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Some properties of multivalent analytic functions associated with an integral operator

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Abstract

Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(p \in N = \{1, 2, 3, \cdots\})$ which are analytic in the open unit disk $U = \{z : z \in C$ and $|z| < 1\}$. By making use of the Noor integral operator, we obtain some interesting properties of multivalent analytic functions.

1 Introduction

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic in the open unit disk $U = \{z\colon z \in C \text{ and } |z| < 1\}$.

For $f \in A(p)$, we denote by $D^{n+p-1} : A(p) \to A(p)$ the operator defined by

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \quad (n > -p) \tag{2}$$

or, equivalently, by

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!},$$

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where n is any integer greater than $-p$ and the symbol $(*)$ stands for the Hadamard product (or convolution). If $f(z)$ is given by (1.1), then from (1.2) it follows that

$$D^{n+p-1}f(z) = z^p + \sum_{k=p+1}^{\infty} \binom{n+k-1}{k-p} a_k z^k \quad (p \in N; n > -p).$$

The symbol D^{n+p-1} when $p = 1$ was introduced by Ruscheweyh [12] and the symbol D^{n+p-1} was introduced by Goel and Sohi [2].

Recently, analogous to D^{n+p-1}, Liu and Noor [4] introduced an integral operator $I_{n,p} : A(p) \to A(p)$ as following.

Let $f_{n,p}(z) = z^p/(1-z)^{n+p}$ $(n > -p)$, and let $f_{n,p}^{(+)}(z)$ be defined such that

$$f_{n,p}(z) * f_{n,p}^{(+)}(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{3}$$

Then

$$\begin{aligned} I_{n,p}f(z) &= f_{n,p}^{(+)}(z) * f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{(p+1)(p+2)\cdots k}{(n+p)(n+p+1)\cdots(n+k-1)} a_k z^k. \end{aligned} \tag{4}$$

It follows from (1.4) that

$$z(I_{n+1,p}f(z))' = (n+p)I_{n,p}f(z) - nI_{n+1,p}f(z). \tag{5}$$

We also note that $I_{0,p}f(z) = zf'(z)/p$ and $I_{1,p}f(z) = f(z)$. Moreover, the operator $I_{n,p}f(z)$ defined by (1.4) is called as the Noor integral operator of $(n+p-1)th$ order of f [4]. For $p = 1$, the operator $I_{n,1}f(z) \equiv I_n f$ was introduced by Noor [7] and Noor and Noor [9]. Many interesting subclasses of analytic functions, associated with the Noor integral operator $I_{n,p}$ and its many special cases, were investigated recently by Cho [1], Liu [3], Liu and Noor [4,5], Noor [7,8], Noor and Noor [9,10] and others. In the present sequel to these earlier works, we shall derive certain interesting properties of the Noor integral operator.

2 Main results

In order to give our theorems, we need the following lemma.

Lemma. (see [6]). Let Ω be a set in the complex plane C and let b be a complex number such that $Reb > 0$. Suppose that the function $\psi : C^2 \times U \longrightarrow C$ satisfies the condition

$$\psi(ix, y; z) \notin \Omega$$

for all real $x, y \le -|b-ix|^2/(2Reb)$ and all $z \in U$. If the function $p(z)$ defined by $p(z) = b + b_1 z + b_2 z^2 + \cdots$ is analytic in U and if

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then $Rep(z) > 0$ in U.

We now prove our first result given by Theorem 1 below.

Theorem 1. Let $n > -p+1, \lambda \ge 0$ and $\gamma > 1$. Suppose that $f(z) \in A(p)$, then

$$Re\left\{(1-\lambda)\frac{I_{n,p}f(z)}{I_{n+1,p}f(z)} + \lambda\frac{I_{n-1,p}f(z)}{I_{n,p}f(z)}\right\} < \gamma \quad (z \in U) \tag{6}$$

implies

$$Re\left\{\frac{I_{n,p}f(z)}{I_{n+1,p}f(z)}\right\} < \beta \quad (z \in U), \tag{7}$$

where $\beta \in (1, +\infty)$ is the positive root of the equation

$$2(n+p+\lambda-1)x^2 - [\lambda + 2\gamma(n+p-1)]x - \lambda = 0. \tag{8}$$

Proof. Let

$$p(z) = \frac{1}{\beta-1}\left[\beta - \frac{I_{n,p}f(z)}{I_{n+1,p}f(z)}\right], \tag{9}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (2.4) and using (1.5), we obtain

