3) $a_1 < 0$ *and* $\arg a_1 \neq \arg a_2$ *or* (4) $-\frac{1}{n}\left(|a_2| - m\right) < a_1 < 0$, $|a_2| > m$ *and* $\arg a_1 = \arg a_2$, *then every solution* $f \not\equiv 0$ *of the equation*

$$f'' + Q\left(e^{-z}\right) f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right)^n f = 0 \tag{1.1}$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

Theorem 1.2 *Let* $A_j(z)$ $(j = 1, 2)$, $Q(z)$, a_1, a_2, n *satisfy the additional hypotheses of Theorem 1.1. If* $\varphi \not\equiv 0$ *is an entire function of order* $\sigma(\varphi) < +\infty$, *then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies*

$$\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty,$$

$$\overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = 1.$$

Theorem 1.3 *Let* $A_j(z)$ $(j = 1, 2)$, $Q(z)$, a_1, a_2, n *satisfy the additional hypotheses of Theorem 1.1. If* $\varphi \not\equiv 0$ *is an entire function of order* $\sigma(\varphi) < 1$, *then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies*

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = +\infty.$$

Furthermore, if (i) $(2n+2)\,a_1 \neq (2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, 2;\ k = 0, 1, \cdots, 2m)$, $(n+2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, \cdots, n+2;\ k = 0, 1, \cdots, m)$ *or* (ii) $(2n+2)\,a_2 \neq (2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, 2;\ k = 0, 1, \cdots, 2m)$, $(n+2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, \cdots, n+2;\ k = 0, 1, \cdots, m)$, *then*

$$\overline{\lambda}(f'' - \varphi) = +\infty.$$

Corollary 1.1 *Let* $A_j(z)$ $(j = 1, 2)$, $Q(z)$, a_1, a_2, n *satisfy the additional hypotheses of Theorem 1.1. If* $f \not\equiv 0$ *is any solution of equation* (1.1), *then* f, f' *all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f) = \overline{\tau}(f') = \infty.$$

Furthermore, if (i) $(2n+2)\,a_1 \neq (2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, 2;\ k = 0, 1, \cdots, 2m)$, $(n+2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, \cdots, n+2;\ k = 0, 1, \cdots, m)$ *or* (ii) $(2n+2)\,a_2 \neq (2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, 2;\ k = 0, 1, \cdots, 2m)$, $(n+2-p)\,a_1 + pa_2 - k$ $(p = 0, 1, \cdots, n+2;\ k = 0, 1, \cdots, m)$, *then* f'' *has infinitely many fixed points and satisfies*

$$\overline{\tau}(f'') = \infty.$$

2 Preliminary lemmas

To prove our theorems, we need the following lemmas.

Lemma 2.1 ([7]) *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$, $H = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, \cdots, q)$ and let $\varepsilon > 0$ be a given constant. Then,*
(i) *there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ with linear measure zero, such that, if $\psi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$, such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_0$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.1}$$

(ii) *there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| \notin E_2 \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}, \tag{2.2}$$

(iii) *there exists a set $E_3 \subset (0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma+\varepsilon)}. \tag{2.3}$$

Lemma 2.2 ([4]) *Suppose that $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geqslant 1$, that $A(z)$ $(\not\equiv 0)$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \diagdown (E_4 \cup E_5)$, there is $R > 0$, such that for $|z| = r > R$, we have*
(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta) r^n\} \leqslant \left|g\left(re^{i\theta}\right)\right| \leqslant \exp\{(1+\varepsilon)\delta(P,\theta) r^n\}; \tag{2.4}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta) r^n\} \leqslant \left|g\left(re^{i\theta}\right)\right| \leqslant \exp\{(1-\varepsilon)\delta(P,\theta) r^n\}, \tag{2.5}$$

where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3 ([11]) *Suppose that $n \geqslant 1$ is a positive entire number. Let $P_j(z) = a_{jn}z^n + \cdots$ $(j = 1, 2)$ be nonconstant polynomials, where a_{jq} $(q = 1, \cdots, n)$ are complex numbers and $a_{1n}a_{2n} \neq 0$. Set $z = re^{i\theta}$, $a_{jn} = |a_{jn}| e^{i\theta_j}$, $\theta_j \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\delta(P_j, \theta) = |a_{jn}| \cos(\theta_j + n\theta)$, then there is a set $E_6 \subset \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right)$ that has linear measure zero. If $\theta_1 \neq \theta_2$, then there exists a ray $\arg z = \theta$, $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$, such that*

$$\delta(P_1, \theta) > 0, \ \delta(P_2, \theta) < 0 \tag{2.6}$$

or

$$\delta(P_1, \theta) < 0, \ \delta(P_2, \theta) > 0, \tag{2.7}$$

where $E_7 = \left\{\theta \in \left[-\frac{\pi}{2n}, \frac{3\pi}{2n}\right) : \delta(P_j, \theta) = 0\right\}$ is a finite set, which has linear measure zero.

Remark 2.1 ([11]) In Lemma 2.3, if $\theta \in \left(-\frac{\pi}{2n}, \frac{\pi}{2n}\right) \setminus (E_6 \cup E_7)$ is replaced by $\theta \in \left(\frac{\pi}{2n}, \frac{3\pi}{2n}\right) \setminus (E_6 \cup E_7)$, then we obtain the same result.

