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Necessary and sufficient conditions for the boundedness of the anisotropic Riesz potential in anisotropic modified Morrey spaces

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Abstract

We prove that the anisotropic fractional maximal operator $M_{\alpha,\sigma}$ and the anisotropic Riesz potential operator $I_{\alpha,\sigma}$, $0<\alpha<|\sigma|$ are bounded from the anisotropic modified Morrey space $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ to the weak anisotropic modified Morrey space $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ if and only if, $\alpha/|\sigma|\le 1-1/q\le \alpha/(|\sigma|(1-b))$ and from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ if and only if, $\alpha/|\sigma|\le 1/p-1/q\le \alpha/((1-b)|\sigma|)$. In the limiting case $\frac{|\sigma|(1-b)}{\alpha}\le p\le \frac{|\sigma|}{\alpha}$ we prove that the operator $M_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$ and the modified anisotropic Riesz potential operator $\widetilde{I}_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $BMO_\sigma(\mathbb{R}^n)$.

1 Introduction

For $x\in\mathbb{R}^n$ and $t>0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t)=\mathbb{R}^n\backslash B(x,t)$. Let $0\le b\le 1$, $\sigma=(\sigma_1,\cdots,\sigma_n)$ with $\sigma_i>0$ for $i=1,\cdots,n$, $|\sigma|=\sigma_1+\cdots+\sigma_n$ and $t^\sigma x\equiv(t^{\sigma_1}x_1,\ldots,t^{\sigma_n}x_n)$ for $t>0$. For $x\in\mathbb{R}^n$ and $t>0$, let $E_\sigma(x,t)=\prod_{i=1}^n(x_i-t^{\sigma_i},x_i+t^{\sigma_i})$ denote the open parallelepiped centered at x of side length $2t^{\sigma_i}$ for $i=1,\cdots,n$.

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By [3, 10], the function $F(x,\rho)=\sum_{i=1}^{n} x_i^2\rho^{-2\sigma_i}$, considered for any fixed $x\in\mathbb{R}^n$, is a decreasing one with respect to $\rho>0$ and the equation $F(x,\rho)=1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. Define $\rho(x)=\rho$ and $\rho(0)=0$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x,\, y\in\mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ , defines a homogeneous metric space ([3, 5, 10]). Note that $\rho(x)$ is equivalent to $|x|_\sigma=\max\limits_{1\le i\le n}|x_i|^{\frac{1}{\sigma_i}}$.

One of the most important variants of the anisotropic maximal function is the so-called anisotropic fractional maximal function defined by the formula

$$M_{\alpha,\sigma}f(x)=\sup_{t>0}|E_\sigma(x,t)|^{-1+\alpha/|\sigma|}\int_{E_\sigma(x,t)}|f(y)|dy,\quad 0\le\alpha<|\sigma|,$$

where $|E_\sigma(x,t)|=2^n t^{|\sigma|}$ is the Lebesgue measure of the parallelepiped $E_\sigma(x,t)$.

It coincides with the anisotropic maximal function $M_\sigma f\equiv M_{0,\sigma}f$ and is intimately related to the anisotropic Riesz potential operator

$$I_{\alpha,\sigma}f(x)=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|_\sigma^{|\sigma|-\alpha}},\qquad 0<\alpha<|\sigma|.$$

If $\sigma=\mathbf{1}$, then $M_\alpha\equiv M_{\alpha,\mathbf{1}}$ and $I_\alpha\equiv I_{\alpha,\mathbf{1}}$ is the fractional maximal operator and Riesz potential, respectively. The operators M_α , $M_{\alpha,\sigma}$, I_α and $I_{\alpha,\sigma}$ play important role in real and harmonic analysis (see, for example [4] and [35]).

Definition 1.1. *Let $0\le b\le 1$, $1\le p<\infty$ and $[t]_1=\min\{1,t\}$. We denote by $L_{p,b,\sigma}(\mathbb{R}^n)$ anisotropic Morrey space, and by $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ the modified anisotropic Morrey space, the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,b,\sigma}}=\sup_{x\in\mathbb{R}^n,\,t>0}\left(t^{-b|\sigma|}\int_{E_\sigma(x,t)}|f(y)|^p dy\right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,b,\sigma}}=\sup_{x\in\mathbb{R}^n,\,t>0}\left([t]_1^{-b|\sigma|}\int_{E_\sigma(x,t)}|f(y)|^p dy\right)^{1/p}$$

respectively.

Remark 1.1. *Note that $L_{p,0,\sigma}=L_p(\mathbb{R}^n)$ and $L_{p,1,\sigma}=L_\infty(\mathbb{R}^n)$. If $b<0$ or $b>1$, then $L_{p,b,\sigma}=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. In the case $\sigma\equiv\mathbf{1}=(1,\dots,1)$ and $b=\frac{\lambda}{n}$ we get the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)=L_{p,\frac{\lambda}{n},\mathbf{1}}(\mathbb{R}^n)$, $0\le\lambda\le n$.*

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [25]). Later, Morrey spaces found important applications to Navier-Stokes ([22], [36]) and Schrödinger ([26], [28], [29], [31], [32]) equations, elliptic problems with discontinuous coefficients ([8], [11]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the book [20].

The modified Morrey space $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ firstly was defined and investigated by [19] (see also [4]).

Note that

$$\widetilde{L}_{p,0,\sigma}(\mathbb{R}^n) = L_{p,0,\sigma}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) \subset_{\succ} L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and}$$
$$\max\{\|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p}\} \leq \|f\|_{\widetilde{L}_{p,b,\sigma}} \tag{1.1}$$

and if $b < 0$ or $b > 1$, then $L_{p,b,\sigma}(\mathbb{R}^n) = \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) = \Theta$.

