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RAD-$\bigoplus$-SUPPLEMENTED MODULES

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Abstract

In this paper we provide various properties of *Rad*-$\oplus$-supplemented modules. In particular, we prove that a projective module M is *Rad*-$\oplus$-supplemented if and only if M is $\oplus$-supplemented, and then we show that a commutative ring R is an artinian serial ring if and only if every left R-module is *Rad*-$\oplus$-supplemented. Moreover, every left R-module has the property (P^*) if and only if R is an artinian serial ring and $J^2 = 0$, where J is the Jacobson radical of R. Finally, we show that every *Rad*-supplemented module is *Rad*-$\oplus$-supplemented over dedekind domains.

1 Introduction

Throughout this paper, it is assumed that R is an associative ring with identity and all modules are unital left R-modules. A submodule N of an R-module M will be denoted by $N \leq M$. A submodule $L \leq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \leq M$. Dually, a submodule N of M is called *small* (in M) and denoted by $N << M$, if $N + L \neq M$ for every proper submodule L of M. The Jacobson radical of M will be denoted by $Rad(M)$. Equivalently, $Rad(M)$ is the sum of all small submodules of M.

A nonzero module M is said to be *hollow* if every proper submodule is small in M, and it is said to be *local* if it is hollow and is finitely generated. M is local if and only if it is finitely generated and $Rad(M)$ is maximal (see

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[6, 2.12§2.15]). A ring R is said to be *local* if J is maximal, where J is the Jacobson radical of R.

For any ring R, an R-module M is called *supplemented* if every submodule N of M has a *supplement*, that is a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ [17]. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a proper generalization of artinian modules.

Mohamed and Müller [12] call a module M *$\oplus$-supplemented* if every submodule N of M has a supplement that is a direct summand of M [12]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [12, Lemma A.4 (2)]). It is shown in [12, Proposition A.7 and Proposition A.8] that if R is a dedekind domain, every supplemented R-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented. Characterizations and the structure of supplemented and $\oplus$-supplemented modules are extensively studied by many authors. We specifically mention [8, 10, 12, 17, 19] among papers concerning supplemented and $\oplus$-supplemented modules.

A module M is *lifting* if every submodule N of M contains a direct summand L of M such that $M = L \oplus K$ and $N \cap K << K$ (see [6]). Every projective module over a left artinian ring is lifting, and lifting modules are $\oplus$-supplemented. In addition, every π-projective supplemented module is lifting (see [17, 41.15]). Here a module M over an arbitrary ring is called *π-projective* if for every two submodules U, V of M such that $U + V = M$, there exists an endomorphism f of M with $f(M) \leq U$ and $(1 - f)(M) \leq V$ [17]. For example, projective modules are π-projective.

Let M be a module. Weakening the "supplement" condition, one calls a submodule K of M *Rad-supplement* of N in M (in [18], *generalized supplement*) if $M = N + K$ and $N \cap K \leq Rad(K)$ [6, pp. 100]. Adapting the concept of supplemented modules, we say that M is *Rad-supplemented* if every submodule has a *Rad*-supplement in M, and M is *Rad-$\oplus$-supplemented* if every submodule has a *Rad*-supplement that is a direct summand of M [4, 7]. Under given definitions, we clearly have the following implication on modules:

$$\begin{tikzcd}
& lifting \arrow[d] & \\
& \oplus - supplemented \arrow[dl] \arrow[dr] & \\
Rad - \oplus - supplemented \arrow[dr] & & supplemented \arrow[dl] \\
& Rad - supplemented &
\end{tikzcd}$$

Let $f : P \longrightarrow M$ be an epimorphism. Xue [18] calls f a *(generalized) cover* if ($Ker(f) \leq Rad(P)$) $Ker(f) << P$, and calls a (generalized) cover f a *(generalized) projective cover* if P is a projective module. In the spirit of [18], a module M is said to be *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. Every (generalized) semiperfect module is (*Rad*-) supplemented.