Lemma 2.4([5]) *Suppose that $k \geqslant 2$ and $B_0, B_1, \cdots, B_{k-1}$ are entire functions of finite order and let $\sigma = \max\{\sigma(B_j) : j = 0, \cdots, k-1\}$. Then every solution f of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_1f' + B_0f = 0 \tag{2.8}$$

satisfies $\sigma_2(f) \leqslant \sigma$.

Lemma 2.5 ([7]) *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_8 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \leqslant i < j \leqslant k)$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant B\left\{\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\right\}^{j-i}. \tag{2.9}$$

Lemma 2.6([2]) *Let $A_0, A_1, \cdots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\sigma(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F, \tag{2.10}$$

then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = +\infty.$$

Lemma 2.7 ([1]) *Let $A_0, A_1, \cdots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation (2.10) with $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma$, then f satisfies*

$$\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma. \tag{2.11}$$

Lemma 2.8([6], [13]) *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)\ (n \geqslant 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *are entire functions satisfying the following conditions:*
(i) $\sum\limits_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leqslant j < k \leqslant n$;
(iii) For $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$, $T(r, f_j) = o\left\{T\left(r, e^{g_h(z)-g_k(z)}\right)\right\}$ $(r \to \infty$, $r \notin E_9)$, *where* E_9 *is a set with finite linear measure.*
Then $f_j(z) \equiv 0\ (j = 1, \cdots, n)$.

Lemma 2.9 ([12]) *Suppose that* $f_1(z), f_2(z), \cdots, f_n(z)\ (n \geqslant 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *are entire functions satisfying the following conditions:*
(i) $\sum\limits_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv f_{n+1}$;
(ii) *If* $1 \leqslant j \leqslant n+1, 1 \leqslant k \leqslant n$, *the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \geqslant 2$, $1 \leqslant j \leqslant n+1$, $1 \leqslant h < k \leqslant n$, *and the order of* f_j *is less than the order of* $e^{g_h - g_k}$. *Then* $f_j(z) \equiv 0\ (j = 1, 2, \cdots, n+1)$.

3 Proof of Theorem 1.1

Assume that $f\ (\not\equiv 0)$ is a solution of equation (1.1).
First step: We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. We rewrite (1.1) as

$$\frac{f''}{f} + Q\left(e^{-z}\right)\frac{f'}{f} + A_1^n e^{na_1 z} + A_2^n e^{na_2 z} + \sum_{p=1}^{n-1} C_n^p A_1^{n-p} e^{(n-p)a_1 z} A_2^p e^{pa_2 z} = 0. \tag{3.1}$$

By Lemma 2.1, for any given ε,

$$0 < \varepsilon < \min\left\{\frac{|a_2| - n\,|a_1|}{2\left[(2n-1)\,|a_2| + n\,|a_1|\right]}, \frac{1}{2\,(2n-1)}\right\},$$

there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geqslant R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant r^{j(\sigma-1+\varepsilon)} \quad (j = 1, 2)\,. \tag{3.2}$$

Let $z = re^{i\theta}$, $a_1 = |a_1|\, e^{i\theta_1}$, $a_2 = |a_2|\, e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. We know that $\delta(pa_1 z, \theta) = p\delta(a_1 z, \theta)$ and $\delta(pa_2 z, \theta) = p\delta(a_2 z, \theta)$, where $p > 0$.

Case 1: Assume that $\arg a_1 \neq \pi$ and $\arg a_1 \neq \arg a_2$, which is $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$.

By Lemma 2.2 and Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ (where E_6 and E_7 are defined as in Lemma 2.3, $E_1 \cup E_6 \cup E_7$ is of the linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0,\ \delta(a_2 z, \theta) < 0$$

or

$$\delta(a_1 z, \theta) < 0,\ \delta(a_2 z, \theta) > 0.$$

$a)$ When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we get by Lemma 2.2

$$|A_1^n e^{n a_1 z}| \geqslant \exp\{(1-\varepsilon) n \delta(a_1 z, \theta) r\}, \tag{3.3}$$

$$|A_2^n e^{n a_2 z}| \leqslant \exp\{(1-\varepsilon) n \delta(a_2 z, \theta) r\} < 1, \tag{3.4}$$

$$\left|A_1^{n-p} e^{(n-p) a_1 z}\right| \leqslant \exp\{(1+\varepsilon)(n-p)\delta(a_1 z, \theta) r\}$$
$$\leqslant \exp\{(1+\varepsilon)(n-1)\delta(a_1 z, \theta) r\},\ p = 1, \cdots, n-1, \tag{3.5}$$

$$|A_2^p e^{p a_2 z}| \leqslant \exp\{(1-\varepsilon) p \delta(a_2 z, \theta) r\} < 1,\ p = 1, \cdots, n-1. \tag{3.6}$$

