



Improving the computation of invariants of plane curve singularities

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Abstract

In this article we present an algorithm to compute the incidence matrix of the resolution graph, the total multiplicities, the strict multiplicities and the Milnor number of a reduced plane curve singularity and its implemetation in Singular.

1 Introduction

In this article we describe how to improve the computation of the resolution graph and related invariants of a reduced plane curve singularity. We were inspired by the use of the computer algebra system SINGULAR ([2],[4]). The SINGULAR libraries hnoether.lib and alexpoly.lib contain procedures to compute the incidence matrix of the resolution graph (for details see section-2 respectively [5],[1]), the multiplicity sequence and other related invariants of a plane curve singularity $(C, 0)$ defined by a polynomial $f \in \mathbb{Q}[x, y]$. If f is not irreducible over \mathbb{Q} then the computation can be very time consuming compared to the computation of the corresponding invariants for the singularities defined by the factors* of f (see section-4). There are combinatorial formulae which relate the invariants of the branches of a curve together with some additional informations, so called contact numbers of the branches (see section-2 for a definition), to the invariants of the curve. Therefore it is much faster to compute the invariants for the branches and some additional invariants and obtain on this basis the invariants of the curve we started with.

Key Words: Resolution Graph, Total and Strict Multiplicities, Hamburger-Noether Expansion.

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*The factors of f may be not analytically irreducible. In this case the splitting to the branches will be computed by the procedures of the SINGULAR-library mentioned above.

2 The resolution graph and the multiplicity sequences of a plane curve singularity

The following definitions and results can be found in [5],[1],[3].

Let $C = \bigcup_{i=1}^r C_i \subset \mathbb{C}^2$ be a small representative of a reducible plane curve singularity with branches C_1, \dots, C_r $r \geq 2$. Assume that $X_i \xrightarrow{\pi_1} \dots \xrightarrow{\pi_i} X_1 \xrightarrow{\pi_1} \mathbb{C}^2$ is a sequence of blowing up points. Denote by $E^{(i)} = (\pi_1 \circ \dots \circ \pi_i)^{-1}(0)$ the exceptional divisor, $C^{(i)} = (\pi_1 \circ \dots \circ \pi_i)^{-1}(C \setminus \{0\})$ the strict transform and $(\pi_1 \circ \dots \circ \pi_i)^{-1}(C)$ the total transform of C . Let $X_{i+1} \xrightarrow{\pi_{i+1}} X_i$ be the blowing up of X_i in all points of $C^{(i)} \cap E^{(i)}$ which are still singular on $C^{(i)}$ or non-transversal intersection of $C^{(i)}$ with $E^{(i)}$ that is the points with intersection multiplicity of $C^{(i)}$ and $E^{(i)}$ greater than one or where two exceptional divisors and $C^{(i)}$ meet.

Definition 2.1. (i) $X_k \xrightarrow{\pi_k} \dots \xrightarrow{\pi_1} X_1 \rightarrow \mathbb{C}^2$ is called a standard resolution of $(C, 0)$ if all branches of $C^{(k)}$ are smooth, do not intersect each other, do intersect just one component of $E^{(k)}$ and do intersect this component transversally. (ii) For some i and k assume that $C_i^{(j)} \cap C_k^{(j)} \neq \emptyset$ but $C_i^{(j+1)} \cap C_k^{(j+1)} = \emptyset$ then $\gamma(C_i, C_k) := j + 1$ is called the contact number of the branches C_i and C_k .

We consider the following weighted graph, the resolution graph of C .

- (i) To each component of $E^{(k)}$ a point \bullet is associated.
- (ii) To each component of $C^{(k)}$ a point $*$ is associated.
- (iii) Two points are connected by an edge if the corresponding components intersect.
- (iv) The points of type(i) are weighted. Let E be a component of $E^{(k)}$. We give to the corresponding point the weight i if E is created in the i -th level of the blowing ups that is i is minimal such that $\pi_{i+1} \circ \dots \circ \pi_n(E)$ is not a point.

The multiplicity sequence corresponding to the strict transform of C is called strict multiplicity sequence and the multiplicity sequence corresponding to the total transform of C is called total multiplicity sequence.

Now we want to recall the Hamburger-Noether expansion of an irreducible plane curve singularity. We assume that the curve has a singular point at 0 and $x = x(t), y = y(t)$ is a parametrization. $x(t), y(t) \in \mathbb{C}[[t]]$ and $\dim_{\mathbb{C}} \mathbb{C}[[t]] / \langle x(t), y(t) \rangle < \infty$. We further assume that $\text{ord}(x(t)) \leq \text{ord}(y(t))$.