$$\begin{aligned}
&(1-\lambda)\frac{I_{n,p}f(z)}{I_{n+1,p}f(z)} + \lambda\frac{I_{n-1,p}f(z)}{I_{n,p}f(z)} \\
&= \beta + \frac{\lambda(\beta-1)}{n+p-1} - \frac{(\beta-1)(n+p+\lambda-1)}{n+p-1}p(z) - \frac{\lambda(\beta-1)}{n+p-1}\cdot\frac{zp'(z)}{\beta-(\beta-1)p(z)} \\
&= \psi(p(z), zp'(z)),
\end{aligned}$$

where

$$\psi(r, s) = \beta + \frac{\lambda(\beta-1)}{n+p-1} - \frac{(\beta-1)(n+p+\lambda-1)}{n+p-1}r - \frac{\lambda(\beta-1)}{n+p-1}\cdot\frac{s}{\beta-(\beta-1)r}. \tag{10}$$

Using (2.2) and (2.5), we have

$$\{\psi(p(z), zp'(z)) : z \in U\} \subset \Omega = \{w \in C : Rew < \gamma\}.$$

Now for all real $x, y \le -(1+x^2)/2$, we have

$$\begin{aligned} Re\{\psi(ix,y)\} &= \beta + \frac{\lambda(\beta-1)}{n+p-1} - \frac{\lambda(\beta-1)}{n+p-1}\cdot\frac{\beta y}{\beta^2+(\beta-1)^2x^2} \\ &\ge \beta + \frac{\lambda(\beta-1)}{n+p-1} + \frac{\lambda\beta(\beta-1)}{2(n+p-1)}\cdot\frac{1+x^2}{\beta^2+(\beta-1)^2x^2} \\ &\ge \beta + \frac{\lambda(\beta-1)}{n+p-1} + \frac{\lambda(\beta-1)}{2\beta(n+p-1)} \\ &= \beta + \frac{\lambda(\beta-1)(2\beta+1)}{2\beta(n+p-1)} = \gamma, \end{aligned}$$

where β is the positive root of the equation (2.3).

Note that $n > -p+1, \lambda \ge 0, \gamma > 1$ and let

$$g(x) = 2(n+p+\lambda-1)x^2 - [\lambda + 2\gamma(n+p-1)]x - \lambda,$$

then $g(0) = -\lambda \le 0$ and $g(1) = -2(n+p-1)(\gamma-1) < 0$. This shows $\beta \in (1, +\infty)$. Hence for each $z \in U$, $\psi(ix, y) \notin \Omega$. By Lemma, we get $Rep(z) > 0$. This proves (2.2).

Theorem 2. Let $\lambda \ge 0, \gamma > 1$ and $0 \le \delta < 1$. Let $g(z) \in A(p)$ satisfy

$$Re\left\{\frac{I_{n+1,p}g(z)}{I_{n,p}g(z)}\right\} > \delta \quad (z \in U). \tag{11}$$

If $f(z) \in A(p)$ satisfies

$$Re\left\{(1-\lambda)\frac{I_{n+1,p}f(z)}{I_{n+1,p}g(z)} + \lambda\frac{I_{n,p}f(z)}{I_{n,p}g(z)}\right\} < \gamma \quad (z \in U), \tag{12}$$

then

$$Re\left\{\frac{I_{n+1,p}f(z)}{I_{n+1,p}g(z)}\right\} < \frac{2\gamma(n+p)+\lambda\delta}{2(n+p)+\lambda\delta} \quad (z \in U). \tag{13}$$

Proof. Let $\beta = \frac{2\gamma(n+p)+\lambda\delta}{2(n+p)+\lambda\delta} (\beta > 1)$ and consider the function

$$p(z) = \frac{1}{\beta-1}\left[\beta - \frac{I_{n+1,p}f(z)}{I_{n+1,p}g(z)}\right]. \tag{14}$$

