DOI: 10.2478/auom-2013-0011
An. Şt. Univ. Ovidius Constanţa
VERSITA
Vol. 21(1), 2013, 183–200

Strong convergence theorems for a sequence of nonexpansive mappings with gauge functions

Prasit Cholamjiak, Yeol Je Cho, Suthep Suantai

Abstract

In this paper, we first prove a path convergence theorem for a nonexpansive mapping in a reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function on $[0, \infty)$. Using this result, strong convergence theorems for common fixed points of a countable family of nonexpansive mappings are established.

1 Introduction

Let K be a nonempty, closed and convex subset of a real Banach space E. Let $T : K \to K$ be a nonlinear mapping. We denote by $F(T)$ the fixed points set of T, that is, $F(T) = \{x \in K : \ x = Tx\}$. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.$$

One classical way to study convergence of nonexpansive mappings is to use path convergence for approximating the fixed point of mappings [3, 18, 27]. For any $t \in (0, 1)$, we define the mapping $T_t : K \to K$ as follows:

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in K, \tag{1.1}$$

Key Words: Common fixed point, Gauge function, Modified Mann iteration, Nonexpansive mapping, Reflexive Banach space.
2010 Mathematics Subject Classification: 47H09, 47H10.
Received: October, 2011.
Accepted: August, 2012.

where $u \in K$ is fixed. Banach's contraction principle ensures that T_t has a unique fixed point x_t in K satisfying

$$x_t = tu + (1-t)Tx_t. \tag{1.2}$$

Browder [3] first proved that, if E is a real Hilbert space, then $\{x_t\}$ converges strongly to a fixed point of T. Reich [18] showed that Browder's results also valid in a uniformly smooth Banach space. In 2006, Xu [27] proved that Browder's result holds in a reflexive Banach space which has a weakly continuous duality mapping.

On the other hand, Gossez-Lami gave in [9] some geometric properties related to the fixed point theory for nonexpansive mappings. They proved that a space with a weakly continuous duality mapping satisfies Opial's condition [14]. It is also known that all Hilbert spaces and ℓ^p $(1 < p < \infty)$ satisfy the Opial's condition. However, the L^p $(1 < p < \infty)$ spaces do not unless $p = 2$. In this connection, we focus our aim to study a path convergence of (1.2) in a different setting, a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm concerning a gauge function [4]. We note that our class of Banach spaces includes the spaces L^p, ℓ^p $(1 < p < \infty)$ and the Sobolev spaces W^p_m $(1 < p < \infty)$. Moreover, the duality mappings associated with gauge functions also include the generalized and the normalized duality mappings as special cases.

In 1953, Mann [11] introduced the iterative scheme $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\} \subset (0,1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [18]). Since 1953, many authors have constructed and proposed the modified version of algorithm (1.3) in order to get strong convergence results (see [5, 6, 10, 13, 16, 24, 26, 29, 30] and the references cited therein). Several applications related to the Mann iterative scheme can be found in [17].

Kim-Xu [10] introduced the following modified Mann's iteration as follows:

$$\begin{cases} x_0 = x \in K, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where T is a nonexpansive mapping of K into itself and $u \in K$ is fixed. They proved, in a uniformly smooth Banach space, that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of T if the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Recently, Qin et al. [16] introduced the following iteration:

$$\begin{cases} x_0 = x \in K, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where W_n is the W-mapping [20] generated by nonexpansive self mappings $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$ and $u \in K$ is fixed. They proved, in a reflexive strictly convex Banach space which has a weakly continuous duality mapping j_φ, that the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$ if the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Let K be a nonempty, closed and convex subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty} : K \to K$ be a sequence of nonexpansive mappings.