In this paper, we study the properties of *Rad*-⊕-supplemented modules. We prove that a projective module M is *Rad*-⊕-supplemented if and only if it is ⊕-supplemented. It follows that a ring R is left perfect if and only if every projective left R-module is *Rad*-⊕-supplemented. Every π-projective *Rad*-⊕-supplemented module M has the property (P^*), i.e., for every submodule $N \leq M$, there exists a decomposition $M = K \oplus L$ such that $K \leq N$ and $N \cap L \leq Rad(L)$. We prove that every left R-module has the property (P^*) if and only if R is an artinian serial ring and $J^2 = 0$, where J is the Jacobson radical of R. We show that the class of weakly distributive *Rad*-⊕-supplemented modules is closed under factor modules, and we prove that a commutative ring R is an artinian serial ring if and only if every left R-module is *Rad*-⊕-supplemented. We also prove that over dedekind domains every *Rad*-supplemented module is *Rad*-⊕-supplemented. Finally, we completely determine the structure of *Rad*-⊕-supplemented modules over local dedekind domains.

2 Rad-⊕-Supplemented Modules

Every ⊕-supplemented module is *Rad*-⊕-supplemented; however, the converse is not always true (see [9, Example 3.11]). Now we prove that every projective Rad-⊕-supplemented module is ⊕-supplemented. We start with the following key Lemma.

Lemma 2.1. *Let M be a projective module. If M is Rad-$\oplus$-supplemented, then $Rad(M) << M$.*

Proof. Let $M = Rad(M) + N$ for some submodule N of M. Since M is *Rad*-$\oplus$-supplemented, there exists a direct summand V of M such that $M = N + V$ and $V \cap N \leq Rad(V)$. Then, it follows from [11, Theorem 5.3.4 (b)] that V is projective. Now, for all $v \in V$,

$$\alpha : V \longrightarrow \frac{M}{N}, \quad \text{defined by} \quad \alpha(v) := v + N$$

is an epimorphism and $Ker(\alpha) = N \cap V$. That is, α is a generalized cover since $Ker(\alpha) = N \cap V \leq Rad(V)$. From $M = Rad(M) + N$, it follows immediately that $Rad(\frac{M}{N}) = \frac{M}{N}$. Then, since $\frac{M}{N}$ has a generalized projective cover, it is easy to see that $\frac{M}{N} = 0$. That is, $M = N$. Hence we obtain that $Rad(M) << M$. □

Theorem 2.2. *Let M be a projective module. M is Rad-$\oplus$-supplemented if and only if it is $\oplus$-supplemented.*

Proof. Suppose that M is *Rad*-$\oplus$-supplemented. Since M is projective, it follows from Lemma 2.1 that $Rad(M) << M$. Then, by [7, Proposition 2.1], M is $\oplus$-supplemented. The converse is clear. □

A ring R is called *left perfect* if every left R-module has a projective cover [17, 43.9]. It is well known that R is left perfect if and only if every projective left R-module is $\oplus$-supplemented. Using this fact along with the above Theorem we obtain the following:

Corollary 2.3. *Let R be a ring. R is left perfect if and only if every projective left R-module is Rad-$\oplus$-supplemented.*

Recall that a module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. We denote by $P(M)$ the sum of all radical submodules of M. It is easy to see that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, M is called *reduced*. Note that $\frac{M}{P(M)}$ is reduced for every left R-module M.

Proposition 2.4. *Let M be a module. If M is Rad-$\oplus$-supplemented, then the factor module $\frac{M}{P(M)}$ of M is $\oplus$-supplemented.*

Proof. Firstly, we have $f(Rad(P(M))) \leq Rad(P(M))$ for each $f \in End_R(M)$ by [6, 2.8 (1) (2)]. Note that $P(M) = Rad(P(M))$. Thus $f(P(M)) \leq P(M)$ for each $f \in End_R(M)$. Since M is *Rad*-$\oplus$-supplemented, it follows from [9, Proposition 3.5.(1)] that $\frac{M}{P(M)}$ is *Rad*-$\oplus$-supplemented. Let $P(M) \leq U \leq M$.

Then there exists a direct summand $\frac{V}{P(M)}$ of $\frac{M}{P(M)}$ such that $\frac{M}{P(M)} = \frac{U}{P(M)} + \frac{V}{P(M)}$ and $\frac{U\cap V}{P(M)} \leq Rad(\frac{V}{P(M)})$. Since $\frac{M}{P(M)}$ is reduced, it follows from [4, Theorem 4.6] that $\frac{M}{P(M)}$ is coatomic, so $Rad(\frac{M}{P(M)}) << \frac{M}{P(M)}$. Thus $\frac{U\cap V}{P(M)} << \frac{M}{P(M)}$ and therefore $\frac{U\cap V}{P(M)} << \frac{V}{P(M)}$ by [17, 19.3.(5)]. This means that $\frac{V}{P(M)}$ is a supplement of $\frac{U}{P(M)}$ in $\frac{M}{P(M)}$. Hence $\frac{M}{P(M)}$ is $\oplus$-supplemented. □

We say that a module M is *completely Rad-$\oplus$-supplemented* if every direct summand of M is *Rad*-$\oplus$-supplemented as in [15].