For $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have

$$\left|Q\left(e^{-z}\right)\right| = \left|q_m e^{-mz} + \cdots + q_1 e^{-z} + q_0\right|$$
$$\leqslant |q_m| \left|e^{-mz}\right| + \cdots + |q_1| \left|e^{-z}\right| + |q_0|$$
$$\leqslant |q_m| e^{-mr\cos\theta} + \cdots + |q_1| e^{-r\cos\theta} + |q_0| \leqslant M, \tag{3.7}$$

where $M > 0$ is a some constant. By (3.1) – (3.7), we get

$$\exp\{(1-\varepsilon) n \delta(a_1 z, \theta) r\} \leqslant |A_1^n e^{n a_1 z}|$$
$$\leqslant \left|\frac{f''}{f}\right| + \left|Q\left(e^{-z}\right)\right| \left|\frac{f'}{f}\right| + |A_2^n e^{n a_2 z}| + \sum_{p=1}^{n-1} C_n^p \left|A_1^{n-p} e^{(n-p) a_1 z}\right| |A_2^p e^{p a_2 z}|$$
$$\leqslant r^{2(\sigma-1+\varepsilon)} + M r^{\sigma-1+\varepsilon} + 2^n \exp\{(1+\varepsilon)(n-1)\delta(a_1 z, \theta) r\}$$
$$\leqslant M_1 r^{M_2} \exp\{(1+\varepsilon)(n-1)\delta(a_1 z, \theta) r\}, \tag{3.8}$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. By $0 < \varepsilon < \frac{1}{2(2n-1)}$ and (3.8), we have

$$\exp\left\{\frac{1}{2}\delta(a_1 z, \theta) r\right\} \leqslant M_1 r^{M_2}. \tag{3.9}$$

By $\delta(a_1 z,\theta) > 0$ we know that (3.9) is a contradiction.

b) When $\delta(a_1 z,\theta) < 0$, $\delta(a_2 z,\theta) > 0$, using a proof similar to the above, we can also get a contradiction.

Case 2: Assume that $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ and $|a_2| > n|a_1|$, which is $\theta_1 \neq \pi$ and $\theta_1 = \theta_2$ and $|a_2| > n|a_1|$.

By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_1 z,\theta) > 0$. Since $|a_2| > n|a_1|$ and $n \geqslant 2$, then $|a_2| > |a_1|$, thus $\delta(a_2 z,\theta) > \delta(a_1 z,\theta) > 0$. For sufficiently large r, we have by using Lemma 2.2

$$|A_2^n e^{na_2 z}| \geqslant \exp\{(1-\varepsilon)\, n\delta(a_2 z,\theta)\, r\}, \tag{3.10}$$

$$|A_1^n e^{na_1 z}| \leqslant \exp\{(1+\varepsilon)\, n\delta(a_1 z,\theta)\, r\}, \tag{3.11}$$

$$\left|A_1^{n-p} e^{(n-p)a_1 z}\right| \leqslant \exp\{(1+\varepsilon)(n-1)\,\delta(a_1 z,\theta)\, r\},\ p = 1,\cdots,n-1, \tag{3.12}$$

$$|A_2^p e^{pa_2 z}| \leqslant \exp\{(1+\varepsilon)(n-1)\,\delta(a_2 z,\theta)\, r\},\ p = 1,\cdots,n-1. \tag{3.13}$$

By (3.1), (3.2), (3.7) and $(3.10) - (3.13)$ we get

$$\begin{aligned}
&\exp\{(1-\varepsilon)\, n\delta(a_2 z,\theta)\, r\} \leqslant |A_2^n e^{na_2 z}| \\
&\leqslant \left|\frac{f''}{f}\right| + \left|Q\left(e^{-z}\right)\right|\left|\frac{f'}{f}\right| + |A_1^n e^{na_1 z}| + \sum_{p=1}^{n-1} C_n^p \left|A_1^{n-p} e^{(n-p)a_1 z}\right| |A_2^p e^{pa_2 z}| \\
&\leqslant r^{2(\sigma-1+\varepsilon)} + Mr^{\sigma-1+\varepsilon} + \exp\{(1+\varepsilon)\, n\delta(a_1 z,\theta)\, r\} \\
&+2^n \exp\{(1+\varepsilon)(n-1)\,\delta(a_1 z,\theta)\, r\} \exp\{(1+\varepsilon)(n-1)\,\delta(a_2 z,\theta)\, r\} \\
&\leqslant M_1 r^{M_2} \exp\{(1+\varepsilon)\, n\delta(a_1 z,\theta)\, r\} \exp\{(1+\varepsilon)(n-1)\,\delta(a_2 z,\theta)\, r\}.
\end{aligned} \tag{3.14}$$

Therefore, by (3.14), we obtain

$$\exp\{\alpha r\} \leqslant M_1 r^{M_2}, \tag{3.15}$$

where

$$\alpha = [1-\varepsilon(2n-1)]\,\delta(a_2 z,\theta) - (1+\varepsilon)\, n\delta(a_1 z,\theta).$$

Since $0 < \varepsilon < \frac{|a_2| - n|a_1|}{2[(2n-1)|a_2| + n|a_1|]}$, $\theta_1 = \theta_2$ and $\cos(\theta_1 + \theta) > 0$, then

$$\begin{aligned}
\alpha &= [1-\varepsilon(2n-1)]\,|a_2|\cos(\theta_2+\theta) - (1+\varepsilon)\, n|a_1|\cos(\theta_1+\theta) \\
&= \{|a_2| - n|a_1| - \varepsilon[(2n-1)|a_2| + n|a_1|]\}\cos(\theta_1+\theta)
\end{aligned}$$

$$> \frac{|a_2| - n\,|a_1|}{2} \cos(\theta_1 + \theta) > 0.$$

Hence (3.15) is a contradiction.