Motivated by the works mentioned above, we consider the following modified Mann-type iteration:

$$\begin{cases} u, x_1 \in K, \\ y_n = \beta_n x_n + (1-\beta_n) T_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In this paper, we first prove a path convergence for a nonexpansive mapping in a real reflexive and strictly convex Banach space which has a Gâteaux differentiable norm and admits the duality mapping associated with a gauge function. Then we discuss strong convergence of the modified Mann-type iteration process (1.6) for a countable family of nonexpansive mappings. Our results improve and extend the recent ones announced by many authors.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \ \|x\|, \|y\| \le 1, \ \|x-y\| \ge \epsilon\Big\}, \quad \forall \epsilon \in [0,2].$$

It is known that a Banach space E is uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \le 2$ and every uniformly convex Banach space is strictly convex and reflexive.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for any $x, y \in S(E)$. In this case, E is called *smooth.* The norm of E is said to be *uniformly Gâteaux differentiable* if, for any $y \in S(E)$, the limit is attained uniformly for all $x \in S(E)$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \ x \in S(E), \ \|y\| \le t\Big\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$ (see [1, 7, 23] for more details).

We recall the following definitions and results which can be found in [1, 4, 7].

Definition 2.1. A continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is called the *gauge function* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2. Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \big\{f^* \in E^* : \ \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|)\big\}, \quad \forall x \in E,$$

is called the *duality mapping* with gauge function φ.

In particular, if $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping.* If $\varphi(t) = t^{q-1}$ for any $q > 1$, then the duality mapping $J_\varphi = J_q$ is called the *generalized duality mapping.*

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ and $J_q(x) = \|x\|^{q-2} J(x)$ for any $q > 1$.

Remark 2.3. [1] For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.1}$$

is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Remark 2.4. [1, 7] For any x in a Banach space E, $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the sub-differential.

We know the following subdifferential inequality:

$$\Phi\big(\|x+y\|\big) \le \Phi\big(\|x\|\big) + \big\langle y, j_\varphi(x+y)\big\rangle, \quad \forall j_\varphi(x+y) \in J_\varphi(x+y). \tag{2.2}$$

We also know the following facts (see [1]):
(1) J_φ is a nonempty, closed and convex set in E^* for any $x \in E$.
(2) J_φ is a function when E^* is strictly convex.
(3) If J_φ is single-valued, then

$$J_\varphi(\lambda x) = \frac{sign(\lambda)\varphi(\|\lambda x\|)}{\varphi(\|x\|)} J_\varphi(x), \quad \forall x \in E,\ \lambda \in \mathbb{R},$$

and

$$\langle x-y, J_\varphi(x) - J_\varphi(y)\rangle \ge \big(\varphi(\|x\|) - \varphi(\|y\|)\big)\big(\|x\| - \|y\|\big), \quad \forall x, y \in E.$$

If E is a smooth Banach space, then J_φ is single-valued and also denoted by j_φ.

Remark 2.5. [8] Suppose E has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Then j_φ is uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E.

We next give the definition of Banach limit.

Definition 2.6. Let μ be a continuous linear functional on ℓ^∞ and let $(a_0, a_1, \cdots) \in \ell^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for each $(a_0, a_1, \cdots) \in \ell^\infty$.

For a Banach limit μ, we know that

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$$

for all $a = (a_0, a_1, \cdots) \in \ell^\infty$. Therefore, if $a = (a_0, a_1, \cdots) \in \ell^\infty, b = (b_0, b_1, \cdots) \in \ell^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, then we have $\mu_n(a_n) = \mu_n(b_n)$ (see [1, 7, 23, 25]).

In the sequel, we need the following crucial lemmas:

Lemma 2.7. [21] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \quad \forall n \geq 1,$$

where $\{\beta_n\}$ is a real sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. If $\limsup_{n\to\infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.8. [28] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

To deal with a family of mappings, we consider the following condition:

Let K be a subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [2] if, for any bounded subset B of K,

$$\sum_{n=1}^{\infty} \sup \big\{\|T_{n+1}z - T_n z\| : z \in B\big\} < \infty.$$

Lemma 2.9. [2] *Let K be a nonempty and closed subset of a Banach space E and $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition. Then, for any $x \in K$, $\{T_n x\}$ converges strongly to a point in K. Moreover, let the mapping T be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in K.$$