Corollary 2.5. *$P(M)$ is completely Rad-$\oplus$-supplemented for every R-module M.*

Proof. Let M be a module and let N be a direct summand of $P(M)$. Note that every radical module is *Rad*-$\oplus$-supplemented. Therefore it suffices to show that N is radical. Since N is a direct summand of $P(M)$, we can write $P(M) = N \oplus L$ for some submodule L of $P(M)$. By [17, 21.6.(5)], we have $P(M) = Rad(P(M)) = Rad(N \oplus L) = Rad(N) \oplus Rad(L)$. By the modular law, $N = N \cap P(M) = N \cap (Rad(N) \oplus Rad(L)) = Rad(N) \oplus Rad(L) \cap N = Rad(N)$, i.e., N is radical. Hence $P(M)$ is completely *Rad*-$\oplus$-supplemented. □

Proposition 2.6. *Let M be a Rad-$\oplus$-supplemented module. If every Rad-supplement in M is a direct summand of M, then M is completely Rad-$\oplus$-supplemented.*

Proof. Let N be a direct summand of M. Then we can write $M = N \oplus L$ for some submodule L of M. Since M is *Rad*-$\oplus$-supplemented, it is *Rad*-supplemented and therefore N is *Rad*-supplemented by [2, 2.2 (2)]. Let $U \leq N$, then U has a *Rad*-supplement V in N. Now we argue that V is a direct summand of N. Note that

$$M = N \oplus L = (U + V) + L = (U + L) + V\,,$$

and

$$(U + L) \cap V \leq (U + V) \cap L + (L + V) \cap U = (L + V) \cap U \leq U\,.$$

Then $(U+L)\cap V \leq U \cap V \leq Rad(V)$. This means that V is a *Rad*-supplement of $(U + L)$ in M. By our assumption, we can write $M = V \oplus V'$ for some submodule V' of M. It follows by the modular law that $N = V \oplus V' \cap N$. This completes the proof. □

Let M be a module. M is said to have *the property* (P^*) if for every submodule $N \leq M$ there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \leq Rad(\frac{M}{K})$ [1]. Equivalently, for every submodule $N \leq M$ there exists a decomposition $M = K \oplus L$ such that $K \leq N$ and $N \cap L \leq Rad(L)$.

Proposition 2.7. *Let M be a module. If M has the property (P^*), then M is completely Rad-$\oplus$-supplemented.*

Proof. Let N be a direct summand of M and let $U \leq N$. Since M has the property (P^*), there exists a submodule X of U such that $M = X \oplus X^{'}$ and $U \cap X^{'} \leq Rad(X^{'})$ for some submodule $X^{'}$ of M. By the modular law, we can write $N = X \oplus N \cap X^{'}$. This means that $N \cap X^{'}$ is a direct summand of N. Therefore $N = U + N \cap X^{'}$.

Next, we prove that $U \cap (N \cap X^{'}) = U \cap X^{'} \leq Rad(N \cap X^{'})$. Let m be any element of $U \cap X^{'}$. Since $U \cap X^{'} \leq Rad(X^{'})$, by [11, 9.1.3.(a)], we get $Rm << X^{'}$ so that $Rm << M$. Applying [17, 19.3.(5)] twice, we first obtain $Rm << N$ and then $Rm << N \cap X^{'}$. By [11, 9.1.3.(a)], we have $U \cap X^{'} \leq Rad(N \cap X^{'})$. □

Recall that a π-projective module M is $\oplus$-supplemented if and only if the module is lifting [17, 41.15]. Now we shall prove analogous characterization for *Rad*-$\oplus$-supplemented modules.