Definition 2.2. A Hamburger-Noether expansion is a set of expressions of type

$$\begin{aligned}
 y &= a_{01}x + a_{02}x^2 + \dots + a_{0,h}x^h + x^h z_1 \\
 x &= a_{12}z_1^2 + \dots + a_{1,h_1}z_1^{h_1} + z_1^{h_1} z_2 \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 z_{r-2} &= a_{r-1,2}z_{r-1}^2 + \dots + a_{r-1,h_{r-1}}z_{r-1}^{h_{r-1}} + z_{r-1}^{h_{r-1}} z_r \\
 z_{r-1} &= a_{r,2}z_r^2 + \dots
 \end{aligned}$$

Where $a_{ji} \in \mathbb{C}, z_j \in \mathbb{C}[[t]]$
 $1 \leq \text{ord}(z_r) < \dots < \text{ord}(z_1) < \text{ord}(x) \leq \text{ord}(y)$

Next we would like to recall the properties of the intersection multiplicity of two irreducible plane curves.

Let $(C, 0)$ be given by the parametrization $x = x(t), y = y(t)$ and $(D, 0) = (V(g), 0)$, $g \in \mathbb{C}[[x, y]]$ then the intersection multiplicity is defined by

$$(C, 0) \cdot (D, 0) = \text{ord}(g(x(t), y(t))).$$

One can use the Hamburger-Noether expansion to compute the intersection multiplicity as follows:

Let

$$\begin{aligned}
 z_{j-1} &= \sum_i a_{ji} z_j^i + z_j^{h_j} z_{j+1} \quad , \quad 0 \leq j \leq r \\
 z_{j-1}^{(1)} &= \sum_i a_{ji}^{(1)} z_j^{(1)i} + z_j^{(1)h_j^{(1)}} z_{j+1}^{(1)} \quad , \quad 0 \leq j \leq r^{(1)}
 \end{aligned}$$

Let s be the greatest integer for which $h = h^{(1)}, h_1 = h_1^{(1)}, \dots, h_{s-1} = h_{s-1}^{(1)}$ and $a_{jk} = a_{jk}^{(1)}$ for $j < s$ and $k \leq h_j$. Let i be the least index such that $a_{si} \neq a_{si}^{(1)}$ ($i \leq h_s + 1, i \leq h_s^{(1)} + 1$). Finally set

$$S = \sum_{j=0}^s h_j \cdot n_j \cdot n_j^{(1)}$$

where $n_j = \text{ord}(z_j)$ and $n_j^{(1)} = \text{ord}(z_j^{(1)})$.

Then we have:

(i) if $i \leq h_s$ and $i \leq h_s^{(1)}$

$$(C, D) = S + i \cdot n_s \cdot n_s^{(1)}$$

(ii) if $i = h_s^{(1)} + 1$

$$(C, D) = S + h_s \cdot n_s \cdot n_s^{(1)} + n_{s+1}^{(1)} \cdot n_s$$

(iii) if $i = h_s + 1$

$$(C, D) = S + h_s^{(1)} \cdot n_s^{(1)} \cdot n_s + n_{s+1} \cdot n_s^{(1)}$$

Definition 2.3. Let C be a reduced curve singularity then its milnor number is defined as $\mu(C) = \dim \mathbb{C}[[x, y]] / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

The aim of this paper is to reduce the computation of the resolution graph, the multiplicity sequences and Milnor number of any plane curve singularity to that of the irreducible branches. This can be done because of the following proposition.

Proposition 2.4. Let $(C, 0) = \bigcup_{i=1}^r (C_i, 0)$ be a plane curve singularity. Then the following data determine each other:

- (i) The resolution graph of C .
- (ii) The resolution graph of branches $(C_i, 0)$ together with the contact numbers $\gamma(C_i, C_j)$ for $i \neq j$.

To compute the contact numbers we use the following proposition.

Proposition 2.5. Let $(C, 0)$ respectively $(D, 0)$ be irreducible plane curve singularities with multiplicity sequences (m_0, \dots, m_g) respectively (n_0, \dots, n_h) . For $k \geq g + 1$ define $m_k = 1$ and similarly for $k \geq h + 1$ define $n_k = 1$ then

$$(C, 0) \cdot (D, 0) = \sum_{i=0}^{\gamma(C, D)} m_i \cdot n_i$$

The proof of above two propositions can be found in book [5].

Proposition 2.6. Let $\mu(C)$ be the milnor number of a reduced germ C . If $(C_i)_{i=1 \dots k}$ is decomposition of C . Then

$$\mu(C) = \sum_{i=1}^k \mu(C_i) + 2 \sum_{1 \leq i < j \leq k} (C_i, C_j) - k + 1$$

This proposition can be found in [3].

3 The Algorithms

In this section we present the algorithm to compute the incidence matrix of resolution graph, total multiplicities, strict multiplicities and Milnor number of f by applying the results of section-2.

— Algorithm-1 `contactNumber`

— **Input:** $a \in \mathbb{Z}$, v_1, v_2 vectors of integers.

(the intersection multiplicity respectively the multiplicity sequence of two branches)

Output: an integer d , the contact number of two branches.

