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On Rad-D_{12} Modules

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Abstract

Let M be a right R-module. We call M Rad-D_{12}, if for every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \to M/N$ such that $Ker\alpha \subseteq \text{Rad}(K)$. We show that a direct summand of a Rad-D_{12} module need not be a Rad-D_{12} module. We investigate completely Rad-D_{12} modules (modules for which every direct summand is a Rad-D_{12} module). We also show that a direct sum of Rad-D_{12} modules need not be a Rad-D_{12} module. Then we deal with some cases of direct sums of Rad-D_{12} modules.

1 Introduction

Throughout this paper, we assume that all rings are associative with identity and all modules are unital right modules. Let M be a module. The symbols, "$\leq$", "$\ll$" and "Rad(M)" will denote a submodule, a small submodule and the Jacobson radical of M, respectively. The module M is said to have (D_{12}) (or is a (D_{12})-*module*) if for every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \longrightarrow M/N$ such that $Ker\alpha \ll K$ (see [7]). In this paper we define Rad-D_{12} modules. The module M is said to have Rad-D_{12} (or is a Rad-D_{12} *module*) if for every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \longrightarrow M/N$ such that $Ker\alpha \subseteq \text{Rad}(K)$. It is easy to see that every radical module M (i.e.

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$\mathrm{Rad}(M) = M$) is a Rad-D_{12} module. Therefore the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$ is Rad-D_{12}, but it is not a (D_{12})-module.

Let M be a module. A submodule N of M is called a *weak* Rad-*supplement* (Rad-*supplement*) of a submodule L of M if $M = N + L$ and $N \cap L \subseteq \mathrm{Rad}(M)$ ($M = N + L$ and $N \cap L \subseteq \mathrm{Rad}(N)$). The module M is called *weakly* Rad-*supplemented* (Rad-*supplemented*) if every submodule N of M has a weak Rad-supplement (Rad-supplement). Rad-supplement submodule is defined in [13]. This new concept is also studied in [12] and [3]. According to [5], M is called Rad-$\oplus$-*supplemented* if every submodule of M has a Rad-supplement that is a direct summand of M.

In Section 2, we investigate some properties of Rad-D_{12} modules. We prove that the class of Rad-D_{12} modules contains strictly the class of Rad-$\oplus$-supplemented modules. In Section 3, we will be concerned with direct summands of Rad-D_{12} modules. We provide a characterization of direct summands having Rad-D_{12}. Section 4 deals with direct sums of Rad-D_{12} modules. We show that a direct sum of Rad-D_{12} modules is Rad-D_{12} if the direct sum is a duo module.

2 Rad-D_{12} modules

In this section we will show that the class of Rad-D_{12} modules contains properly the class of Rad-$\oplus$-supplemented modules.

Proposition 2.1. *Let M be a Rad-$\oplus$-supplemented module. Then M is Rad-D_{12}.*

Proof. Let N be a submodule of M. Since M is Rad-$\oplus$-supplemented, then there exist direct summands K and K' of M such that $M = N + K = K \oplus K'$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Now we have the epimorphism g from K to M/N which is defined by $k \mapsto k + N$ with $Ker g = N \cap K \subseteq \mathrm{Rad}(K)$. Hence M is a Rad-D_{12} module. $\square$

Example 2.2. [7, Examples 4.5 and 4.6] Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $dim(_QW) = 2$ and $dim(W_Q) = 1$. Consider the indecomposable injective right R-module $U = [(R \oplus R)/D]$ with $W = Ru + Rv$ and $D = \{(ur, -vr) \mid r \in R\}$. By [7, Example 4.5], U is not Rad-D_{12}. Note that U is Rad-supplemented. Now let $S = R/W$, the simple R-module, and $M = U \oplus S$. By [7, Example 4.6], M is Rad-D_{12}, but not Rad-$\oplus$-supplemented.

Example 2.3. Let $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^n\mathbb{Z})$ where p is a prime number and n is a nonzero positive integer. By [6, Corollary 1.6] and Proposition 2.1, M is Rad-D_{12}.