Definition 1.2. *[6] Let $1 \leq p < \infty$, $0 \leq b \leq 1$. We denote by $WL_{p,b,\sigma}(\mathbb{R}^n)$ the weak anisotropic Morrey space and by $W\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ the weak modified anisotropic Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norms*

$$\|f\|_{WL_{p,b,\sigma}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \Big(t^{-b|\sigma|} \; |\{y \in E_\sigma(x,t) : \; |f(y)| > r\}|\Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,b,\sigma}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \Big([t]_1^{-b|\sigma|} \; |\{y \in E_\sigma(x,t) : \; |f(y)| > r\}|\Big)^{1/p}$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0,\sigma}(\mathbb{R}^n) = W\widetilde{L}_{p,0,\sigma}(\mathbb{R}^n),$$
$$L_{p,b,\sigma}(\mathbb{R}^n) \subset WL_{p,b,\sigma}(\mathbb{R}^n) \;\text{ and }\; \|f\|_{WL_{p,b,\sigma}} \leq \|f\|_{L_{p,b,\sigma}},$$
$$\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) \subset W\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) \;\text{ and }\; \|f\|_{W\widetilde{L}_{p,b,\sigma}} \leq \|f\|_{\widetilde{L}_{p,b,\sigma}}.$$

The anisotropic result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then $I_{\alpha,\sigma}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma|\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, $I_{\alpha,\sigma}$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma|\left(1 - \frac{1}{q}\right)$. Spanne (see [33]) and Adams

[1] studied boundedness of the Riesz potential I_α for $0<\alpha<n$ in Morrey spaces $L_{p,\lambda}$. Later on Chiarenza and Frasca [9] was reproved boundedness of the Riesz potential I_α in these spaces. By more general results of Guliyev [13] (see also [14, 17]) one can obtain the following generalization of the results in [1, 9, 33] to the anisotropic case.

Theorem A. *Let* $0<\alpha<|\sigma|$ *and* $0\le b<1$ *,* $1\le p<\frac{(1-b)|\sigma|}{\alpha}$ *.*

1) If $1<p<\frac{(1-b)|\sigma|}{\alpha}$ *, then condition* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $I_{\alpha,\sigma}$ *from* $L_{p,b,\sigma}(\mathbb{R}^n)$ *to* $L_{q,b,\sigma}(\mathbb{R}^n)$ *.*

2) If $p=1$ *, then condition* $1-\frac{1}{q}=\frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $I_{\alpha,\sigma}$ *from* $L_{1,b,\sigma}(\mathbb{R}^n)$ *to* $WL_{q,b,\sigma}(\mathbb{R}^n)$ *.*

If $\alpha=(1-b)|\sigma|\big(\frac{1}{p}-\frac{1}{q}\big)$, then $b=0$ and the statement of Theorem A reduces to the aforementioned anisotropic result by Hardy-Littlewood-Sobolev.

Recall that, for $0<\alpha<|\sigma|$,

$$M_{\alpha,\sigma}f(x)\le 2^{n(\frac{\alpha}{|\sigma|}-1)}\,I_{\alpha,\sigma}(|f|)(x), \tag{1.2}$$

hence Theorem A also implies the boundedness of the fractional maximal operator $M_{\alpha,\sigma}$. It is known that the anisotropic maximal operator M_σ is also bounded from $L_{p,b,\sigma}$ to $L_{p,b,\sigma}$ for all $1<p<\infty$ and $0<b<1$, which isotropic case proved by F. Chiarenza and M. Frasca [9].

In this paper we study the fractional maximal integral and the Riesz potential in the modified Morrey space. In the case $p=1$ we prove that the operators $M_{\alpha,\sigma}$ and $I_{\alpha,\sigma}$ are bounded from $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ if and only if, $\alpha/|\sigma|\le 1-1/q\le\alpha/((1-b)|\sigma|)$. In the case $1<p<\frac{(1-b)|\sigma|}{\alpha}$ we prove that the operators $M_{\alpha,\sigma}$ and $I_{\alpha,\sigma}$ are bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ if and only if, $\alpha/|\sigma|\le 1/p-1/q\le\alpha/((1-b)|\sigma|)$. In the limiting case $\frac{|\sigma|(1-b)}{\alpha}\le p\le\frac{|\sigma|}{\alpha}$ we prove that the operator $M_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$ and the modified anisotropic Riesz potential operator $\widetilde{I}_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $BMO_\sigma(\mathbb{R}^n)$.

The structure of the paper is as follows. In section 2 the boundedness of the anisotropic maximal operator in anisotropic modified Morrey space $\widetilde{L}_{p,b,\sigma}$ is proved. The main result of the paper is the Hardy-Littlewood-Sobolev inequality in anisotropic modified Morrey space for the anisotropic Riesz potential, established in section 3. In section 4 we prove that the operator $\widetilde{I}_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $BMO_\sigma(\mathbb{R}^n)$ for $\frac{|\sigma|(1-b)}{\alpha}\le p\le\frac{|\sigma|}{\alpha}$.

By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2 $\widetilde{L}_{p,b,\sigma}$ -boundedness of the maximal operator

Define $f_{t^\sigma}(x) =: f(t^\sigma x)$ and $[t]_{1,+} = \max\{1,t\}$. Then

$$\begin{aligned} \|f_{t^\sigma}\|_{L_p} &= t^{-\frac{|\sigma|}{p}} \|f\|_{L_p}, \\ \|f_{t^\sigma}\|_{L_{p,b,\sigma}} &= t^{-\frac{|\sigma|}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} \left(r^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{\frac{(b-1)|\sigma|}{p}} \|f\|_{L_{p,b,\sigma}}, \end{aligned}$$

and

$$\begin{aligned} \|f_{t^\sigma}\|_{\widetilde{L}_{p,b,\sigma}} = & \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-b|\sigma|} \int_{E_\sigma(x,r)} |f_{t^\sigma}(y)|^p \, dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{b|\sigma|/p} \sup_{x\in\mathbb{R}^n,\, r>0} \left([tr]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |f(y)|^p \, dy \right)^{1/p} \\ &= t^{-\frac{|\sigma|}{p}} [t]_{1,+}^{\frac{b|\sigma|}{p}} \|f\|_{\widetilde{L}_{p,b,\sigma}} . \end{aligned} \tag{2.1}$$

In this section we study the $\widetilde{L}_{p,b,\sigma}$ -boundedness of the maximal operator M_σ .