The function $p(z)$ is analytic in U and $p(0) = 1$. Set

$$B(z) = \frac{I_{n+1,p}g(z)}{I_{n,p}g(z)},$$

then $Re\{B(z)\} > \delta$ $(z \in U)$. Differentiating (2.9) and using (1.5), we have

$$\begin{aligned}&(1-\lambda)\frac{I_{n+1,p}f(z)}{I_{n+1,p}g(z)}+\lambda\frac{I_{n,p}f(z)}{I_{n,p}g(z)}\\&=\beta-(\beta-1)p(z)-\frac{\lambda(\beta-1)}{n+p}B(z)\cdot zp'(z).\end{aligned}$$

Let

$$\psi(r,s)=\beta-(\beta-1)r-\frac{\lambda(\beta-1)}{n+p}B(z)\cdot s,$$

then from (2.7), we deduce that

$$\{\psi(p(z),zp'(z)) : z\in U\}\subset\Omega=\{w\in C : Rew<\gamma\}.$$

Now for all real $x, y \le -(1+x^2)/2$ we have

$$\begin{aligned}Re\{\psi(ix,y)\} &= \beta-\frac{\lambda(\beta-1)y}{n+p}Re\{B(z)\}\\&\ge \beta+\frac{\lambda\delta(\beta-1)}{2(n+p)}(1+x^2)\\&\ge \beta+\frac{\lambda\delta(\beta-1)}{2(n+p)}=\gamma.\end{aligned}$$

Hence for each $z \in U$, $\psi(ix, y) \notin \Omega$. Thus by Lemma, $Rep(z) > 0$ in U. The proof of the theorem is complete.

Finally, we prove the following result.

Theorem 3. Let $\beta \ge 1$ and $\gamma > 0$. Let $f(z) \in A(p)$, then

$$Re\left\{\frac{I_{n,p}f(z)}{I_{n+1,p}f(z)}\right\}<\frac{n+p+\gamma}{n+p}\quad(z\in U)\tag{15}$$

implies

$$Re\left\{\left(\frac{I_{n+1,p}f(z)}{z^p}\right)^{-1/2\beta\gamma}\right\}>2^{-1/\beta}\quad(z\in U)\tag{16}$$

The bound $2^{-1/\beta}$ is best possible.

Proof. From (1.5) and (2.10), we have

$$Re\left\{\frac{z(I_{n+1,p}f(z))'}{I_{n+1,p}f(z)}\right\}<p+\gamma\quad(z\in U).$$

That is,

$$\frac{1}{2\gamma}\left(\frac{z(I_{n+1,p}f(z))'}{I_{n+1,p}f(z)}-p\right)\prec\frac{z}{1+z}.\tag{17}$$

Let

$$p(z) = \left(\frac{I_{n+1,p}f(z)}{z^p}\right)^{-1/2\gamma},$$

then (2.12) may be written as

$$z\,(logp(z))' \prec z\left(log\frac{1}{1+z}\right)' \tag{18}$$

By using a well-known result [13] to (2.13), we obtain that

$$p(z) \prec \frac{1}{1+z},$$

that is, that

$$\left(\frac{I_{n+1,p}f(z)}{z^p}\right)^{-1/2\beta\gamma} = \left(\frac{1}{1+w(z)}\right)^{1/\beta}, \tag{19}$$

where $w(z)$ is analytic in U, $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

According to $Re\left(t^{1/\beta}\right) \geq (Ret)^{1/\beta}$ for $Ret > 0$ and $\beta \geq 1$, (2.14) yields

$$\begin{aligned} Re\left\{\left(\frac{I_{n+1,p}f(z)}{z^p}\right)^{-1/2\beta\gamma}\right\} &\geq \left(Re\left(\frac{1}{1+w(z)}\right)\right)^{1/\beta} \\ &> 2^{-1/\beta} \quad (z \in U). \end{aligned}$$

To see that the bound $2^{-1/\beta}$ cannot be increased, we consider the function

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(n+p+1)\cdots(n+k)}{(p+1)(p+2)\cdots k} \cdot \frac{2\gamma(2\gamma-1)\cdots(2\gamma-k+p+1)}{(k-p)!} z^k$$

Since $g(z)$ satisfies

$$\frac{I_{n+1,p}g(z)}{z^p} = (1+z)^{2\gamma},$$

we easily have that $g(z)$ satisfies (2.10) and

$$Re\left\{\left(\frac{I_{n+1,p}g(z)}{z^p}\right)^{-1/2\beta\gamma}\right\} \to 2^{-1/\beta}$$

as $z = Rez \to 1^-$. The proof of the theorem is complete.

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