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λ-statistical convergence in n-normed spaces

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Abstract

In this paper, we introduce the concept of λ-statistical convergence in n-normed spaces. Some inclusion relations between the sets of statistically convergent and λ-statistically convergent sequences are established. We find its relations to statistical convergence, $(C,1)$-summability and strong (V,λ)-summability in n-normed spaces.

1 Introduction

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [28] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Fridy [10], Šalát [26]), topological groups (Çakalli [1], [2]), topological spaces (Di Maio and Kočinac[20]), function spaces (Caserta and Kočinac [3], Caserta, Di Maio and Kočinac [4]), locally convex spaces (Maddox[19]), measure theory (Cheng et al., [5], Connor and Swardson [6], Millar[21]) , fuzzy mathematics (Nuray and Savaş [24], Savaş [27]). In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions [6]. Mursaleen [23], introduced the λ-statistical convergence for real sequences. In this article, we consider only sequences of real numbers, so

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that "a sequence" means "a sequence of real numbers".

The notion of statistical convergence depends on the (natural or asymptotic) density of subsets of $\mathbf{N}$. A subset of $\mathbf{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $(x_k) \xrightarrow{S} \ell$ and S denotes the set of all statistically convergent sequences.

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1.$$

The collection of such sequences λ will be denoted by Δ.

The generalized de la Vallée-Poussin mean is defined by

$$t_m(x) = \frac{1}{\lambda_m} \sum_{k\in I_m} x_k,$$

where $I_m = [m - \lambda_m + 1, m]$.

Definition 1.2.[17] A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number ℓ if

$$t_m(x) \to \ell, \quad \text{as} \quad m \to \infty.$$

If $\lambda_m = m$, then (V, λ)-summability reduces to $(C, 1)$-summability. We write

$$[C, \lambda] = \left\{ x = (x_k) : \exists\ \ell \in \mathbf{R}, \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_k - \ell| = 0 \right\}$$

and

$$[V, \lambda] = \left\{ x = (x_k) : \exists\ \ell \in \mathbf{R}, \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} |x_k - \ell| = 0 \right\}$$

for the sets of sequences $x = (x_k)$ which are *strongly Cesàro summable* (see [9]) and *strongly* (V, λ)-*summable* to ℓ, i.e. $(x_k) \xrightarrow{[C,1]} \ell$ and $(x_k) \xrightarrow{[V,\lambda]} \ell$, respectively.

Definition 1.3. [23] A sequence $x = (x_k)$ is said to be λ-satistically convergent or S_λ-convergent to ℓ if for every $\varepsilon > 0$

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : |x_k - \ell| \geq \varepsilon\}| = 0.$$

In this case we write $S_\lambda - \lim x = \ell$ or $(x_k) \xrightarrow{S_\lambda} \ell$ and

$$S_\lambda = \{x = (x_k) : \exists\ \ell \in \mathbf{R}, S_\lambda - \lim x = \ell\}.$$

It is clear that if $\lambda_m = m$, then S_λ is same as S.

The concept of 2-normed space was initially introduced by Gähler[12], in the mid of 1960's, while that of n-normed spaces can be found in Misiak [22]. Since then, many others authors have studied this concept and obtained various results (see, for instance, Gunawan[14] ,Gähler[11], Gunawan and Mashadi ([13], [15]), Lewandowska[18], Dutta [7]).

2 Definitions and Preliminaries

Let n be a non negative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $||., ..., .||$ from X^n into $\mathbf{R}$ satisfying the following conditions:

(1) $||x_1, x_2, ..., x_n|| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(2) $||x_1, x_2, ..., x_n||$ is invariant under permutation,
(3) $||\alpha x_1, x_2, ..., x_n|| = |\alpha|||x_1, x_2, ..., x_n||$, for any $\alpha \in \mathbf{R}$,
(4) $||x + \overline{x}, x_2, ..., x_n|| \leq ||x, x_2, ..., x_n|| + ||\overline{x}, x_2, ..., x_n||$
is called an n-norm on X and the pair $(X, ||., ..., .||)$ is called an n-normed space.

A trivial example of an n-normed space is $X = \mathbf{R}^n$, equipped with the Euclidean n-norm $||x_1, x_2, ..., x_n||_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, ..., x_n$ which may be given expicitly by the formula

$$||x_1, x_2, ..., x_n||_E = |\det(x_{ij})| = abs\left(\det(< x_i, x_j >)\right),$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in \mathbf{R}^n$ for each $i = 1, 2, 3..., n$.