Lemma 2.1. *Let* $1 \le p < \infty$, $0 \le b \le 1$. *Then*

$$\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) = L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{p,b,\sigma}} = \max\left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}.$$

Proof. Let $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$. Then from (1.1) we have that $f \in L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and $\max\left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\} \le \|f\|_{\widetilde{L}_{p,b,\sigma}}$.

Let now $f \in L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Then

$$\|f\|_{\widetilde{L}_{p,b,\sigma}} = \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p}$$
$$= \max\left\{ \sup_{x\in\mathbb{R}^n, 0<t\le 1} \left(t^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p}, \sup_{x\in\mathbb{R}^n, t>1} \left(\int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \right\}$$
$$\le \max\left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}.$$

Therefore, $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ and the embedding

$$L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \subset\!\!\succ \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$$

is valid.

Thus

$$\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) = L_{p,b,\sigma}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \text{ and}$$
$$\|f\|_{\widetilde{L}_{p,b,\sigma}} = \max\left\{ \|f\|_{L_{p,b,\sigma}}, \|f\|_{L_p} \right\}.$$

□

Analogously proved the following statement.

Lemma 2.2. *Let* $1 \le p < \infty$, $0 \le b \le 1$. *Then*

$$W\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n) = WL_{p,b,\sigma}(\mathbb{R}^n) \cap WL_p(\mathbb{R}^n)$$

and

$$\|f\|_{W\widetilde{L}_{p,b,\sigma}} = \max\left\{ \|f\|_{WL_{p,b,\sigma}}, \|f\|_{WL_p} \right\}.$$

To prove our main result in this section we need the following statement.

Theorem 2.1. *[23] 1. If* $f \in L_{1,b,\sigma}(\mathbb{R}^n)$, $0 \le b < 1$, *then* $M_\sigma f \in WL_{1,b,\sigma}(\mathbb{R}^n)$ *and*

$$\|M_\sigma f\|_{WL_{1,b,\sigma}} \le C_{b,\sigma} \|f\|_{L_{1,b,\sigma}},$$

where $C_{b,\sigma}$ *depends only on* n , b *and* σ .

2. If $f \in L_{p,b,\sigma}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le b < 1$, *then* $M_\sigma f \in L_{p,b,\sigma}(\mathbb{R}^n)$ *and*

$$\|M_\sigma f\|_{L_{p,b,\sigma}} \le C_{p,b,\sigma} \|f\|_{L_{p,b,\sigma}},$$

where $C_{p,b,\sigma}$ *depends only on* n , p , b *and* σ .

Our main theorem in this section is the following statement:

Theorem 2.2. *1. If* $f \in \widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$, $0 \le b < 1$, *then* $M_\sigma f \in W\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ *and*

$$\|M_\sigma f\|_{W\widetilde{L}_{1,b,\sigma}} \le \overline{C}_{1,b,\sigma} \|f\|_{\widetilde{L}_{1,b,\sigma}},$$

where $C_{1,b,\sigma}$ *depends only on* b *and* σ .

2. If $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le b < 1$, *then* $M_\sigma f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ *and*

$$\|M_\sigma f\|_{\widetilde{L}_{p,b,\sigma}} \le \overline{C}_{p,b,\sigma} \|f\|_{\widetilde{L}_{p,b,\sigma}},$$

where $C_{p,b,\sigma}$ *depends only on* p , b *and* σ .

Proof. It is obvious that (see Lemmas 2.1 and 2.2)

$$\|M_\sigma f\|_{\widetilde{L}_{p,b,\sigma}} = \max\left\{\|M_\sigma f\|_{L_{p,b,\sigma}}, \|M_\sigma f\|_{L_p}\right\}$$

for $1 < p < \infty$ and

$$\|M_\sigma f\|_{W\widetilde{L}_{1,b,\sigma}} = \max\left\{\|M_\sigma f\|_{WL_{1,b,\sigma}}, \|M_\sigma f\|_{WL_1}\right\}$$

for $p = 1$.

Let $1 < p < \infty$. By the boundedness of M_σ on $L_p(\mathbb{R}^n)$ and from Theorem 2.1 we have

$$\|M_\sigma f\|_{\widetilde{L}_{p,b,\sigma}} \le \max\{C_{p,\sigma}, C_{p,b,\sigma}\} \|f\|_{\widetilde{L}_{p,b,\sigma}} .$$

Let $p = 1$. By the boundedness of M_σ from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ and from Theorem 2.1 we have

$$\|M_\sigma f\|_{W\widetilde{L}_{1,b,\sigma}} \le \max\{C_{1,\sigma}, C_{1,b,\sigma}\} \|f\|_{\widetilde{L}_{1,b,\sigma}} .$$

□

3 Hardy-Littlewood-Sobolev inequality in modified Morrey spaces

The following Hardy-Littlewood-Sobolev inequality in modified Morrey spaces is valid.

Theorem 3.3. *Let* $0<\alpha<|\sigma|$, $0\le b<1-\frac{\alpha}{|\sigma|}$ *and* $1\le p<\frac{(1-b)|\sigma|}{\alpha}$.

1) If $1<p<\frac{(1-b)|\sigma|}{\alpha}$ *, then condition* $\frac{\alpha}{|\sigma|}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $I_{\alpha,\sigma}$ *from* $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$.

2) If $p=1<\frac{(1-b)|\sigma|}{\alpha}$ *, then condition* $\frac{\alpha}{|\sigma|}\le 1-\frac{1}{q}\le\frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $I_{\alpha,\sigma}$ *from* $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$.