A module M is called *hereditary*, if every submodule of M is projective. Recall from [13] that a module M is called *generalized semiperfect* if for every factor module of M, namely M/N, there exist a projective module P and an epimorphism $f : P \longrightarrow M/N$ such that $Kerf \subseteq \mathrm{Rad}(P)$. In this case f is a *generalized projective cover* of M/N.

Theorem 2.4. *The following are equivalent for a hereditary module M:*
(1) M is generalized semiperfect;
(2) M is Rad-D_{12};
(3) M is Rad-$\oplus$-supplemented;
(4) M is Rad-supplemented.

Proof. (1) $\Rightarrow$ (4) By [13, Proposition 2.1].
(4) $\Rightarrow$ (3) It is by [11, Lemma 2.1].
(3) $\Rightarrow$ (2) By Proposition 2.1.
(2) $\Rightarrow$ (1) Clear. □

Let M be a module and $U \leq M$. Then U is called *QSL* in M if $(A+U)/U$ is a direct summand of M/U, then there exists a direct summand P of M such that $P \leq A$ and $A + U = P + U$ (see [1]).

Proposition 2.5. *Let M be a weakly Rad-supplemented module with Rad(M) QSL in M. Then M is Rad-D_{12}.*

Proof. Let $N \leq M$. Since M is weakly Rad-supplemented, $(N+\mathrm{Rad}(M))/\mathrm{Rad}(M)$ is a direct summand of $M/\mathrm{Rad}(M)$. Since $\mathrm{Rad}(M)$ is QSL in M, there exists a decomposition $M = K \oplus L$ such that $K \leq N$ and $N+\mathrm{Rad}(M) = K+\mathrm{Rad}(M)$. Now consider the epimorphism $\alpha : L \to M/N$ defined by $\alpha(l) = l + N$ $(l \in L)$. It is easy to see that $Ker\alpha \subseteq \mathrm{Rad}(L)$. Hence M is Rad-D_{12}. □

Let M be a module. We say that M is *w-local* if M has a unique maximal submodule. Clearly M is w-local if and only if $\mathrm{Rad}(M)$ is maximal in M.

Lemma 2.6. *Let M be a Rad-D_{12} module. If Rad$(M) \neq M$, then M has a nonzero w-local direct summand.*

Proof. Let N be a maximal submodule of M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \longrightarrow M/N$ such that $Ker\alpha \subseteq \mathrm{Rad}(K)$. Clearly, $K \neq 0$ and $Ker\alpha$ is a maximal submodule of K. Therefore $Ker\alpha = \mathrm{Rad}(K)$ and hence K is a nonzero w-local direct summand of M. □

Corollary 2.7. *If M is a Rad-D_{12} module with Rad$(M) \ll M$, then M contains a local direct summand.*

Proof. Since $\mathrm{Rad}(M) \ll M$, M is a (D_{12})-module. Now apply the proof of Lemma 2.6. □

3 Direct summands of Rad-D_{12} modules

The following example exhibits a Rad-D_{12} module that contains a direct summand which is not a Rad-D_{12} module.

Example 3.1. Consider the right R-module $M = U \oplus S$ in Example 2.2. The module M is Rad-D_{12}, but the submodule U is not Rad-D_{12}.

Let M be a module. We will say that M is *completely* Rad-D_{12} if every direct summand of M is Rad-D_{12}.

Recall from [2] that a module M is said to have (P^*) *property* if for any submodule N of M there exists a direct summand D of M such that $D \subseteq N$ and $N/D \subseteq \mathrm{Rad}(M/D)$, equivalently, for every submodule N of M there exists a decomposition $M = K \oplus K'$ such that $K \subseteq N$ and $N \cap K' \subseteq \mathrm{Rad}(K')$. It is easy to check that every module with (P^*) is Rad-$\oplus$-supplemented and hence Rad-D_{12} by Proposition 2.1.