Let $(X, ||., ..., .||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then the function $||., ..., .||_\infty$

from X^{n-1} into $\mathbf{R}$ defined by

$$||x_1, x_2, ..., x_{n-1}||_\infty = \max_{1 \leq i \leq n} \{||x_1, x_2, ..., x_{n-1}, a_i||\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, ...a_n\}$ and this is known as the derived $(n-1)$-norm (for details see [13]).

The standard n-norm on a real inner product space of dimension $d \geq n$ is as follows:

$$||x_1, x_2, ..., x_n||_S = [\det(< x_i, x_j >)]^{\frac{1}{2}},$$

where $<, >$ denotes the inner product on X. If we take $X = \mathbf{R}^n$ then this n-norm is exactly the same as the Euclidean n-norm $||x_1, x_2, ..., x_n||_E$ mentioned earlier. For $n = 1$ this n-norm is the usual norm $||x_1|| = \sqrt{< x_1, x_1 >}$(for further details see [13]).

Definition 2.1. A sequence (x_k) in an n-normed space $(X, ||., ..., .||)$ is said to be *convergent* to $\ell \in X$ with respect to the n-norm if for each $\varepsilon > 0$ there exists an positive integer n_0 such that $||x_k - \ell, z_1, z_2, ..., z_{n-1}|| < \varepsilon$, for all $k \geq n_0$ and for every $z_1, z_2, ..., z_{n-1} \in X$.

Definition 2.2. A sequence (x_k) in an n-normed space $(X, ||., ..., .||)$ is said to be *Cauchy* with respect to the n-norm if for each $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $||x_k - x_m, z_1, z_2, ..., z_{n-1}|| < \varepsilon$, for all $k, m \geq n_0$ and for every $z_1, z_2, ..., z_{n-1} \in X$.

If every Cauchy sequence in X converges to some $\ell \in X$, then X is said to be *complete* with respect to the n-norm. Any complete n-normed space is said to be an n-*Banach* space.

Definition 2.3. A sequence (x_k) in an n-normed space $(X, ||., ..., .||)$ is said to be *statistically-convergent* to some $\ell \in X$ with respect to the n-norm if for each $\varepsilon > 0$ the set $\{k \in \mathbf{N} : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}$ has natural density zero, for every $z_1, z_2, ..., z_{n-1} \in X$.

In other words the sequence (x_k) *statistical converges* to ℓ an n-normed space X if

$$\lim_{m \to \infty} \frac{1}{m} |\{k \in \mathbf{N} : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}| = 0,$$

for each $z_1, z_2, ..., z_{n-1} \in X$. Let $S^{nN}(X)$ denotes the set of all statistically convergent sequences in n-normed space X.

Recently, Gürdal and Pehlivan [16] studied statistical convergence in 2-normed spaces. B.S. Reddy [25] extended this idea to n-normed space and studied some properties.

In the present paper we study λ-statistical convergence in n-normed spaces. We show that some properties of λ-statistical convergence of real numbers also hold for sequences in n-normed spaces. We find some relations related to statistical convergent, λ-statistical convergent sequences, $(C,1)$-summability and strong (V,λ)-summability in n-normed spaces.

3 λ-statistical convergent sequences in n-normed space X

In this section we define λ-statistically convergent sequences in n-normed linear space X. Also, we obtained some basic properties of this notion in n-normed spaces.

Definition 3.1. A sequence $x = (x_k)$ in an n-normed space $(X, ||., ..., .||)$ is said to be λ-satistically convergent or S_λ-convergent to $\ell \in X$ with respect to the n-norm if for every $\varepsilon > 0$

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}| = 0,$$

for each $z_1, z_2, ..., z_{n-1} \in X$. In this case we write $S_\lambda^{nN} - \lim x = \ell$ or $(x_k) \xrightarrow{S_\lambda^{nN}} \ell$ and

$$S_\lambda^{nN}(X) = \{x = (x_k) : \exists\ \ell \in \mathbf{R}, S_\lambda^{nN} - \lim x = \ell\}.$$

Let $S_\lambda^{nN}(X)$ denotes the set of all λ-statistically convergent sequences in the n-normed space X.