Proof. 1) *Sufficiency.* Let $0<\alpha<|\sigma|$, $0<b<1-\frac{\alpha}{|\sigma|}$, $f\in\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ and $1<p<\frac{(1-b)|\sigma|}{\alpha}$. Then

$$I_{\alpha,\sigma}f(x)=\left(\int_{E_\sigma(x,t)}+\int_{\complement E_\sigma(x,t)}\right)f(y)|x-y|_\sigma^{\alpha-|\sigma|}dy\equiv A(x,t)+C(x,t).$$

For $A(x,t)$ we have

$$\begin{aligned}|A(x,t)|&\le\int_{E_\sigma(x,t)}|x-y|_\sigma^{\alpha-|\sigma|}\,|f(y)|\,dy\\&\le\sum_{j=1}^{\infty}\left(2^{-j}t\right)^{\alpha-|\sigma|}\int_{E_\sigma(x,2^{-j+1}t)\setminus E_\sigma(x,2^{-j}t)}|f(y)|\,dy.\end{aligned}$$

Hence
$$|A(x,t)|\lesssim t^\alpha Mf(x). \tag{3.1}$$

In the second integral by the Hölder's inequality we have

$$\begin{aligned}|C(x,t)|&\le\left(\int_{\complement E_\sigma(x,t)}|x-y|_\sigma^{-\beta}|f(y)|^p dy\right)^{1/p}\\&\times\left(\int_{\complement E_\sigma(x,t)}|x-y|_\sigma^{\left(\frac{\beta}{p}+\alpha-|\sigma|\right)p'}dy\right)^{1/p'}=J_1\cdot J_2.\end{aligned}$$

For J_2 we obtain

$$J_2\lesssim\left(\int_t^\infty r^{|\sigma|-1+\left(\frac{\beta}{p}+\alpha-|\sigma|\right)p'}dr\right)^{\frac{1}{p'}}\approx t^{\frac{\beta}{p}+\alpha-\frac{|\sigma|}{p}}, \tag{3.2}$$

where $\beta<|\sigma|-\alpha p$.

Let $b|\sigma| < \beta < |\sigma| - \alpha p$. For J_1 we get

$$
\begin{aligned}
J_1 &= \Big(\sum_{j=0}^{\infty} \int_{E_\sigma(x,2^{j+1}t)\setminus E_\sigma(x,2^j t)} |x-y|_\sigma^{-\beta} |f(y)|^p \, dy \Big)^{1/p} \\
&\le t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} \, \Big(\sum_{j=0}^{\infty} 2^{-\beta j} [2^{j+1} t]_1^{b|\sigma|} \Big)^{1/p} \\
&= t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} \begin{cases} \Big(2^{b|\sigma|} \, t^{b|\sigma|} \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(b|\sigma|-\beta)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-\beta j} \Big)^{1/p}, & 0 < t < \frac{1}{2}, \\ \Big(\sum\limits_{j=0}^{\infty} 2^{-\beta j} \Big)^{1/p}, & t \ge \frac{1}{2} \end{cases} \\
&\approx t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} \begin{cases} \Big(t^{b|\sigma|} + t^{\beta} \Big)^{1/p}, & 0 < t < \frac{1}{2}, \\ 1, & t \ge \frac{1}{2} \end{cases} \\
&\approx \|f\|_{\widetilde{L}_{p,b,\sigma}} \begin{cases} t^{\frac{b|\sigma|-\beta}{p}}, & 0 < t < \frac{1}{2}, \\ t^{-\frac{\beta}{p}}, & t \ge \frac{1}{2} \end{cases} \\
&= [2t]_1^{\frac{b|\sigma|}{p}} \; t^{-\frac{\beta}{p}} \; \|f\|_{\widetilde{L}_{p,b,\sigma}} .
\end{aligned}
\tag{3.3}
$$

From (3.2) and (3.3) we have

$$
|C(x,t)| \lesssim [t]_1^{\frac{b|\sigma|}{p}} \; t^{\alpha - \frac{|\sigma|}{p}} \; \|f\|_{\widetilde{L}_{p,b,\sigma}} . \tag{3.4}
$$

Thus for all $t > 0$ we get

$$
|I_{\alpha,\sigma} f(x)| \lesssim t^{\alpha} M_\sigma f(x) + [t]_1^{\frac{b|\sigma|}{p}} \; t^{\alpha - \frac{|\sigma|}{p}} \; \|f\|_{\widetilde{L}_{p,b,\sigma}}
$$

$$
\le \min \Big\{ t^{\alpha} M_\sigma f(x) + t^{\alpha - \frac{|\sigma|}{p}} \; \|f\|_{\widetilde{L}_{p,b,\sigma}} \, , \; t^{\alpha} M_\sigma f(x) + t^{\alpha - \frac{(1-b)|\sigma|}{p}} \; \|f\|_{\widetilde{L}_{p,b,\sigma}} \Big\} .
$$

Minimizing with respect to t , at

$$
t = \Big[(M_\sigma f(x))^{-1} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} \Big]^{p/((1-b)|\sigma|)}
$$

and

$$
t = \Big[(M_\sigma f(x))^{-1} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} \Big]^{p/|\sigma|}
$$

we get

$$
|I_{\alpha,\sigma} f(x)| \lesssim \min \left\{ \left(\frac{M_\sigma f(x)}{\|f\|_{\widetilde{L}_{p,b,\sigma}}} \right)^{1 - \frac{p\alpha}{(1-b)|\sigma|}} , \left(\frac{M_\sigma f(x)}{\|f\|_{\widetilde{L}_{p,b,\sigma}}} \right)^{1 - \frac{p\alpha}{|\sigma|}} \right\} \; \|f\|_{\widetilde{L}_{p,b,\sigma}} .
$$

Then

$$|I_{\alpha,\sigma}f(x)| \lesssim (M_\sigma f(x))^{p/q} \, \|f\|^{1-p/q}_{\widetilde{L}_{p,b,\sigma}} .$$

Hence, by Theorem 2.2, we have

$$\begin{aligned}\int_{E_\sigma(x,t)} |I_{\alpha,\sigma}f(y)|^q \, dy &\lesssim \|f\|^{q-p}_{\widetilde{L}_{p,b,\sigma}} \int_{E_\sigma(x,t)} (M_\sigma f(y))^p \, dy \\ &\lesssim [t]_1^{b|\sigma|} \, \|f\|^q_{\widetilde{L}_{p,b,\sigma}} ,\end{aligned}$$

which implies that $I_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$.