Then, for each bounded subset B of K,

$$\lim_{n\to\infty} \sup \big\{\|Tz - T_n z\| : z \in B\big\} = 0.$$

In the sequel, we write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.9 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Example 2.10. Let $T_1, T_2, \cdots,$ be an infinite family of nonexpansive mappings of K into itself and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_i < 1$ for all $i \in \mathbb{N}$. Moreover, let W_n and W be the W-mappings [20] generated by $T_1, T_2, \cdots, T_n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$, and $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$. Then $\big(\{W_n\}, W\big)$ satisfies the AKTT-condition (see [15, 20]).

Example 2.11. Let $T_1, T_2, \cdots$ be an infinite family of nonexpansive mappings of K into itself. For each $n \in \mathbb{N}$, define the mapping $V_n : K \to K$ by

$$V_n x = \sum_{i=1}^{n} \lambda_n^i T_i x, \quad \forall x \in K,$$

where $\{\lambda_n^i\}$ is a family of nonnegative numbers satisfying the following conditions:

(a) $\sum_{i=1}^{n} \lambda_n^i = 1$ for each $n \in \mathbb{N}$;
(b) $\lambda^i := \lim_{n\to\infty} \lambda_n^i > 0$ for each $i \in \mathbb{N}$;
(c) $\sum_{n=1}^{\infty} \sum_{i=1}^{n} |\lambda_{n+1}^i - \lambda_n^i| < \infty$.

Let $V : K \to K$ be the mapping defined by

$$Vx = \sum_{i=1}^{\infty} \lambda^i T_i x, \quad \forall x \in K.$$

Then $\big(\{V_n\}, V\big)$ satisfies the AKTT-condition (see [2]).

3 Path convergence theorem

Now, we denote the subset K' of K by

$$K' = \Big\{x \in K : \ \mu_n \Phi\big(\|x_n - x\|\big) = \inf_{y \in K} \mu_n \Phi\big(\|x_n - y\|\big)\Big\},$$

where Φ is the function defined by (2.1).

Proposition 3.1. [8] *Let K be a nonempty, closed and convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Suppose that $\{x_n\}$ is a bounded sequence of K. Let μ_n be a Banach limit and $z \in K$. Then $z \in K'$ if and only if*

$$\mu_n \big\langle y - z, j_\varphi(x_n - z)\big\rangle \le 0, \quad \forall y \in K.$$

Proposition 3.2. *Let K be a nonempty, closed and convex subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a bounded sequence in K with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F(T) \cap K' \neq \emptyset$.*

Proof. Set $g(y) = \mu_n \Phi\big(\|x_n - y\|\big)$ for all $y \in K$. Then g is convex and continuous since Φ is convex and continuous. Further, $g(y_m) \to \infty$ as $\|y_m\| \to \infty$ since $\varphi(\|y_m\|) \to \infty$ as $\|y_m\| \to \infty$. Since E is reflexive, by Theorem 1.3.11 in [23], there exists $z \in K$ such that $g(z) = \inf_{y\in K} g(y)$. Hence K' is nonempty. Further, K' is closed and convex since g is continuous and convex. For any $x \in K'$, we have

$$\begin{aligned} g(Tx) &= \mu_n \Phi\big(\|x_n - Tx\|\big) \\ &\leq \mu_n \Phi\big(\|x_n - Tx_n\| + \|Tx_n - Tx\|\big) \\ &\leq \mu_n \Phi\big(\|x_n - x\|\big) \\ &= g(x). \end{aligned}$$

Therefore, $Tx \in K'$ for all $x \in K'$.