Definition 3.2. A sequence $x = (x_k)$ in an n-normed space $(X, ||., ..., .||)$ is said to be (V, λ)-summable to $\ell \in X$ with respect to the n-norm if

$$t_m(x) \to \ell, \quad \text{as} \quad m \to \infty.$$

If $\lambda_m = m$, then (V, λ)-summability reduces to $(C, 1)$-summability with respect to the n-norm. We write

$$[C, \lambda]^{nN}(X) = \left\{ x = (x_k) : \exists\ \ell \in \mathbf{R}, \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} ||x_k - \ell, z_1, ..., z_{n-1}|| = 0 \right\}$$

and

$$[V,\lambda]^{nN}(X)=\left\{x=(x_k):\exists\ \ell\in\mathbf{R},\ \lim_{m\to\infty}\frac{1}{\lambda_m}\sum_{k\in I_m}||x_k-\ell,z_1,...,z_{n-1}||=0\right\}$$

for the sets of X-valued sequences $x=(x_k)$ which are *strongly Cesàro summable* and *strongly (V,λ)-summable* to ℓ with respect to the n-norm, i.e., $(x_k)\overset{[C,1]^{nN}}{\longrightarrow}\ell$ and $(x_k)\overset{[V,\lambda]^{nN}}{\longrightarrow}\ell$, respectively.

Theorem 3.1. *Let X be an $n-$normed space and $\lambda=(\lambda_n)\in\Delta$. If (x_k) is a sequence in X such that $S_\lambda^{nN}-\lim x_k=\ell$ exists, then it is unique.*

Proof. Suppose that there exist elements ℓ_1,ℓ_2 $(\ell_1\neq\ell_2)$ in X such that

$$S_\lambda^{nN}-\lim_{k\to\infty}x_k=\ell_1;S_\lambda^{nN}-\lim_{k\to\infty}x_k=\ell_2.$$

Since $\ell_1\neq\ell_2$, then $\ell_1-\ell_2\neq 0$, so there exist $z_1,z_2,...,z_{n-1}\in X$ such that $\ell_1-\ell_2$ and $z_1,z_2,...,z_{n-1}$ are linearly independent. Therefore,

$$||\ell_1-\ell_2,z_1,z_2,...z_{n-1}||=2\varepsilon>0.$$

Since $S_\lambda^{nN}-\lim_{k\to\infty}x_k=\ell_1$ and $S_\lambda^{nN}-\lim_{k\to\infty}x_k=\ell_2$ it follows that

$$\lim_{m\to\infty}\frac{1}{\lambda_m}|\{k\in I_m:||x_k-\ell_1,z_1,z_1,...z_{n-1}||\geq\varepsilon\}|=0$$

and

$$\lim_{m\to\infty}\frac{1}{\lambda_m}|\{k\in I_m:||x_k-\ell_2,z_1,z_2,...z_{n-1}||\geq\varepsilon\}|=0.$$

There is $k\in I_m$ such that

$$||x_k-\ell_1,z_1,z_1,...z_{n-1}||<\varepsilon \text{ and } ||x_k-\ell_2,z_1,z_1,...z_{n-1}||<\varepsilon.$$

Further, for this k we have

$$||\ell_1-\ell_2,z_1,z_1,...z_{n-1}||\leq||x_k-\ell_1,z_1,z_1,...z_{n-1}||+||x_k-\ell_2,z_1,z_1,...z_{n-1}||<2\varepsilon$$

which is a contradiction. This completes the proof.

The next theorem gives the algebraic characterization of λ-statistical convergence on n-normed spaces.

Theorem 3.2. *Let X be an n-normed space, $\lambda = (\lambda_n) \in \Delta$, $x = (x_k)$ and $y = (y_k)$ be two sequences in X.*
(a) *If $S_\lambda^{nN} - \lim_{k\to\infty} x_k = \ell$ and $c(\neq 0) \in \mathbf{R}$, then $S_\lambda^{nN} - \lim_{k\to\infty} cx_k = c\ell$.*
(b) *If $S_\lambda^{nN} - \lim_{k\to\infty} x_k = \ell_1$ and $S_\lambda^{nN} - \lim_{k\to\infty} y_k = \ell_2$, then $S_\lambda^{nN} - \lim_{k\to\infty}(x_k + y_k) = \ell_1 + \ell_2$.*

Proof of the theorem is straightforward, thus omitted.