Case 3: Assume that $a_1 < 0$ and $\arg a_1 \neq \arg a_2$, which is $\theta_1 = \pi$ and $\theta_2 \neq \pi$.

By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos\theta > 0$, we have $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta < 0$. For sufficiently large r, we obtain by Lemma 2.2

$$|A_2^n e^{n a_2 z}| \geqslant \exp\{(1-\varepsilon)\, n\delta(a_2 z, \theta)\, r\}, \tag{3.16}$$

$$|A_1^n e^{n a_1 z}| \leqslant \exp\{(1-\varepsilon)\, n\delta(a_1 z, \theta)\, r\} < 1, \tag{3.17}$$

$$\left|A_1^{n-p} e^{(n-p) a_1 z}\right| \leqslant \exp\{(1-\varepsilon)(n-p)\,\delta(a_1 z, \theta)\, r\} < 1,\ p = 1, \cdots, n-1, \tag{3.18}$$

$$|A_2^p e^{p a_2 z}| \leqslant \exp\{(1+\varepsilon)(n-1)\,\delta(a_2 z, \theta)\, r\},\ p = 1, \cdots, n-1. \tag{3.19}$$

Using the same reasoning as in Case 1(a), we can get a contradiction.

Case 4. Assume that $-\frac{1}{n}(|a_2| - m) < a_1 < 0$, $|a_2| > m$ and $\arg a_1 = \arg a_2$, which is $\theta_1 = \theta_2 = \pi$ and $|a_1| < \frac{1}{n}(|a_2| - m)$, then $|a_2| > n\,|a_1| + m$, hence $|a_2| > n\,|a_1|$.

By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, then $\cos\theta < 0$, $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta > 0$, $\delta(a_2 z, \theta) = |a_2| \cos(\theta_2 + \theta) = -|a_2| \cos\theta > 0$. Since $|a_2| > n\,|a_1|$ and $n \geqslant 2$, then $|a_2| > |a_1|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$, for sufficiently large r, we get $(3.10) - (3.13)$ hold. For $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ we have

$$\left|Q\left(e^{-z}\right)\right| \leqslant M e^{-mr\cos\theta}. \tag{3.20}$$

By (3.1), (3.2), $(3.10) - (3.13)$ and (3.20), we get

$$\exp\{(1-\varepsilon)\, n\delta(a_2 z, \theta)\, r\} \leqslant |A_2^n e^{n a_2 z}|$$

$$\leqslant \left|\frac{f''}{f}\right| + \left|Q\left(e^{-z}\right)\right| \left|\frac{f'}{f}\right| + |A_1^n e^{n a_1 z}| + \sum_{p=1}^{n-1} C_n^p \left|A_1^{n-p} e^{(n-p) a_1 z}\right| |A_2^p e^{p a_2 z}|$$

$$\leqslant r^{2(\sigma - 1 + \varepsilon)} + M r^{\sigma - 1 + \varepsilon} e^{-mr\cos\theta} + \exp\{(1+\varepsilon)\, n\delta(a_1 z, \theta)\, r\}$$

$$+ 2^n \exp\{(1+\varepsilon)(n-1)\,\delta(a_1 z, \theta)\, r\} \exp\{(1+\varepsilon)(n-1)\,\delta(a_2 z, \theta)\, r\}$$

$$\leqslant M_1 r^{M_2} e^{-mr\cos\theta} \exp\left\{(1+\varepsilon) n\delta(a_1 z,\theta) r\right\} \exp\left\{(1+\varepsilon)(n-1)\delta(a_2 z,\theta) r\right\}. \tag{3.21}$$

Therefore, by (3.21), we obtain

$$\exp\{\beta r\} \leqslant M_1 r^{M_2}, \tag{3.22}$$

where

$$\beta = [1-\varepsilon(2n-1)]\,\delta(a_2 z,\theta) - (1+\varepsilon) n\delta(a_1 z,\theta) + m\cos\theta.$$

Since $|a_2| - n|a_1| - m > 0$, then

$$2\left[(2n-1)|a_2| + n|a_1|\right] > |a_2| - n|a_1| - m > 0.$$

Therefore,

$$\frac{|a_2| - n|a_1| - m}{2\left[(2n-1)|a_2| + n|a_1|\right]} < 1.$$

Then, we can take $0 < \varepsilon < \frac{|a_2|-n|a_1|-m}{2[(2n-1)|a_2|+n|a_1|]}$. Since $0 < \varepsilon < \frac{|a_2|-n|a_1|-m}{2[(2n-1)|a_2|+n|a_1|]}$, $\theta_1 = \theta_2 = \pi$ and $\cos\theta < 0$, then

$$\beta = -\cos\theta\left\{|a_2| - n|a_1| - m - \varepsilon\left[(2n-1)|a_2| + n|a_1|\right]\right\}$$

$$> -\frac{1}{2}\left(|a_2| - n|a_1| - m\right)\cos\theta > 0.$$

Hence, (3.22) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = +\infty$.