Necessity. Let $1 < p < \frac{(1-b)|\sigma|}{\alpha}$, $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ and $I_{\alpha,\sigma}$ bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$. Then from (2.1) we have

$$\|f_{t^\sigma}\|_{\widetilde{L}_{p,b,\sigma}} = t^{-\frac{|\sigma|}{p}} [t]_{1,+}^{\frac{b|\sigma|}{p}} \, \|f\|_{\widetilde{L}_{p,b,\sigma}} ,$$

and

$$I_{\alpha,\sigma} f_{t^\sigma}(x) = t^{-\alpha} I_{\alpha,\sigma} f(t^\sigma x), \tag{3.5}$$

$$\begin{aligned}\|I_{\alpha,\sigma} f_{t^\sigma}\|_{\widetilde{L}_{q,b,\sigma}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} \left([r]_1^{-b|\sigma|} \int_{E_\sigma(x,r)} |I_{\alpha,\sigma} f(t^\sigma y)|^q \, dy \right)^{1/q} \\ &= t^{-\alpha-\frac{|\sigma|}{q}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{b|\sigma|/q} \sup_{x\in\mathbb{R}^n,\, r>0} \left([tr]_1^{-b|\sigma|} \int_{E_\sigma(t^\sigma x, tr)} |I_{\alpha,\sigma} f(y)|^q \, dy \right)^{1/q} \\ &= t^{-\alpha-\frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \, \|I_{\alpha,\sigma} f\|_{\widetilde{L}_{q,b,\sigma}} .\end{aligned}$$

By the boundedness of $I_{\alpha,\sigma}$ from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$

$$\begin{aligned}\|I_{\alpha,\sigma} f\|_{\widetilde{L}_{q,b,\sigma}} &= t^{\alpha+\frac{|\sigma|}{q}} \; [t]_{1,+}^{-\frac{b|\sigma|}{q}} \; \|I_{\alpha,\sigma} f_{t^\sigma}\|_{\widetilde{L}_{q,b,\sigma}} \\ &\le t^{\alpha+\frac{|\sigma|}{q}} \; [t]_{1,+}^{-\frac{b|\sigma|}{q}} \; \|f_{t^\sigma}\|_{\widetilde{L}_{p,b,\sigma}} \\ &= t^{\alpha+\frac{|\sigma|}{q}-\frac{|\sigma|}{p}} [t]_{1,+}^{\frac{b|\sigma|}{p}-\frac{b|\sigma|}{q}} \|f\|_{\widetilde{L}_{p,b,\sigma}},\end{aligned}$$

where $C_{p,q,b,\sigma}$ depends only on p , q , b and σ .

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|\sigma|}$, then in the case $t \to 0$ we have $\|I_{\alpha,\sigma} f\|_{\widetilde{L}_{q,b,\sigma}} = 0$ for all $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{(1-b)|\sigma|}$, then at $t \to \infty$ we obtain $\|I_{\alpha,\sigma} f\|_{\widetilde{L}_{q,b,\sigma}} = 0$ for all $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{|\sigma|} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{(1-b)|\sigma|}$.

2) *Sufficiency.* Let $f \in \widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$. We have

$$\begin{aligned} &|\{y \in E_\sigma(x,t) \ : \ |I_{\alpha,\sigma} f(y)| > 2\beta\}| \\ &\le |\{y \in E_\sigma(x,t) \ : \ |A(y,t)| > \beta\}| \\ &+ |\{y \in E_\sigma(x,t) \ : \ |C(y,t)| > \beta\}| . \end{aligned}$$

Then

$$\begin{aligned} C(y,t) &= \sum_{j=0}^{\infty} \int_{E_\sigma(y,2^{j+1}t)\setminus E_\sigma(y,2^j t)} |f(z)||y-z|_\sigma^{\alpha-|\sigma|} dz \\ &\le t^{\alpha-|\sigma|} \, \|f\|_{\widetilde{L}_{1,b,\sigma}} \sum_{j=0}^{\infty} 2^{-(|\sigma|-\alpha)j} [2^{j+1}t]_1^{b|\sigma|} = t^{\alpha-|\sigma|} \, \|f\|_{\widetilde{L}_{1,b,\sigma}} \\ &\times \begin{cases} 2^{b|\sigma|}\, t^{b|\sigma|} \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(b|\sigma|-|\sigma|+\alpha)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-(|\sigma|-\alpha)j}, & 0<t<\frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{-(|\sigma|-\alpha)j}, & t \ge \frac{1}{2} \end{cases} \\ &\approx t^{\alpha-|\sigma|} \, \|f\|_{\widetilde{L}_{1,b,\sigma}} \begin{cases} t^{b|\sigma|} + t^{|\sigma|-\alpha}, & 0<t<\frac{1}{2}, \\ 1, & t \ge \frac{1}{2} \end{cases} \\ &\approx \|f\|_{\widetilde{L}_{1,b,\sigma}} \begin{cases} t^{b|\sigma|+\alpha-|\sigma|}, & 0<t<\frac{1}{2}, \\ t^{\alpha-|\sigma|}, & t \ge \frac{1}{2} \end{cases} \\ &= [2t]_1^{b|\sigma|} \, t^{\alpha-|\sigma|} \ \|f\|_{\widetilde{L}_{1,b,\sigma}} . \end{aligned}$$

Taking into account inequality (3.1) and Theorem 2.2, we have

$$\begin{aligned} &|\{y \in E_\sigma(x,t) \ : \ |A(y,t)| > \beta\}| \\ &\le \left| \left\{ y \in E_\sigma(x,t) \ : \ Mf(y) > \frac{\beta}{C_1 t^\alpha} \right\} \right| \\ &\le \frac{C_2 t^\alpha}{\beta} \cdot [t]_1^{b|\sigma|} \, \|f\|_{\widetilde{L}_{1,b,\sigma}} , \end{aligned}$$

where $C_2 = C_1 \cdot C_{1,b,\sigma}$ and thus if $C_2 [2t]_1^{b|\sigma|} \, t^{\alpha-|\sigma|} \ \|f\|_{\widetilde{L}_{1,b,\sigma}} = \beta$, then $|C(y,t)| \le \beta$ and consequently, $|\{y \in E_\sigma(x,t) \ : \ |C(y,t)| > \beta\}| = 0$.