Theorem 3.3. *$S_\lambda^{nN}(X) \cap \ell_\infty(X)$ is a closed subset of $\ell_\infty(X)$, if X is an n-Banach space.*

Proof. Suppose that $(x^i)_{i\in\mathbf{N}}$, $x^i = (x_k^i)_{k\in\mathbf{N}}$, is a convergent sequence in $S_\lambda^{nN}(X) \cap \ell_\infty(X)$ converging to $x = (x_k) \in \ell_\infty(X)$. We need to prove that $x \in S_\lambda^{nN}(X) \cap \ell_\infty(X)$. Assume that $(x_k^i)_k \overset{S_\lambda^{nN}}{\to} \ell_i$, for all $i \in \mathbf{N}$. Take a positive decreasing convergent sequence $(\varepsilon_i)_{i\in\mathbf{N}}$, where $\varepsilon_i = \frac{\varepsilon}{2^i}$, for a given $\varepsilon > 0$. Clearly $(\varepsilon_i)_{i\in\mathbf{N}}$ converges to 0. Choose a positive integer i such that $||x - x^i, z_1, z_2, ..., z_{n-1}||_\infty < \frac{\varepsilon_i}{4}$, for every $z_1, z_2, ..., z_{n-1} \in X$. Then we have

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_i}{4}\}| = 0$$

and

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\}| = 0.$$

Since,

$$\frac{1}{\lambda_m}\Big|\Big\{k \in I_m \quad : \quad ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_i}{4} \quad \vee \\ ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\Big\}\Big| < 1$$

and for $m \in \mathbf{N}$

$$\Big\{k \in I_m : ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_i}{4}\Big\} \quad \cap \\ \Big\{k \in I_m : ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\Big\}$$

is infinite. Hence there must exists a $k \in I_m$ for which we have simultaneously,

$$||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| < \frac{\varepsilon_i}{4} \text{ and } ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ..., z_{n-1}|| < \frac{\varepsilon_{i+1}}{4}.$$

Then it follows that

$$||\ell_i - \ell_{i+1}, z_1, z_2, ..., z_{n-1}||$$
$$\leq ||\ell_i - x_k^i, z_1, z_2, ...z_{n-1}|| + ||x_k^i - x_k^{i+1}, z_1, z_2, ...z_{n-1}|| +$$
$$+ ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ...z_{n-1}||$$
$$\leq ||x_k^i - \ell_i, z_1, z_2, ...z_{n-1}|| + ||x_k^{i+1} - \ell_{i+1}, z_1, z_2, ...z_{n-1}||$$
$$+ ||x - x^i, z_1, z_2, ...z_{n-1}||_\infty + ||x - x^{i+1}, z_1, z_2, ...z_{n-1}||_\infty$$
$$< \frac{\varepsilon_i}{4} + \frac{\varepsilon_{i+1}}{4} + \frac{\varepsilon_i}{4} + \frac{\varepsilon_{i+1}}{4} < \varepsilon_i.$$

This implies that (ℓ_i) is a Cauchy sequence in X and there is an element $\ell \in X$ such that $\ell_i \to \ell$ as $i \to \infty$. We need to prove that $(x_k) \overset{S_\lambda^{nN}}{\longrightarrow} \ell$.
For any $\varepsilon > 0$, choose $i \in \mathbf{N}$ such that $\varepsilon_i < \frac{\varepsilon}{4}$,

$$||x_k - x_k^i, z_1, z_2, ..., z_{n-1}||_\infty < \frac{\varepsilon}{4}, ||\ell_i - \ell, z_1, z_2, ..., z_{n-1}|| < \frac{\varepsilon}{4}.$$

Then

$$\frac{1}{\lambda_m}|\{k \in I_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}|$$
$$\leq \frac{1}{\lambda_m}|\{k \in I_m : ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}||$$
$$+ ||x_k - x_k^i, z_1, z_2, ..., z_{n-1}||_\infty + ||\ell_i - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}|$$
$$\leq \frac{1}{\lambda_m}|\{k \in I_m : ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon\}|$$
$$\leq \frac{1}{\lambda_m}|\{k \in I_m : ||x_k^i - \ell_i, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon}{2}\}| \to 0 \text{ as } m \to \infty.$$

This gives that $(x_k) \overset{S_\lambda^{nN}}{\longrightarrow} \ell$, which completes the proof.