Second step: We prove that $\sigma_2(f) = 1$. By

$$\max\{\sigma(Q(e^{-z})), \sigma((A_1 e^{a_1 z} + A_2 e^{a_2 z})^n)\} = 1$$

and the Lemma 2.4, we get $\sigma_2(f) \leqslant 1$. By Lemma 2.5, we know that there exists a set $E_8 \subset (1,+\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_8$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T(2r,f)\right]^{j+1} \quad (j = 1,2). \tag{3.23}$$

Case 1: $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_1 z,\theta) > 0, \delta(a_2 z,\theta) < 0 \text{ or } \delta(a_1 z,\theta) < 0, \delta(a_2 z,\theta) > 0.$$

a) When $\delta(a_1z,\theta) > 0$, $\delta(a_2z,\theta) < 0$, for sufficiently large r, we get $(3.3)-(3.7)$ holds. By (3.1), $(3.3)-(3.7)$ and (3.23), we obtain

$$\exp\{(1-\varepsilon)n\delta(a_1z,\theta)r\} \leqslant |A_1^n e^{na_1z}|$$
$$\leqslant \left|\frac{f''}{f}\right| + \left|Q\left(e^{-z}\right)\right| \left|\frac{f'}{f}\right| + |A_2^n e^{na_2z}| + \sum_{p=1}^{n-1} C_n^p \left|A_1^{n-p} e^{(n-p)a_1z}\right| |A_2^p e^{pa_2z}|$$
$$\leqslant B\left[T(2r,f)\right]^3 + MB\left[T(2r,f)\right]^2 + 2^n \exp\{(1+\varepsilon)(n-1)\delta(a_1z,\theta)r\}$$
$$\leqslant M_1 \exp\{(1+\varepsilon)(n-1)\delta(a_1z,\theta)r\}\left[T(2r,f)\right]^3. \tag{3.24}$$

By $0 < \varepsilon < \frac{1}{2(2n-1)}$ and (3.24), we have

$$\exp\left\{\frac{1}{2}\delta(a_1z,\theta)r\right\} \leqslant M_1\left[T(2r,f)\right]^3. \tag{3.25}$$

By $\delta(a_1z,\theta) > 0$ and (3.25), we have $\sigma_2(f) \geqslant 1$, then $\sigma_2(f) = 1$.

b) When $\delta(a_1z,\theta) < 0$, $\delta(a_2z,\theta) > 0$, using a proof similar to the above, we can also get $\sigma_2(f) = 1$.

Case 2: $\theta_1 \neq \pi$, $\theta_1 = \theta_2$ and $|a_2| > n|a_1|$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2z,\theta) > \delta(a_1z,\theta) > 0$$

and for sufficiently large r, we get (3.7) and $(3.10)-(3.13)$ hold. By (3.1), (3.7), $(3.10)-(3.13)$ and (3.23), we get

$$\exp\{\alpha r\} \leqslant M_1\left[T(2r,f)\right]^3, \tag{3.26}$$

where
$$\alpha = [1-\varepsilon(2n-1)]\,\delta(a_2z,\theta) - (1+\varepsilon)n\delta(a_1z,\theta) > 0.$$

By $\alpha > 0$ and (3.26), we have $\sigma_2(f) \geqslant 1$, then $\sigma_2(f) = 1$.

Case 3: $a_1 < 0$ and $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2z,\theta) > 0 \text{ and } \delta(a_1z,\theta) < 0$$

and for sufficiently large r, we get $(3.16)-(3.19)$ hold. Using the same reasoning as in second step (**Case 1** (a)), we can get $\sigma_2(f) = 1$.

Case 4: $-\frac{1}{n}(|a_2|-m) < a_1 < 0$, $|a_2| > m$ and $\theta_1 = \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus (E_1 \cup E_6 \cup E_7)$, satisfying

$$\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$$

and for sufficiently large r, we get $(3.10)-(3.13)$ hold. By (3.1), $(3.10)-(3.13)$, (3.20) and (3.23) we obtain

$$\exp\{\beta r\} \leqslant M_1 [T(2r, f)]^3, \tag{3.27}$$

where

$$\beta = [1-\varepsilon(2n-1)]\,\delta(a_2 z, \theta) - (1+\varepsilon)\,n\delta(a_1 z, \theta) + m\cos\theta > 0.$$

By $\beta > 0$ and (3.27), we have $\sigma_2(f) \geqslant 1$, then $\sigma_2(f) = 1$. Concluding the above proof, we obtain $\sigma_2(f) = 1$. The proof of Theorem 1.1 is complete.

Example 1.1 Consider the differential equation

$$f'' + \left(-4e^{-3z} - 4ie^{-z} - 1\right) f' + \left(ie^{z} + 2e^{-z}\right)^2 f = 0, \tag{3.28}$$

where $Q(z) = -4z^3 - 4iz - 1$, $a_1 = 1$, $a_2 = -1$, $A_1(z) = i$ and $A_2(z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.28).

Example 1.2 Consider the differential equation

$$f'' + \left(-8e^{-2z} - 12e^{i\frac{\pi}{3}} e^{-z} - 1 - 6e^{i\frac{2\pi}{3}}\right) f' + \left(e^{i\frac{\pi}{3}} e^{\frac{2}{3}z} + 2e^{-\frac{1}{3}z}\right)^3 f = 0, \tag{3.29}$$

where $Q(z) = -8z^2 - 12e^{i\frac{\pi}{3}} z - 1 - 6e^{i\frac{2\pi}{3}}$, $a_1 = \frac{2}{3}$, $a_2 = -\frac{1}{3}$, $A_1(z) = e^{i\frac{\pi}{3}}$ and $A_2(z) = 2$. Obviously, the conditions of Theorem 1.1 (1) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.29).