Theorem 2.8. *A π-projective module M is Rad-$\oplus$-supplemented if and only if M has the property (P^*).*

Proof. $(\Longrightarrow)$ Let U be a submodule of M. Then, we have the sum $M = U + V$, where V is a direct summand of M. Since M is a π-projective module, we can write $M = X \oplus V$ for some submodule X of M by [6, 4.14.(1)]. It follows that, for $U \leq M$, there exists a decomposition $M = X \oplus V$ such that $X \leq U$ and $U \cap V \leq Rad(V)$. This means that M has the property (P^*).

$(\Longleftarrow)$ By Proposition 2.7. □

Clearly lifting modules has the property (P^*), but the converse is not true in general. For example, the left $\mathbb{Z}$-module $\mathbb{Q}$ has the property (P^*) but it is not lifting. If a module M is projective, then we have the following fact.

Proposition 2.9. *Let M be a module. If M is projective and has the property (P^*), then M is lifting.*

Proof. By Proposition 2.7, M is *Rad*-$\oplus$-supplemented. Applying Theorem 2.2, we obtain that M is $\oplus$-supplemented. Since M is projective, it is π-projective and thus M is lifting by [17, 41.15]. □

Before giving the following corollary which summarizes the combined results of Theorem 2.2, Theorem 2.8 and Proposition 2.9, we recall some known definitions. For a module M, consider the following conditions:

(D_2) If N is a submodule of M such that $\frac{M}{N}$ is isomorphic to a direct summand of M, then N is a direct summand of M.

(D_3) For every direct summands K and L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

In [12], a module M is called *discrete* if M is lifting and satisfies the property(D_2). This is equivalent to M is supplemented, π-projective and direct projective (see [6, 27.1]). The module M is called *quasi-discrete* if it is lifting and satisfies the property (D_3). We know that M is quasi-discrete if and only if it is supplemented and π-projective (see [6, 26.6]).

Corollary 2.10. *For a projective module M, the following conditions are equivalent.*

1. *M is supplemented.*
2. *M is $\oplus$-supplemented.*
3. *M is Rad-$\oplus$-supplemented.*
4. *M has the property (P^*).*
5. *M is lifting.*
6. *M is (quasi-) discrete.*

Proof. (1) $\Longrightarrow$ (2) It is obvious according to [17, 41.15].

(2) $\Longrightarrow$ (3) By Theorem 2.2.

(3) $\Longrightarrow$ (4) It follows from Theorem 2.8.

(4) $\Longrightarrow$ (5) It is proven in Proposition 2.9.

(5) $\Longrightarrow$ (6) Clear since projective modules are direct projective and π-projective.

(6) $\Longrightarrow$ (1) Trivial. □

Now, we shall characterize the rings whose modules have the property (P^*) in the following Corollary.

Corollary 2.11. *The following statements are equivalent for a ring R.*

1. *Every left R-module has the property (P^*).*
2. *Every left R-module is lifting.*
3. *R is an artinian serial ring and $J^2 = 0$, where J is the Jacobson radical of R.*

Proof. (1) $\Longrightarrow$ (2) Observe first that R is a left perfect ring. Let F be any projective R-module. By the hypothesis, F has the property (P^*). Since F is projective, it is π-projective and so F is *Rad*-$\oplus$-supplemented by Theorem 2.8. It follows from Corollary 2.3 that R is left perfect.

For any module M, let $U \leq M$. By assumption, there exists a decomposition $M = U \oplus V$ such that $K \leq U$ and $U \cap V \leq Rad(V)$. Since R is left perfect, we have that $U \cap V << V$. This means that M is lifting.

(2) $\Longleftrightarrow$ (3) See [6, 29.10].

(2) $\Longrightarrow$ (1) Clear. □

A module M is called *weakly distributive* if every submodule N of M is weak distributive, i.e., $N = U \cap N + V \cap N$ whenever $M = U + V$ (see [5]). It follows from [7, Example 4.1] that factor modules of a *Rad*-$\oplus$-supplemented module need not be *Rad*-$\oplus$-supplemented, in general. For weakly distributive modules we have the following fact:

Theorem 2.12. *Every factor module of a weakly distributive Rad-$\oplus$-supplemented module is Rad-$\oplus$-supplemented.*