Theorem 3.4. *Let X be an n-normed space and let $\lambda = (\lambda_m) \in \Delta$. Then*
(i) $(x_k) \overset{[V,\lambda]^{nN}}{\longrightarrow} \ell \Rightarrow (x_k) \overset{S_\lambda^{nN}}{\longrightarrow} \ell$,
(ii) $[V, \lambda]^{nN}(X)$ *is a proper subset of* $S_\lambda^{nN}(X)$,
(iii) $x \in \ell_\infty(X)$ *and* $(x_k) \overset{S_\lambda^{nN}}{\longrightarrow} \ell$, *then* $(x_k) \overset{[V,\lambda]^{nN}}{\longrightarrow} \ell$ *and hence* $(x_k) \overset{[C,1]^{nN}}{\longrightarrow} \ell$, *provided $x = (x_k)$ is not eventually constant,*
(iv)$S_\lambda^{nN}(X) \cap \ell_\infty(X) = [V, \lambda]^{nN}(X) \cap \ell_\infty(X)$.

Proof. (i) If $\varepsilon > 0$ and $(x_k) \overset{[V,\lambda]^{nN}}{\longrightarrow} \ell$, we can write

$$\sum_{k\in I_m} ||x_k-\ell, z_1, z_2, ..., z_{n-1}|| \geq$$

$$\geq \sum_{k\in I_m, ||x_k-\ell, z_1, z_2, ..., z_{n-1}||\geq\varepsilon} ||x_k-\ell, z_1, z_2, ..., z_{n-1}|| \geq$$

$$\geq \varepsilon|\{k\in I_m : ||x_k-\ell, z_1, z_2, ...z_{n-1}|| \geq \varepsilon\}|$$

and so

$$\frac{1}{\varepsilon\lambda_m}\sum_{k\in I_m} ||x_k-\ell, z_1, z_2, ..., z_{n-1}|| \quad \geq$$

$$\frac{1}{\lambda_m}|\{k\in I_m : ||x_k-\ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}|.$$

This proves the result.

(ii) In order to establish that the inclusion $[V,\lambda]^{nN}(X) \subset S_\lambda^{nN}(X)$ is proper, we define a sequence $x=(x_k)$ by

$$x_k = \begin{cases} k, & \text{if } m-[\sqrt{\lambda_m}]+1 \leq k \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \notin \ell_\infty$ and for every $\varepsilon > 0 (0 < \varepsilon < 1)$,

$$\frac{1}{\lambda_m}\sum_{k\in I_m} ||x_k-0, z_1, z_2, ..., z_{n-1}|| \leq \frac{[\sqrt{\lambda_m}]}{\lambda_m} \to 0, \quad \text{as} \quad m\to\infty,$$

i.e. $(x_k) \xrightarrow{S_\lambda^{nN}} 0$. On the other hand,

$$\frac{1}{\lambda_m}|\{k\in I_m : ||x_k-0, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}| \to \infty, \quad \text{as} \quad m\to\infty,$$

i.e. (x_k) does not converge to 0 in $[V,\lambda]^{nN}(X)$.

(iii) Suppose that $(x_k) \xrightarrow{S_\lambda^{nN}} \ell$ and $(x_k) \in \ell_\infty(X)$. Then there exists a $M > 0$ such that $||x_k-\ell, z_1, z_2, ...z_{n-1}|| \leq M$ for all $k\in\mathbf{N}$. Given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_m}\sum_{k\in I_m} ||x_k-\ell, z_1, z_2, ..., z_{n-1}|| =$$

$$\frac{1}{\lambda_m}\sum_{k\in I_m, ||x_k-\ell, z_1, z_2, ..., z_{n-1}||\geq\frac{\varepsilon}{2}} ||x_k-\ell, z_1, z_2, ..., z_{n-1}||$$

$$+\frac{1}{\lambda_m}\sum_{k\in I_m, ||x_k-\ell,z_1,z_2,...,z_{n-1}||<\frac{\varepsilon}{2}} ||x_k-\ell,z_1,z_2,...,z_{n-1}||$$
$$\leq \frac{M}{\lambda_m}|\{k\in I_m : ||x_k-\ell,z_1,z_2,...,z_{n-1}|| \geq \frac{\varepsilon}{2}\}| + \frac{\varepsilon}{2},$$

This shows that $(x_k) \overset{[V,\lambda]^{nN}}{\longrightarrow} \ell$.