Then

$$\begin{aligned} |\{y \in E_\sigma(x,t) \ : \ |I_{\alpha,\sigma} f(y)| > 2\beta\}| &\lesssim \frac{1}{\beta} [t]_1^{b|\sigma|} \, t^\alpha \ \|f\|_{\widetilde{L}_{1,b,\sigma}} \\ &\lesssim [t]_1^{b|\sigma|} \left(\frac{\|f\|_{\widetilde{L}_{1,b,\sigma}}}{\beta} \right)^{\frac{(1-b)|\sigma|}{(1-b)|\sigma|-\alpha}} , \ \text{if } 2t<1 \end{aligned}$$

and

$$|\{y \in E_\sigma(x,t) \ : \ |I_{\alpha,\sigma} f(y)| > 2\beta\}| \lesssim \frac{1}{\beta}\, [t]_1^{b|\sigma|}\, t^\alpha \, \|f\|_{\widetilde{L}_{1,b,\sigma}}$$

$$\lesssim [t]_1^{b|\sigma|} \left(\frac{\|f\|_{\widetilde{L}_{1,b,\sigma}}}{\beta}\right)^{\frac{|\sigma|}{|\sigma|-\alpha}}, \text{ if } 2t \geq 1.$$

Finally we have

$$\begin{aligned} & |\{y \in E_\sigma(x,t) \ : \ |I_{\alpha,\sigma} f(y)| > 2\beta\}| \\ & \lesssim [t]_1^{b|\sigma|} \min\left\{\left(\frac{\|f\|_{\widetilde{L}_{1,b,\sigma}}}{\beta}\right)^{\frac{(1-b)|\sigma|}{(1-b)|\sigma|-\alpha}}, \left(\frac{\|f\|_{\widetilde{L}_{1,b,\sigma}}}{\beta}\right)^{\frac{|\sigma|}{|\sigma|-\alpha}}\right\} \\ & \leq [t]_1^{b|\sigma|} \left(\frac{1}{\beta}\, \|f\|_{\widetilde{L}_{1,b,\sigma}}\right)^q . \end{aligned}$$

Necessity. Let $I_{\alpha,\sigma}$ is bounded from $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$. From (3.5) we have

$$\begin{aligned} \|I_{\alpha,\sigma} f_{t^\sigma}\|_{W\widetilde{L}_{q,b,\sigma}} &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([\tau]_1^{-b|\sigma|} \int_{\{y\in E_\sigma(x,\tau) \ : \ |I_{\alpha,\sigma} f_{t^\sigma}(y)|>r\}} dy\right)^{1/q} \\ &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([\tau]_1^{-b|\sigma|} \int_{\{y\in E_\sigma(x,\tau) \ : \ |I_{\alpha,\sigma} f(t^\sigma y)|>rt^\alpha\}} dy\right)^{1/q} \\ &= t^{-\alpha-\frac{|\sigma|}{q}} \sup_{\tau>0} \left(\frac{[t\tau]_1}{[\tau]_1}\right)^{b|\sigma|/q} \sup_{r>0} rt^\alpha \\ &\times \sup_{x\in\mathbb{R}^n,\, \tau>0} \left([t\tau]_1^{-b|\sigma|} \int_{\{y\in E_\sigma(t^\sigma x,t\tau) \ : \ |I_{\alpha,\sigma} f(y)|>rt^\alpha\}} dy\right)^{1/q} \\ &= t^{-\alpha-\frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \, \|I_{\alpha,\sigma} f\|_{W\widetilde{L}_{q,b,\sigma}} . \end{aligned}$$

By the boundedness of $I_{\alpha,\sigma}$ from $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ and from (2.1) we get

$$\begin{aligned} \|I_{\alpha,\sigma} f\|_{W\widetilde{L}_{q,b,\sigma}} &= t^{\alpha+\frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \, \|I_{\alpha,\sigma} f_{t^\sigma}\|_{W\widetilde{L}_{q,b,\sigma}} \\ &\lesssim t^{\alpha+\frac{|\sigma|}{q}} [t]_{1,+}^{-\frac{b|\sigma|}{q}} \, \|f_{t^\sigma}\|_{\widetilde{L}_{1,b,\sigma}} \\ &\lesssim t^{\alpha+\frac{|\sigma|}{q}-|\sigma|} [t]_{1,+}^{b|\sigma|-\frac{b|\sigma|}{q}} \, \|f\|_{\widetilde{L}_{1,b,\sigma}}. \end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{|\sigma|}$, then in the case $t \to 0$ we have $\|I_{\alpha,\sigma} f\|_{W\widetilde{L}_{q,b,\sigma}} = 0$ for all $f \in \widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{(1-b)|\sigma|}$, then for $t \to \infty$ we obtain $\|I_{\alpha,\sigma} f\|_{W\widetilde{L}_{q,b,\sigma}} = 0$ for all $f \in \widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$.

Therefore $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$. □

Corollary 3.1. *Let* $0 < \alpha < |\sigma|$, $0 \le b < 1 - \frac{\alpha}{|\sigma|}$ *and* $1 \le p \le \frac{|\sigma|}{\alpha}$.

1) If $1 < p < \frac{(1-b)|\sigma|}{\alpha}$ *, then condition* $\frac{\alpha}{|\sigma|} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $M_{\alpha,\sigma}$ *from* $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{(1-b)|\sigma|}{\alpha}$ *, then condition* $\frac{\alpha}{|\sigma|} \le 1 - \frac{1}{q} \le \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operator* $M_{\alpha,\sigma}$ *from* $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$.

3) If $\frac{(1-b)|\sigma|}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$ *, then the operator* $M_{\alpha,\sigma}$ *is bounded from* $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ *to* $L_\infty(\mathbb{R}^n)$.