Proposition 3.2. *A module with (P^*) property is completely Rad-D_{12}.*

Proof. By [2, Lemma 16], every direct summand of a module with (P^*) has (P^*). Now the result follows from the fact that every module with (P^*) is Rad-D_{12}. □

Example 3.3. (i) Let F be a field and R the upper triangular matrix ring $\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $M = A \oplus (R/B)$. By [8, Lemma 3], M has (P^*). So by Proposition 3.2, M is completely Rad-D_{12}.

(ii) Let $M = \mathbb{Z}(_{p_1^\infty}) \oplus ... \oplus \mathbb{Z}(_{p_n^\infty})$ where $p_1, \dots, p_n$ are distinct prime integers. By [9, Example 2.16], M has (P^*). Hence M is completely Rad-D_{12}.

The converse of Proposition 3.2 is not true as we see in following example.

Example 3.4. Let M be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Since M is finitely generated, M does not have (P^*) by [8, Example 10]. By [6, Theorem 1.4], M is $\oplus$-supplemented and hence Rad-$\oplus$-supplemented. By [10, Example 2.10], every direct summand of M is $\oplus$-supplemented and hence Rad-$\oplus$-supplemented. Therefore by Proposition 2.1, M is completely Rad-D_{12}.

A module M is called *refinable* if for any submodules U, V of M with $M = U + V$, there exists a direct summand U' of M with $U' \subseteq U$ and $M = U' + V$ (see [4, 11. 26]). It is easy to prove that M is refinable iff every submodule of M is QSL.

Proposition 3.5. *Let M be a weakly Rad-supplemented refinable module. Then M is Rad-D_{12}.*

Proof. By Proposition 2.5. $\square$

Corollary 3.6. *Every weakly Rad-supplemented refinable module is completely Rad-D_{12}.*

Proof. This is a consequence of Proposition 3.5 and the fact that every direct summand of a weakly Rad-supplemented refinable module is weakly Rad-supplemented refinable. $\square$

Let M be an R-module. By $P(M)$ we denote the sum of radical submodules of M.

Proposition 3.7. *Let M be a Rad-D_{12} module. If $P(M)$ is a direct summand of M, then $P(M)$ is a Rad-D_{12} module.*

Proof. Let $M = P(M) \oplus L$ for some submodule L of M. Let X be a submodule of $P(M)$. By hypothesis, there exist a direct summand K of M and an epimorphism $\alpha : K \longrightarrow M/(X \oplus L)$ such that $Ker\alpha \subseteq \mathrm{Rad}(K)$. It is clear that $M/(X \oplus L) \cong P(M)/X$. Thus $\mathrm{Rad}(K/Ker\alpha) = K/Ker\alpha$, and so $\mathrm{Rad}(K) = K$. Therefore $K \subseteq P(M)$. This means that $P(M)$ is Rad-D_{12}. $\square$

The following result gives a new characterization of direct summands having Rad-D_{12}.

Theorem 3.8. *Let $M = M_1 \oplus M_2$. Then M_2 is a Rad-D_{12} module if and only if for every submodule N of M containing M_1, there exist a direct summand K of M_2 and an epimorphism $\varphi : M \longrightarrow M/N$ such that K is a direct summand Rad-supplement of $Ker\varphi$ in M.*

Proof. Suppose that M_2 is a Rad-D_{12} module. Let $N \leq M$ with $M_1 \subseteq N$. Consider the submodule $N \cap M_2$ of M_2. Then there exist a direct summand K of M_2 and an epimorphism $\alpha : K \longrightarrow M_2/(N \cap M_2)$ such that $Ker\alpha \subseteq \mathrm{Rad}(K)$. Note that $M = N + M_2$ and K is a direct summand of M. Let $M = K \oplus K'$ for some submodule K' of M. Consider the projection map $\eta : M \longrightarrow K$ and the isomorphism $\beta : M_2/(N \cap M_2) \longrightarrow M/N$ defined by $\beta(x + N \cap M_2) = x + N$. Thus $\beta\alpha\eta : M \longrightarrow M/N$ is an epimorphism. Let $\varphi = \beta\alpha\eta$. Clearly, we have $Ker\varphi = Ker\alpha \oplus K'$. Therefore $M = K + Ker\varphi$. Moreover $K \cap Ker\varphi = Ker\alpha \subseteq \mathrm{Rad}(K)$.