Again, we have

$$\frac{1}{m}\sum_{k=1}^{m} ||x_k-\ell,z_1,z_2,...,z_{n-1}||$$
$$\leq \frac{1}{m}\sum_{k=1}^{m-\lambda_m} ||x_k-\ell,z_1,z_2,...,z_{n-1}|| + \frac{1}{m}\sum_{k\in I_m} ||x_k-\ell,z_1,z_2,...,z_{n-1}||$$
$$\leq \frac{1}{\lambda_m}\sum_{k=1}^{m-\lambda_m} ||x_k-\ell,z_1,z_2,...,z_{n-1}|| + \frac{1}{\lambda_m}\sum_{k\in I_m} ||x_k-\ell,z_1,z_2,...,z_{n-1}||$$
$$\leq \frac{2}{\lambda_m}\sum_{k\in I_m} ||x_k-\ell,z_1,z_2,...,z_{n-1}||.$$

Hence $(x_k) \overset{[C,1]^{nN}}{\longrightarrow} \ell$, because $(x_k) \overset{[V,\lambda]^{nN}}{\longrightarrow} \ell$.

(iv) This is an immediate consequence of (i), (ii) and (iii).

Theorem 3.5. *Let X be an n-normed space and let $\lambda = (\lambda_m) \in \Delta$. Then $S^{nN}(X) \subset S_\lambda^{nN}(X)$ if and only if $\liminf_m \frac{\lambda_m}{m} > 0$.*

Proof. Suppose first that $\liminf_m \frac{\lambda_m}{m} > 0$. Then a given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{m}|\{k\leq m : ||x_k-\ell,z_1,z_2,...,z_{n-1}|| \geq \varepsilon\}| &\geq \\ \frac{1}{m}|\{k\in I_m : ||x_k-\ell,z_1,z_2,...,z_{n-1}|| \geq \varepsilon\}| &\geq \\ \frac{\lambda_m}{m}.\frac{1}{\lambda_m}|\{k\in I_m : ||x_k-\ell,z_1,z_2,...,z_{n-1}|| \geq \varepsilon\}|.\end{aligned}$$

It follows that $(x_k) \overset{S^{nN}}{\longrightarrow} \ell \Rightarrow (x_k) \overset{S_\lambda^{nN}}{\longrightarrow} \ell$. Hence $S^{nN}(X) \subset S_\lambda^{nN}(X)$.

Conversely, suppose that $\liminf_m \frac{\lambda_m}{m} = 0$. Then we can select a subsequence $(m(j))_{j=1}^{\infty}$ such that

$$\frac{\lambda_{m(j)}}{m(j)} < \frac{1}{j}.$$

We define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & \text{if } k \in I_{m(j)}, j = 1, 2, 3, ...; \\ 0, & \text{otherwise.} \end{cases}$$

Then x is statistically convergent, so $x \in S^{nN}(X)$. But $x \notin [V, \lambda]^{nN}(X)$. Theorem 3.4(iii) implies that $x \notin S_\lambda^{nN}(X)$. This completes the proof.

Theorem 3.6. *Let X be an n-normed space and let $\lambda = (\lambda_m) \in \Delta$ such that $\lim_m \frac{\lambda_m}{m} = 1$. Then $S_\lambda^{nN}(X) \subset S^{nN}(X)$.*

Proof. Since $\lim_m \frac{\lambda_m}{m} = 1$, then for $\varepsilon > 0$, we observe that

$$\begin{aligned} &\frac{1}{m}|\{k \le m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \ge \varepsilon\}| \\ &\le \frac{1}{m}|\{k \le m - \lambda_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \ge \varepsilon\}| \\ &\quad + \frac{1}{m}|\{k \in I_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \ge \varepsilon\}| \\ &\le \frac{m - \lambda_m}{m} + \frac{1}{m}|\{k \in I_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \ge \varepsilon\}| \\ &= \frac{m - \lambda_m}{m} + \frac{\lambda_m}{m}\frac{1}{\lambda_m}|\{k \in I_m : ||x_k - \ell, z_1, z_2, ..., z_{n-1}|| \ge \varepsilon\}|. \end{aligned}$$

This implies that (x_k) is statistically convergent, if (x_k) is λ-statistically convergent. Hence $S_\lambda^{nN}(X) \subset S^{nN}(X)$.

Remark: We do not know whether the condition $\lim_m \frac{\lambda_m}{m} = 1$ in the Theorem 3.6 is necessary and leave it as an open problem.

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