Proof. Suppose that a module M is weakly distributive *Rad*-$\oplus$-supplemented. Let $N \leq U \leq M$. Then there exist submodules V and L of M such that $M = U + V$, $U \cap V \leq Rad(V)$ and $M = V \oplus L$. By [9, Lemma 3.4], $\frac{V+N}{N}$ is a *Rad*-supplement of $\frac{U}{N}$ of $\frac{M}{N}$. Since M is a weakly distributive module, we conclude that $N = V \cap N + L \cap N$. It follows that

$$(\tfrac{V+N}{N}) \cap (\tfrac{L+N}{N}) = \tfrac{(V+N)\cap L+N}{N} = \tfrac{(V+L\cap N)\cap L+N}{N} = \tfrac{V\cap L+L\cap N+N}{N} = 0.$$

Hence $\frac{V+N}{N}$ is a direct summand of $\frac{M}{N}$. This means that $\frac{M}{N}$ is *Rad*-$\oplus$-supplemented. □

It is proven in [9, Theorem 3.3] that every finite direct sum of *Rad*-$\oplus$-supplemented modules is *Rad*-$\oplus$-supplemented. The following example shows that the class of *Rad*-$\oplus$-supplemented is not closed under infinite direct sums.

Example 2.13. Let R be a local dedekind domain (i.e. DVR) with quotient $K \neq R$ (e.g. the ring $\mathbb{Z}_{(p)}$ containing all rational numbers of the form $\frac{a}{b}$ with $p \nmid b$ for any prime p in $\mathbb{Z}$). Since R is local, it follows that R is $\oplus$-supplemented and therefore R is *Rad*-$\oplus$-supplemented. On the other hand, by Corollary 2.3, there exists a projective R-module which is not *Rad*-$\oplus$-supplemented because R is not field.

A module M is said to be a *duo module* if every submodule N of M is fully invariant [13]. Now we prove that direct sums of *Rad*-$\oplus$-supplemented modules is *Rad*-$\oplus$-supplemented, under a certain condition: namely, when M

is a duo module. The proof of the next result is taken from [16, Theorem 1], but is given for the sake of completeness.

Proposition 2.14. *Let M_i $(i \in I)$ be any infinite collection of Rad-$\oplus$-supplemented modules and let $M = \bigoplus_{i\in I} M_i$. If M is a duo module, then M is Rad-$\oplus$-supplemented.*

Proof. Let $U \leq M$. Since M is a duo module, by [13, Lemma 2.1], $U = \bigoplus_{i\in I}(M_i \cap U)$. By the hypothesis, there exists a submodule V_i of M_i such that $M_i = M_i \cap U + V_i$ and $(M_i \cap U) \cap V_i = U \cap V_i \leq Rad(V_i)$ for every $i \in I$. Let $V = \bigoplus_{i\in I} V_i$. Note that V is a direct summand of M. Then

$$M = U + V$$

and

$$U \cap V = (\bigoplus_{i\in I}(M_i \cap U)) \cap (\bigoplus_{i\in I} V_i) \leq \bigoplus_{i\in I} Rad(V_i) = Rad(V)$$

by [17, 21.6.(5)]. It follows that V is a *Rad*-supplement of U in M. Thus M is *Rad*-$\oplus$-supplemented. □

It is shown in [10, Theorem 1.1] that a commutative ring R is an artinian serial ring if and only if every left R-module is $\oplus$-supplemented. Now we generalize this fact in the next Corollary, characterizing the commutative rings in which modules are *Rad*-$\oplus$-supplemented.

Corollary 2.15. *Let R be any commutative ring. Then R is an artinian serial ring if and only if every left R-module is Rad-$\oplus$-supplemented.*

Proof. Suppose that every left R-module is *Rad*-$\oplus$-supplemented. Then every projective left R-module is *Rad*-$\oplus$-supplemented and so, by Corollary 2.3, R is left perfect. It follows that any module has a small radical. Therefore a *Rad*-$\oplus$-supplemented module over the ring is $\oplus$-supplemented. So every module is $\oplus$-supplemented. Thus, the proof follows from [10, Theorem 1.1]. □

Recall that a module M is called *w-local* if M has a unique maximal submodule. It is clear that M is w-local if and only if $Rad(M)$ is maximal. Every local module is w-local. However, a w-local module is not necessarily local (see [3]). It is clear that if a w-local module M is finitely generated, then it is local.