Proof. Sufficiency of Corollary 3.1 follows from Theorem 3.3 and inequality (1.2).

Necessity. (1) Let $M_{\alpha,\sigma}$ be bounded from $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$ for $1 < p < \frac{(1-b)|\sigma|}{\alpha}$. Then we have

$$M_{\alpha,\sigma} f_{t^\sigma}(x) = t^{-\alpha} M_{\alpha,\sigma} f(t^\sigma x),$$

and

$$\|M_{\alpha,\sigma} f_{t^\sigma}\|_{\widetilde{L}_{q,b,\sigma}} = t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|M_{\alpha,\sigma} f\|_{\widetilde{L}_{q,b,\sigma}} .$$

By the same argument in Theorem 3.3 we obtain $\frac{\alpha}{|\sigma|} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{(1-b)|\sigma|}$.

(2) Let $M_{\alpha,\sigma}$ be bounded from $\widetilde{L}_{1,b,\sigma}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,b,\sigma}(\mathbb{R}^n)$. Then

$$\|M_{\alpha,\sigma} f_{t^\sigma}\|_{W\widetilde{L}_{q,b,\sigma}} = t^{-\alpha - \frac{|\sigma|}{q}} [t]_{1,+}^{\frac{b|\sigma|}{q}} \|M_{\alpha,\sigma} f\|_{W\widetilde{L}_{q,b,\sigma}} .$$

Hence we obtain $\frac{\alpha}{|\sigma|} \le 1 - \frac{1}{q} \le \frac{\alpha}{(1-b)|\sigma|}$.

(3) Let $\frac{(1-b)|\sigma|}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$. Then by the Hölder's inequality we have

$$\begin{aligned}
\|M_{\alpha,\sigma} f\|_{L_\infty} &= 2^{-n} \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-|\sigma|} \int_{E_\sigma(x,t)} |f(y)|\, dy \\
&\le 2^{-\frac{n}{p}} \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \left([t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p\, dy \right)^{1/p} \\
&\le 2^{-\frac{n}{p}} \|f\|_{\widetilde{L}_{p,b,\sigma}} \sup_{t>0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \\
&= 2^{-\frac{n}{p}} \|f\|_{\widetilde{L}_{p,b,\sigma}} \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{|\sigma|(1-b)}{p}}, \sup_{t>1} t^{\alpha-\frac{|\sigma|}{p}} \right\} \\
&= 2^{-\frac{n}{p}} \|f\|_{\widetilde{L}_{p,b,\sigma}} .
\end{aligned}$$

□

4 The modified anisotropic Riesz potential in the spaces $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$

The examples show that the anisotropic Riesz potential $I_{\alpha,\sigma}$ are not defined for all functions $f \in L_{p,b,\sigma}(\mathbb{R}^n)$, $0 \le b < 1 - \frac{\alpha}{|\sigma|}$, if $p \ge \frac{|\sigma|(1-b)}{\alpha}$, and $I_{\alpha,\sigma}$ are not defined for all functions $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$, $0 \le b < 1 - \frac{\alpha}{|\sigma|}$, if $p \ge \frac{|\sigma|(1-b)}{\alpha}$.

We consider the modified Riesz potential

$$\widetilde{I}_{\alpha,\sigma} f(x) = \int_{\mathbb{R}^n} \left(|x-y|_\sigma^{\alpha-|\sigma|} - |y|_\sigma^{\alpha-|\sigma|} \chi_{\complement E_\sigma(0,1)}(y) \right) f(y)\, dy.$$

Note that in the limiting case $\frac{|\sigma|(1-b)}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$ statement 1) in Theorem A does not hold. Moreover, there exists $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ such that $I_{\alpha,\sigma} f(x) = \infty$ for all $x \in \mathbb{R}^n$. However, as will be proved, statement 1) holds for the modified anisotropic Riesz potential $\widetilde{I}_{\alpha,\sigma}$ if the space $L_\infty(\mathbb{R}^n)$ is replaced by a wider space $BMO_\sigma(\mathbb{R}^n)$.

The following theorem is our main result in which we obtain conditions ensuring that the modified anisotropic Riesz potential $\widetilde{I}_{\alpha,\sigma}$ is bounded from the space $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ to $BMO_\sigma(\mathbb{R}^n)$.

Theorem 4.4. *Let* $0 < \alpha < |\sigma|$, $0 \le b < 1 - \frac{\alpha}{|\sigma|}$, *and* $\frac{|\sigma|(1-b)}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$, *then the operator* $\widetilde{I}_{\alpha,\sigma}$ *is bounded from* $\widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$ *to* $BMO_\sigma(\mathbb{R}^n)$.

Moreover, if the integral $I_{\alpha,\sigma} f$ *exists almost everywhere for* $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$, $\frac{|\sigma|(1-b)}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$ *, then* $I_{\alpha,\sigma} f \in BMO_\sigma(\mathbb{R}^n)$ *and the following inequality is valid*

$$\|I_{\alpha,\sigma} f\|_{BMO_\sigma} \le C \|f\|_{\widetilde{L}_{p,b,\sigma}},$$

where $C > 0$ *is independent of* f *.*

Proof. For given $t > 0$ we denote

$$f_1(x) = f(x)\chi_{E_\sigma(0,2t)}(y), \quad f_2(x) = f(x) - f_1(x), \tag{4.1}$$

where $\chi_{E_\sigma(0,2t)}$ is the characteristic function of the set $E_\sigma(0,2t)$. Then

$$\widetilde{I}_{\alpha,\sigma} f(x) = \widetilde{I}_{\alpha,\sigma} f_1(x) + \widetilde{I}_{\alpha,\sigma} f_2(x) = F_1(x) + F_2(x), \tag{4.2}$$

where

$$F_1(x) = \int_{E_\sigma(0,2t)} \Big(|x-y|_\sigma^{\alpha-|\sigma|} - |y|_\sigma^{\alpha-|\sigma|} \chi_{\complement_{E_\sigma(0,1)}}(y) \Big) f(y)\, dy,$$

$$F_2(x) = \int_{\complement_{E_\sigma(0,2t)}} \Big(|x-y|_\sigma^{\alpha-|\sigma|} - |y|_\sigma^{\alpha-|\sigma|} \chi_{\complement_{E_\sigma(0,1)}}(y) \Big) f(y)\, dy.$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{E_\sigma(0,2t)\setminus E_\sigma(0,\min\{1,2t\})} |y|_\sigma^{\alpha-|\sigma|} f(y)\, dy$$

is finite.