Example 1.3 Consider the differential equation

$$f'' + \left(-e^{-3z} - 4e^{i\frac{\pi}{4}} e^{-2z} - 6ie^{-z} - 1 - 4e^{i\frac{3\pi}{4}}\right) f' + \left(e^{-\frac{1}{2}z} + e^{i\frac{\pi}{4}} e^{\frac{1}{2}z}\right)^4 f = 0, \tag{3.30}$$

where $Q(z) = -z^3 - 4e^{i\frac{\pi}{4}} z^2 - 6iz - 1 - 4e^{i\frac{3\pi}{4}}$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{2}$, $A_1(z) = 1$ and $A_2(z) = e^{i\frac{\pi}{4}}$. Obviously, the conditions of Theorem 1.1 (3) are satisfied. The entire function $f(z) = e^{e^z}$, with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$, is a solution of (3.30).

4 Proof of Theorem 1.2

We prove that $\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \sigma(f) = +\infty$ and $\overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \sigma_2(f) = 1$. First, setting $\omega = f - \varphi$. Since $\sigma(\varphi) < \infty$, then we have $\sigma(\omega) = \sigma(f) = +\infty$. From (1.1), we have

$$\omega'' + Q\left(e^{-z}\right)\omega' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)^n \omega = H, \tag{4.1}$$

where $H = -\left[\varphi'' + Q\left(e^{-z}\right)\varphi' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)^n \varphi\right]$. Now we prove that $H \not\equiv 0$. In fact if $H \equiv 0$, then

$$\varphi'' + Q\left(e^{-z}\right)\varphi' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)^n \varphi = 0. \tag{4.2}$$

Hence φ is a solution of equation (1.1) with $\sigma(\varphi) = \infty$ and by Theorem 1.1, it is a contradiction. Since $\sigma(f) = \infty$, $\sigma(\varphi) < \infty$ and $\sigma_2(f) = 1$, we get $\sigma_2(\omega) = \sigma_2(f-\varphi) = \sigma_2(f) = 1$. By the Lemma 2.6 and Lemma 2.7, we have $\overline{\lambda}(\omega) = \lambda(\omega) = \sigma(\omega) = \sigma(f) = +\infty$ and $\overline{\lambda}_2(\omega) = \lambda_2(\omega) = \sigma_2(\omega) = \sigma_2(f) = 1$, i.e., $\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \sigma(f) = +\infty$ and $\overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \sigma_2(f) = 1$.

5 Proof of Theorem 1.3

Suppose that $f \not\equiv 0$ is a solution of equation (1.1), then $\sigma(f) = +\infty$ by Theorem 1.1. Since $\sigma(\varphi) < 1$, then by Theorem 1.2, we have $\overline{\lambda}(f-\varphi) = +\infty$. Now we prove that $\overline{\lambda}(f'-\varphi) = \infty$. Set $g_1(z) = f'(z) - \varphi(z)$, then $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$. Set $B(z) = Q\left(e^{-z}\right)$ and $R(z) = A_1e^{a_1z} + A_2e^{a_2z}$, then $B'(z) = -e^{-z}Q'\left(e^{-z}\right)$ and $R' = \left(A_1' + a_1A_1\right)e^{a_1z} + \left(A_2' + a_2A_2\right)e^{a_2z}$. Differentiating both sides of equation (1.1), we have

$$f''' + Bf'' + \left(B' + R^n\right)f' + nR'R^{n-1}f = 0. \tag{5.1}$$

By (1.1), we have

$$f = -\frac{1}{R^n}\left(f'' + Bf'\right). \tag{5.2}$$

Substituting (5.2) into (5.1), we have

$$f''' + \left(B - n\frac{R'}{R}\right)f'' + \left(B' + R^n - nB\frac{R'}{R}\right)f' = 0. \tag{5.3}$$

Substituting $f' = g_1 + \varphi$, $f'' = g_1' + \varphi'$, $f''' = g_1'' + \varphi''$ into (5.3), we get

$$g_1'' + E_1g_1' + E_0g_1 = E, \tag{5.4}$$

where

$$E_1 = B - n\frac{R'}{R}, \quad E_0 = B' + R^n - nB\frac{R'}{R},$$

$$E = -\left\{\varphi'' + \left(B - n\frac{R'}{R}\right)\varphi' + \left(B' + R^n - nB\frac{R'}{R}\right)\varphi\right\}.$$

Now we prove that $E \not\equiv 0$. In fact, if $E \equiv 0$, then we get

$$\frac{\varphi''}{\varphi}R + \frac{\varphi'}{\varphi}\left(BR - nR'\right) + B'R - nBR' + R^{n+1} = 0. \tag{5.5}$$