Lemma 2.16. *Let R be a local commutative ring and let M be a uniform R-module. Suppose that every submodule of M is Rad-$\oplus$-supplemented. Then M is uniserial.*

Proof. By [14, Lemma 6.2], it suffices to show that every finitely generated submodule of M is local. Let K be any finitely generated submodule of M. Then K contains a maximal submodule L. By the assumption, L has a *Rad*-supplement V in K such that $V \oplus V^{'} = K$ for some submodule $V^{'}$ of K. Note that $V^{'} \subseteq M$. It follows from [3, Lemma 3.3] that V has a unique maximal submodule, i.e., V is w-local as in [3]. Therefore V is local. Since M is uniform and L is maximal, we have $V^{'} = 0$. In conclusion $V = K$. $\square$

Corollary 2.17. *Let R be a local commutative ring with a maximal ideal J. Suppose that every submodule of $E(\frac{R}{J})$ is Rad-$\oplus$-supplemented, where $E(\frac{R}{J})$ is the injective hull of the simple module $\frac{R}{J}$. Then R is a uniserial noetherian ring.*

Proof. Since $E(\frac{R}{J})$ is uniform, it follows from Lemma 2.16 and [14, Lemma 6.2 (Corollary)] that R is uniserial. Therefore R is a uniserial noetherian ring by [14, Lemma 6.3]. $\square$

A ring R is called *semilocal* if $\frac{R}{J}$ is semisimple, where J is the Jacobson radical of R. We know that a semilocal ring R is left perfect if and only if R is a left max ring (i.e. every left R-module has a maximal submodule).

Proposition 2.18. *The following conditions on a semilocal ring R are equivalent:*

1. *Every w-local module is semiperfect.*
2. *Every w-local module is generalized semiperfect.*
3. *R is left perfect.*

Proof. Clearly, we have (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2). Finally, it remains to prove the implication (2) $\Longrightarrow$ (3). Let M be any w-local module. Assume that, for $N \leq M$, $Rad(M) + N = M$. Then $Rad(\frac{M}{N})$ has no maximal submodules. It follows from proof of Lemma 2.1 that $Rad(M) << M$. So M is local. By [3, Lemma 3.1], R is left perfect. $\square$

3 Rad-$\oplus$-Supplemented Modules Over Commutative Domains

Throughout this section, we consider only commutative domains. Our aim is to prove that a *Rad*-supplemented module is *Rad*-$\oplus$-supplemented over dedekind domains. To this aim, we need the following key Lemma:

Lemma 3.1. *Let M be a module over a dedekind domain. The following are equivalent:*

1. *M is Rad-$\oplus$-supplemented.*
2. *$\frac{M}{P(M)}$ is Rad-$\oplus$-supplemented.*
3. *$\frac{M}{P(M)}$ is $\oplus$-supplemented.*

Proof. (1) $\Longrightarrow$ (3) It follows from Proposition 2.4.

(3) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (1) Since every radical module over a dedekind domain is injective, the submodule $P(M)$ of M is injective. Therefore, there exists a submodule N of M such that $M = P(M) \oplus N$. From (2), N is *Rad*-$\oplus$-supplemented. By Corollary 2.5 and [9, Theorem 3.3], M is *Rad*-$\oplus$-supplemented. □

Note that Lemma 3.1 is not true for $\oplus$-supplemented modules, in general (see [9, Example 3.11].

Theorem 3.2. *Let R be a dedekind domain. Then every Rad-supplemented R-module is Rad-$\oplus$-supplemented.*

Proof. Let M be any *Rad*-supplemented module over the domain R. Then, by [2, 2.2.(2)], $\frac{M}{P(M)}$ is *Rad*-supplemented. Applying [4, Proposition 7.3], we conclude that $\frac{M}{P(M)}$ is supplemented. Therefore $\frac{M}{P(M)}$ is $\oplus$-supplemented according to [12, Proposition A.7 and Proposition A.8]. Hence M is *Rad*-$\oplus$-supplemented by Lemma 3.1. □

The structure of *Rad*-supplemented modules over local dedekind domains is completely determined in [4, Theorem 7.2]. Using this Theorem along with Theorem 3.2 we obtain:

Corollary 3.3. *Let R be a local dedekind domain with a quotient field K and let M be an R-module. Then M is Rad-$\oplus$-supplemented if and only if $M \cong R^n \oplus K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus N$ for some bounded R-module N. Here n is a nonnegative integer, and I and J are any index sets.*

Proof. This equivalence follows from Theorem 3.2 and [4, Theorem 7.2]. □

Zöschinger proved in [19, Theorem 3.1 (Folgerung)] that every supplemented module over a dedekind domain is the direct sum of hollow modules. Using this fact we obtain a new characterization of dedekind domains.