Note also that

$$\begin{aligned} F_1(x) - a_1 &= \int_{E_\sigma(0,2t)} |x-y|_\sigma^{\alpha-|\sigma|} f(y)\, dy \\ &- \int_{E_\sigma(0,2t)\setminus E_\sigma(0,\min\{1,2t\})} |y|_\sigma^{\alpha-|\sigma|} f(y)\, dy \\ &+ \int_{E_\sigma(0,2t)\setminus E_\sigma(0,\min\{1,2t\})} |y|_\sigma^{\alpha-|\sigma|} f(y)\, dy \\ &= \int_{\mathbb{R}^n} |x-y|_\sigma^{\alpha-|\sigma|} f_1(y)\, dy = I_{\alpha,\sigma} f_1(x). \end{aligned}$$

Therefore

$$|F_1(x) - a_1| \le \int_{\mathbb{R}^n} |y|_\sigma^{\alpha-|\sigma|} |f_1(x-y)|\, dy = \int_{E_\sigma(0,2t)} |y|_\sigma^{\alpha-|\sigma|} |f(x-y)|\, dy.$$

Then

$$\begin{aligned}
&|E_\sigma(x,t)|^{-1}\int_{E_\sigma(x,t)}|F_1(y)-a_1|\,dy\\
&=|E_\sigma(0,t)|^{-1}\int_{E_\sigma(0,t)}|F_1(x-y)-a_1|\,dy\\
&\le |E_\sigma(0,t)|^{-1}\int_{E_\sigma(0,t)}\left(\int_{E_\sigma(0,2t)}|z|_\sigma^{\alpha-|\sigma|}|f(x-y-z)|\,dz\right)dy\\
&=|E_\sigma(0,t)|^{-1}\int_{E_\sigma(0,2t)}\left(\int_{E_\sigma(0,t)}|f(x-y-z)|dy\right)|z|_\sigma^{\alpha-|\sigma|}\,dz\\
&\lesssim t^{-|\sigma|}t^{|\sigma|-\alpha}\|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}}\int_{E_\sigma(0,2t)}|z|_\sigma^{\alpha-|\sigma|}\,dz\\
&\approx \|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}}. \qquad (4.3)
\end{aligned}$$

Denote

$$a_2=\int_{E_\sigma(0,\max\{1,2t\})\setminus E_\sigma(0,2t)}|y|_\sigma^{\alpha-|\sigma|}f(y)\,dy.$$

If $2|x|_\sigma\le|y|_\sigma$, then

$$||x-y|_\sigma^{\alpha-|\sigma|}-|y|_\sigma^{\alpha-|\sigma|}|\le C|x|_\sigma|y|_\sigma^{\alpha-|\sigma|-1}.$$

By the Hölder's inequality we have

$$\begin{aligned}
|F_2(x)-a_2|&\le C|x|_\sigma\int_{\complement E_\sigma(0,2t)}|y|_\sigma^{\alpha-|\sigma|-1}\,|f(y)|\,dy\\
&\lesssim |x|_\sigma\sum_{j=0}^{\infty}\int_{2^{j+1}t\le|y|_\sigma\le 2^{j+2}t}|y|_\sigma^{\alpha-|\sigma|-1}\,|f(y)|\,dy\\
&\lesssim |x|_\sigma\sum_{j=0}^{\infty}(2^{j+1}t)^{\alpha-|\sigma|-1}\int_{|y|_\sigma\le 2^{j+2}t}|f(y)|\,dy\\
&\lesssim |x|_\sigma\,\|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}}\sum_{j=0}^{\infty}(2^{j+2}t)^{\alpha-|\sigma|-1}(2^{j+2}t)^{|\sigma|-\alpha}\\
&\approx |x|_\sigma\,t^{-1}\,\|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}}. \qquad (4.4)
\end{aligned}$$

Therefore, from (4.3) and (4.4) we have

$$\sup_{x,t}\frac{1}{|E_\sigma(0,t)|}\int_{E_\sigma(0,t)}\left|\widetilde{I}_{\alpha,\sigma}f(x-y)-a_f\right|dy\lesssim\|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}}. \qquad (4.5)$$

By the Hölder's inequality we have

$$\begin{aligned}
\|f\|_{L_{1,1-\frac{\alpha}{|\sigma|},\sigma}} &= \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-|\sigma|} \int_{E_\sigma(x,t)} |f(y)|dy \\
&\le 2^{\frac{n}{p'}} \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \left([t]_1^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p} \\
&\le 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,b,\sigma}} \sup_{t>0} t^{\alpha-\frac{|\sigma|}{p}} [t]_1^{\frac{b|\sigma|}{p}} \qquad (4.6)\\
&= 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,b,\sigma}} \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{|\sigma|(1-b)}{p}}, \sup_{t>1} t^{\alpha-\frac{|\sigma|}{p}} \right\} \\
&= 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,b,\sigma}} .
\end{aligned}$$

Finally let $\frac{|\sigma|(1-b)}{\alpha} \le p \le \frac{|\sigma|}{\alpha}$ and $f \in \widetilde{L}_{p,b,\sigma}(\mathbb{R}^n)$, then from (4.5) and (4.6) we get

$$\left\| \widetilde{I}_{\alpha,\sigma} f \right\|_{BMO_\sigma} \le 2 \sup_{x,t} \frac{1}{|E_\sigma(0,t)|} \int_{E_\sigma(0,t)} \left| \widetilde{I}_{\alpha,\sigma} f(x-y) - a_f \right| dy \lesssim \|f\|_{\widetilde{L}_{p,b,\sigma}}.$$

The Theorem 4.4 is proved. □

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