Obviously $\frac{\varphi''}{\varphi}$, $\frac{\varphi'}{\varphi}$ are meromorphic functions with $\sigma\left(\frac{\varphi''}{\varphi}\right) < 1$, $\sigma\left(\frac{\varphi'}{\varphi}\right) < 1$. We can rewrite (5.5) in the form

$$\sum_{k=0}^{m} f_k e^{(a_1-k)z} + \sum_{l=0}^{m} h_l e^{(a_2-l)z} + \sum_{p=1}^{n} C_{n+1}^{p} A_1^{n+1-p} A_2^{p} e^{[(n+1-p)a_1+pa_2]z}$$

$$+A_1^{n+1}e^{(n+1)a_1 z} + A_2^{n+1}e^{(n+1)a_2 z} = 0, \tag{5.6}$$

where f_k $(k = 0, 1, \cdots, m)$ and h_l $(l = 0, 1, \cdots, m)$ are meromorphic functions with $\sigma(f_k) < 1$ and $\sigma(f_l) < 1$. Set $I = \{a_1 - k\ (k = 0, 1, \cdots, m),\ a_2 - l\ (l = 0, 1, \cdots, m),\ (n+1-p)\,a_1 + pa_2\ (p = 1, 2, \cdots, n),\ (n+1)\,a_1,\ (n+1)\,a_2\}$. By the conditions of the Theorem 1.1, it is clear that $(n+1)\,a_1 \neq a_1, (n+1)\,a_2$, $(n+1-p)\,a_1 + pa_2$ $(p = 1, 2, \cdots, n)$.

(i) If $(n+1)\,a_1 \neq a_1 - k$ $(k = 1, \cdots, m)$, $a_2 - l$ $(l = 0, 1, \cdots, m)$, then we write (5.6) in the form

$$A_1^{n+1}e^{(n+1)a_1 z} + \sum_{\beta\in\Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq I \setminus \{(n+1)\,a_1\}$. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, it is a contradiction.

(ii) If $(n+1)\,a_1 = \gamma$ such that $\gamma \in \{a_1 - k\ (k = 1, \cdots, m),\ a_2 - l\ (l = 0, 1, \cdots, m)\}$, then $(n+1)\,a_2 \neq \beta$ for all $\beta \in I \setminus \{(n+1)\,a_2\}$. Hence, we write (5.6) in the form

$$A_2^{n+1}e^{(n+1)a_2 z} + \sum_{\beta\in\Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq I \setminus \{(n+1)\,a_2\}$. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $E \not\equiv 0$ is proved. We know that the functions E_1, E_0 and E are of finite order. By Lemma 2.6 and (5.4), we have $\overline{\lambda}(g_1) = \overline{\lambda}(f' - \varphi) = \infty$.

Now we prove that $\overline{\lambda}\left(f''-\varphi\right)=\infty$. Set $g_2\left(z\right)=f''\left(z\right)-\varphi\left(z\right)$, then $\sigma\left(g_2\right)=\sigma\left(f''\right)=\sigma\left(f\right)=\infty$. Differentiating both sides of equation (1.1), we have

$$f^{(4)}+Bf'''+\left(2B'+R^n\right)f''+\left(B''+2nR'R^{n-1}\right)f'$$
$$+n\left[R''R^{n-1}+\left(n-1\right)R'^2R^{n-2}\right]f=0. \tag{5.7}$$

Combining (5.2) with (5.7), we get

$$f^{(4)}+Bf'''+\left(2B'+R^n-n\frac{R''}{R}-n\left(n-1\right)\frac{R'^2}{R^2}\right)f''$$
$$+\left(B''+2nR'R^{n-1}-nB\frac{R''}{R}-n\left(n-1\right)B\frac{R'^2}{R^2}\right)f'=0. \tag{5.8}$$

Now we prove that $B'+R^n-nB\frac{R'}{R}\not\equiv 0$. Suppose that $B'+R^n-nB\frac{R'}{R}\equiv 0$, then we have

$$B'R+R^{n+1}-nBR'=0. \tag{5.9}$$

We can write (5.9) in the form (5.6), then by the same reasoning as in the proof of $\overline{\lambda}\left(f'-\varphi\right)=\infty$ we get a contradiction. Hence $B'+R^n-nB\frac{R'}{R}\not\equiv 0$ is proved. Set

$$\psi\left(z\right)=B'R+R^{n+1}-nBR', \tag{5.10}$$

$$S_1=2B'R^2+R^{n+2}-nR''R-n\left(n-1\right)R'^2, \tag{5.11}$$

$$S_2=B''R^2+2nR'R^{n+1}-nBR''R-n\left(n-1\right)BR'^2, \tag{5.12}$$

$$S_3=BR-nR'. \tag{5.13}$$

By (5.3), (5.10) and (5.13), we get

$$f'=-\frac{R}{\psi\left(z\right)}\left(f'''+\frac{S_3}{R}f''\right). \tag{5.14}$$

By (5.14), (5.11), (5.12) and (5.8), we obtain

$$f^{(4)}+\left(B-\frac{S_2}{R\psi\left(z\right)}\right)f'''+\left(\frac{S_1}{R^2}-\frac{S_2S_3}{R^2\psi\left(z\right)}\right)f''=0. \tag{5.15}$$