Conversely, suppose that every submodule of M containing M_1 has the stated property. Let H be a submodule of M_2. Consider the submodule $H \oplus M_1$ of M. By hypothesis, there exist a direct summand K of M_2 and an epimorphism $\varphi : M \longrightarrow M/(H \oplus M_1)$ such that $M = K + Ker\varphi$ and $K \cap Ker\varphi \subseteq \mathrm{Rad}(K)$. Let $f : K \longrightarrow M/(H \oplus M_1)$ be the restriction of φ to K. Consider the isomorphism $\eta : M/(H \oplus M_1) \longrightarrow M_2/H$ defined by $\eta(m_1 + m_2 + (H \oplus M_1)) = m_2 + H$. Therefore $\eta f : K \longrightarrow M_2/H$ is an epimorphism. Let $\alpha = \eta f$. Clearly, $Ker\alpha = Kerf = K \cap Ker\varphi$. Thus $Ker\alpha \subseteq \mathrm{Rad}(K)$. Hence M_2 is a Rad-D_{12} module. □

4 Direct sums of Rad-D_{12} modules

We begin this section by giving an example showing that the class of Rad-D_{12} modules is not closed under direct sums.

Example 4.1. Let R be a discrete valuation ring and let K be its quotient field. There exist a free module F and a submodule X of F such that $F/X \cong K$ since every module is a homomorphic image of a free module. Then F is not Rad-$\oplus$-supplemented by [5, Example 2.15]. Since R is a hereditary ring, then F is hereditary. Therefore F cannot be Rad-D_{12} from Theorem 2.4. Note that since $F \cong \oplus_{i \in I} R$ and R is local, F is a direct sum of *Rad*-D_{12}-modules.

Let M be a module. M is called a *duo module* if every submodule of M is fully invariant. We next give a sufficient condition for arbitrary direct sums of Rad-D_{12} modules to be Rad-D_{12}.

Theorem 4.2. *Let $M = \bigoplus_{i\in I} M_i$ be a duo module. If each M_i is Rad-D_{12}, then M is Rad-D_{12}.*

Proof. Let L be a submodule of M. Since M is a duo module we have $L = \bigoplus_{i\in I}(L \cap M_i)$. Let $i \in I$. Because M_i is Rad-D_{12} and $L \cap M_i$ is a submodule of M_i, there exist a direct summand K_i of M_i and an epimorphism $\alpha_i : K_i \to \frac{M_i}{L\cap M_i}$ with $Ker\alpha_i \subseteq \mathrm{Rad}(K_i)$. Now we define the homomorphism $\alpha : \bigoplus_{i\in I} K_i \to \bigoplus_{i\in I}[\frac{M_i}{(L\cap M_i)}] \cong \frac{M}{[\bigoplus_{i\in I}(L\cap M_i)]} = \frac{M}{L}$ by $k_{i_1} + \ldots + k_{i_n} \mapsto \alpha_{i_1}(k_{i_1}) + \ldots + \alpha_{i_n}(k_{i_n})$ with $k_{i_j} \in K_{i_j}$ for every $j = 1, \ldots, n$. It is not hard to check that α is an epimorphism with $Ker\alpha \subseteq \mathrm{Rad}(\bigoplus_{i\in I} K_i)$ and $\bigoplus_{i\in I} K_i$ is a direct summand of M. It follows that M is Rad-D_{12}. □

Recall that a module M has *Summand Intersection Property* (SIP), if the intersection of any two direct summands of M is again a direct summand of M. By [10, Page 969], every duo module has SIP.

Remark 4.3. Being duo module in Theorem 4.2 is not necessary. The module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ in Example 3.4 is not a duo module (M doesn't have SIP). Also $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$ and M are Rad-D_{12}.

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