- if $(size(v_1) \neq size(v_2))$ then add a suitable numbers of 1 to the smaller vector.
 - compute $e := v_1 \cdot v_2$ [†]
 - if $(e < a)$ then *return* $(size(v_1) + a - e)$
 - $c := 0$; $l := 0$
 - while $(c \leq a)$
 - $l := l + 1$
 - $c := c + v_{1,l} \cdot v_{2,l}$
 - return (l)
-

— Algorithm-2 `resGraph`

— **Input:** a polynomial f , (f defines a reduced curve singularity)

Output: list L containing the incidence matrix of the resolution graph, the total multiplicities, the strict multiplicities and the Milnor number of f

- factorize $f = f_1, \dots, f_r$
- compute a list H , the Hamburger-Noether expansion[‡] for all f_i
- compute the matrix of intersection multiplicities[§] corresponding to the irreducible branches of f
- compute the Milnor number m by using the Milnor number of the branches and the intersection multiplicities.
- compute a list of invariants[¶], the characteristic exponents and the multiplic-

[†] $v_1 \cdot v_2$ is the scalar multiplication of v_1 and v_2

[‡]In SINGULAR this can be done by using the procedure *hnexpansion*(f_i) from *hnoether.lib*.

[§]if h_1, h_2 is the output of *hnexpansion* then in SINGULAR the procedure *intersection*(h_1, h_2) computes the intersection multiplicity of the branches corresponding to h_1 and h_2 .

[¶]If h is the output of *hnexpansion* then in SINGULAR the procedure *invariants*(h) gives a list of invariants corresponding to each branch, the first entry of the list contains

ity sequence for each $h \in H$

- compute the matrix of contact numbers corresponding to the irreducible branches of f (Algorithm-1)
- compute a list^{||} L of the incidence matrix of the resolution graph, the total multiplicities and the strict multiplicities of f
- add the Milnor number to the list L
- return(L)

4 Timings and Table

In this section we provide some examples and a table which gives a time comparison between the algorithm totalmultiplicities, algorithm milnor and the algorithm resGraph. Timings are conducted by using Singular 3-1-3 on an Intel®T2400, dual core 1.83 GHz processor, 1 GB RAM under the Window 2007 operating system.

We consider the following polynomials:

$$\begin{aligned}
 f_1 &= ((x^7 - y^6)^4 - x^{20}y^{10} + x^{35}) \cdot ((x^7 - y^6)^3 - x^{24} + y^{25}) \cdot ((x^7 - y^5)^3 - x^{24} - y^{25}) \\
 f_2 &= (x^{14} + x^5y^5 + y^{10} + y^{15}) \cdot (x^{16} + x^{18} + y^{20} + y^{23}) \\
 f_3 &= (x^{14} - y^{12} + x^{10}y^4 + y^{19}) \cdot (x^{12} - y^{14} + x^{15} + y^{15}) \cdot ((x^6 - y^5)^4 - x^{26}y^{12} + \\
 &\quad x^{28} + y^{30}) \cdot (x^{14} + y^{18} + y^{21}) \\
 f_4 &= (x^8 + 2y^{14}) \cdot (x^{10} + 5y^{10}) \cdot (y^2 - x^3) \cdot (x^2 - y^3) \\
 f_5 &= (x^{18} + y^{24} + x^{29}) \cdot (x^{14} + y^{18} + y^{21}) \cdot (x^9 - x^3y^3 + y^{11}) \cdot (x^{15} - y^{10} - y^{19}) \\
 f_6 &= (x^{16} + 3y^{18}) \cdot (x^5 + 7y^5) \\
 f_7 &= ((x^7 - y^6)^4 - x^{20}y^{10} + x^{35}) \cdot ((x^7 - y^6)^3 - x^{24} + y^{25}) \\
 f_8 &= (x^{18} + y^{24} + x^{29}) \cdot (x^{14} + y^{18} + y^{21}) \cdot (x^9 - x^3y^3 + y^{11})
 \end{aligned}$$

the characteristics exponents, semi-group generators, the Puiseux pairs, the degree of the conductor, the sequence of multiplicities and the second entry of the list contains the contact matrix of the branches, the intersection multiplicities of the branches and the delta invariant of f .

^{||}If N is the contact matrix and A is the list of characteristic exponents of all branches of f then in the SINGULAR procedure totalmultiplicities(N, A) from alexpoly.lib gives a list of the incidence matrix of the resolution graph, the total multiplicities and the strict multiplicities of f .

Poly	Timing(totalmultiplicities)	Timing(resGraph)	Timing(milnor)
f_1	27 min	2 sec	not enough memory
f_2	40 min	2 sec	2 sec
f_3	55 min	3 sec	1 min
f_4	>1 hr	3 min	2 sec
f_5	>1 hr	3 sec	not enough memory
f_6	not enough memory	10 sec	2 sec
f_7	1 min	2 sec	6 min
f_8	4 min	2 sec	8 min

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