Substituting $f''=g_2+\varphi$, $f'''=g_2'+\varphi'$, $f^{(4)}=g_2''+\varphi''$ into (5.15) we get

$$g_2''+H_1g_2'+H_0g_2=H, \tag{5.16}$$

where

$$H_1=B-\frac{S_2}{R\psi\left(z\right)},\quad H_0=\frac{S_1}{R^2}-\frac{S_2S_3}{R^2\psi\left(z\right)},$$

$$-H = \varphi'' + \varphi' H_1 + \varphi H_0.$$

We can get

$$H_1 = \frac{L_1(z)}{R\psi(z)},\ H_0 = \frac{L_0(z)}{R\psi(z)}, \tag{5.17}$$

where

$$L_1(z) = B'BR^2 + BR^{n+2} - nB^2R'R - B''R^2 - 2nR'R^{n+1}$$
$$+nBR''R + n(n-1)BR'^2, \tag{5.18}$$
$$L_0(z) = 2B'^2R^2 + 3B'R^{n+2} - 2nB'BR'R + R^{2n+2} - 3nBR'R^{n+1}$$
$$-nB'R''R - nR''R^{n+1} - n(n-1)B'R'^2 + \left(n^2+n\right)R'^2R^n - B''BR^2$$
$$+nB^2R''R + n(n-1)B^2R'^2 + nB''R'R. \tag{5.19}$$

Therefore

$$\frac{-H}{\varphi} = \frac{1}{R\psi(z)}\left(\frac{\varphi''}{\varphi}R\psi(z) + \frac{\varphi'}{\varphi}L_1(z) + L_0(z)\right), \tag{5.20}$$

$$R\psi(z) = B'R^2 + R^{n+2} - nBR'R. \tag{5.21}$$

Now we prove that $-H \not\equiv 0$. In fact, if $-H \equiv 0$, then by (5.20) we have

$$\frac{\varphi''}{\varphi}R\psi(z) + \frac{\varphi'}{\varphi}L_1(z) + L_0(z) = 0. \tag{5.22}$$

Obviously, $\frac{\varphi''}{\varphi}$ and $\frac{\varphi'}{\varphi}$ are meromorphic functions with $\sigma\left(\frac{\varphi''}{\varphi}\right) < 1$, $\sigma\left(\frac{\varphi'}{\varphi}\right) < 1$. By (5.18), (5.19) and (5.21), we can rewrite (5.22) in the form

$$A_1^{2n+2}e^{(2n+2)a_1z} + A_2^{2n+2}e^{(2n+2)a_2z} + \sum_{p=1}^{2n+1} C_{2n+2}^{p}A_1^{2n+2-p}A_2^{p}e^{[(2n+2-p)a_1+pa_2]z}$$
$$+\sum_{\substack{0\leqslant p\leqslant 2\\ 0\leqslant k\leqslant 2m}} f_{p,k}e^{[(2-p)a_1+pa_2-k]z} + \sum_{\substack{0\leqslant p\leqslant n+2\\ 0\leqslant k\leqslant m}} h_{p,k}e^{[(n+2-p)a_1+pa_2-k]z} = 0, \tag{5.23}$$

where $f_{p,k}$ $(0 \leqslant p \leqslant 2, 0 \leqslant k \leqslant 2m)$ and $h_{p,k}$ $(0 \leqslant p \leqslant n+2, 0 \leqslant k \leqslant m)$ are meromorphic functions with $\sigma(f_{p,k}) < 1$ and $\sigma(h_{p,k}) < 1$. Set $J = \{(2n+2)a_1,\ (2n+2)a_2,\ (2n+2-p)a_1 + pa_2\ (p = 1,2,\cdots,2n+1),$ $(2-p)a_1 + pa_2 - k\ (p = 0,1,2;\ k = 0,\cdots,2m),\ (n+2-p)a_1 + pa_2 - k$ $(p = 0,1,\cdots,n+2;\ k = 0,1,\cdots,m)\}$. By the conditions of Theorem 1.3, it is clear that $(2n+2)a_1 \neq (2n+2)a_2$, $(2n+2-p)a_1 + pa_2$ $(p = 1,2,\cdots,2n+1)$,

$2a_1$, $(n+2)\,a_1$ and $(2n+2)\,a_2 \neq (2n+2)\,a_1$, $(2n+2-p)\,a_1 + pa_2$ $(p = 1, 2, \cdots, 2n+1)$, $2a_2$, $(n+2)\,a_2$.

(1) By the conditions of Theorem 1.3 (i), we have $(2n+2)\,a_1 \neq \beta$ for all $\beta \in J \setminus \{(2n+2)\,a_1\}$, hence we write (5.23) in the form

$$A_1^{2n+2} e^{(2n+2)a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq J \setminus \{(2n+2)\,a_1\}$. By Lemma 2.8 and Lemma 2.9, we get $A_1 \equiv 0$, it is a contradiction.

(2) By the conditions of Theorem 1.3 (ii), we have $(2n+2)\,a_2 \neq \beta$ for all $\beta \in J \setminus \{(2n+2)\,a_2\}$, hence we write (5.23) in the form

$$A_2^{2n+2} e^{(2n+2)a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq J \setminus \{(2n+2)\,a_2\}$. By Lemma 2.8 and Lemma 2.9, we get $A_2 \equiv 0$, it is a contradiction. Hence, $H \not\equiv 0$ is proved. We know that the functions H_1, H_0 and H are of finite order. By Lemma 2.6 and (5.16), we have $\overline{\lambda}\,(g_2) = \overline{\lambda}\,(f'' - \varphi) = \infty$. The proof of Theorem 1.3 is complete.

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