Proposition 3.4. *Let R be a local noetherian ring. Every Rad-supplemented R-module is the direct sum of hollow modules if and only if R is a dedekind domain.*

Proof. Suppose that R is a dedekind domain. Let M be any Rad-supplemented R-module. Then we can write $M = P(M) \oplus N$ for some submodule N of M. Since R is a local dedekind domain, $P(M)$ is the direct sum of hollow radical modules. By Lemma 3.1, N is supplemented and therefore N is the direct sum of hollow modules according to [19, Theorem 3.1 (Folgerung)]. It follows immediately that M is the direct sum of hollow modules. The converse is clear by [19, Lemma 3.2]. □

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References

[1] Al-Khazzi, I., and Smith, P.F., "Modules with chain conditions on superfluous submodules, *Communications in Algebra*, Vol. 19(8), pp. 2331-2351, 1991.

[2] Al-Takhman, K., Lomp, C., and Wisbauer, R., "τ-complemented and τ-supplemented modules", *Algebra Discrete Math.*, Vol. 3, pp. 1-16, 2006.

[3] Büyükaşık, E., and Lomp, C., "On a recent generalization of semiperfect rings", *Bull. Aust. Math. Soc.*, Vol. 78(2), pp. 317-325, 2008.

[4] Büyükaşık, E., Mermut, E., and Özdemir S., "Rad-supplemented modules", *Rend. Sem. Mat. Univ. Padova*, Vol. 124, pp. 157-177, 2010.

[5] Büyükaşık, E., and Demirci, Y., "Weakly distributive modules. Applications to supplement submodules", *Proc. Indian Acad. Sci.(Math. Sci.)*, Vol. 120, pp. 525-534, 2010.

[6] Clark, J., Lomp, C., Vanaja, N., and Wisbauer, R., "Lifting Modules. Supplements and projectivity in module theory", *Frontiers in Mathematics*, pp. 406, Birkhäuser-Basel, 2006.

[7] Ecevit, Ş., Koşan M. T., and Tribak, R., "*Rad*-$\oplus$-supplemented modules and cofinitely *Rad*-$\oplus$-supplemented modules", *Algebra Colloquium* (To appear).

[8] Harmancı, A., Keskin, D., and Smith, P.F., "On $\oplus$-supplemented modules", *Acta Math. Hungar.*, Vol. 83(1-2), pp. 161-169, 1999.

[9] Çalışıcı, H., and Türkmen, E., "Generalized $\oplus$-supplemented modules", *Algebra Discrete Math.*, Vol. 10(2), pp. 10-18, 2010.

[10] Idelhadj, A., and Tribak, R., "Modules for which every submodule has a supplement that is a direct summand", *The Arabian Journal for Sciences and Engineering*, Vol. 25(2C), pp. 179-189, 2000.

[11] Kasch, F., "Modules and rings", *Academic Press Inc.*, 1982.

[12] Mohamed, S.H., and Müller, B.J. "Continuous and discrete modules", *London Math. Soc. LNS 147 Cambridge University*, pp. 190, Cambridge, 1990.

[13] Özcan, A. Ç., Harmacı, A., and Smith, P.F., "Duo modules", *Glasgow Math. J.*, Vol. 48, pp. 533-545, 2006.

[14] Sharpe, D. W., and Vamos, P., Injective modules, *Cambridge University Press*, Cambridge, 1972.

[15] Talebi, Y., Hamzekolaei, A.R.M., and Tütüncü, D.K., "On *Rad*-$\oplus$-supplemented modules", *Hadronic J.*, Vol. 32, pp. 505-512, 2010.

[16] Türkmen, E., and Pancar, A., "Some properties of Rad-supplemented modules", *International Journal of the Physical Sciences*, Vol. 6(35), pp. 7904-7909, 2011.

[17] Wisbauer, R., "Foundations of modules and rings", *Gordon and Breach*, 1991.

[18] Xue, W., "Characterizations of semiperfect and perfect modules", *Publicacions Matematiqes*, Vol. 40(1), pp. 115-125, 1996.

[19] Zöschinger, H., "Komplementierte moduln über Dedekindringen", *J. Algebra*, Vol. 29, pp.42–56, 1974.

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