Let $p \in F(T)$. By Day-James's theorem [12], we know that there exists a unique element $v \in K'$ such that

$$\|p - v\| = \inf_{x\in K'} \|p - x\|.$$

Since $p = Tp$ and $Tv \in K'$, we have

$$\|p - Tv\| = \|Tp - Tv\| \leq \|p - v\| \leq \|p - Tv\|.$$

It follows that $v = Tv$ since E is strictly convex. Hence $v \in F(T) \cap K'$. This completes the proof. □

Using Propositions 3.1 and 3.2, we next prove a path convergence theorem, which is important to prove our main theorem.

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a real reflexive and strictly Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $T : K \to K$ be a nonexpansive such that $F(T) \neq \emptyset$. Fix $u \in K$ and let $t \in (0, 1)$. Then the net $\{x_t\}$ defined by (1.2) converges strongly as $t \to 0$ to a fixed point p of T which solves the variational inequality:*

$$\big\langle u - p, j_\varphi(w - p)\big\rangle \leq 0, \quad \forall w \in F(T). \tag{3.1}$$

Proof. First, we prove that the solution of variational inequality (3.1) is unique. Suppose that $p, q \in F(T)$ satisfy (3.1). Then we have

$$\big\langle u - p, j_\varphi(q - p)\big\rangle \le 0, \quad \big\langle u - q, j_\varphi(p - q)\big\rangle \le 0.$$

Adding the above inequalities, we obtain

$$\big\langle p - q, j_\varphi(p - q)\big\rangle \le 0,$$

which implies that

$$\|p - q\|\varphi(\|p - q\|) \le 0$$

and so $p = q$.

Next, we prove that $\{x_t\}$ is bounded in K. For any $w \in F(T)$, we see that

$$\begin{aligned}
& \|x_t - w\|\varphi(\|x_t - w\|) \\
= \; & \big\langle x_t - w, j_\varphi(x_t - w)\big\rangle \\
= \; & t\big\langle u - w, j_\varphi(x_t - w)\big\rangle + (1 - t)\big\langle Tx_t - w, j_\varphi(x_t - w)\big\rangle \\
\le \; & t\big\langle u - w, j_\varphi(x_t - w)\big\rangle + (1 - t)\|x_t - w\|\varphi\big(\|x_t - w\|\big),
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_t - w\|\varphi(\|x_t - w\|) &\le \big\langle u - w, j_\varphi(x_t - w)\big\rangle \\
&\le \|u - w\|\varphi\big(\|x_t - w\|\big). \qquad (3.2)
\end{aligned}$$

Hence $\|x_t - w\| \le \|u - w\|$ and, consequently, $\{x_t\}$ is bounded. So is $\{Tx_t\}$. We see that

$$\|x_t - Tx_t\| = t\|u - Tx_t\| \to 0 \quad (t \to 0).$$

Since E is reflexive, $\{x_t\}$ has a weakly convergent subsequence $\{x_{t_n}\}$. Thus $\{x_{t_n}\}$ is bounded. Putting $x_n := x_{t_n}$, in particular, we also have

$$\|x_n - Tx_n\| \to 0 \quad (n \to \infty).$$

By Proposition 3.2, since $\{x_n\}$ is bounded, there exists $p \in F(T)$ such that

$$\mu_n\Phi\big(\|x_n - p\|\big) = \inf_{y \in K} \mu_n\Phi\big(\|x_n - y\|\big).$$

It follows from Proposition 3.1 that

$$\mu_n\big\langle y - p, j_\varphi(x_n - p)\big\rangle \le 0, \quad \forall y \in K.$$

Since $u \in K$, in particular, we have

$$\mu_n\big\langle u - p, j_\varphi(x_n - p)\big\rangle \le 0. \qquad (3.3)$$

Observe that

$$\Phi(\|y\|) = \int_0^{\|y\|} \varphi(s)ds \le \|y\|\varphi(\|y\|).$$

It follows from (3.2) and (3.3) that

$$\mu_n \Phi\big(\|x_n - p\|\big) \le \mu_n \big\langle u - p, j_\varphi(x_n - p)\big\rangle \le 0$$

and hence

$$\mu_n \Phi\big(\|x_n - p\|\big) = 0. \tag{3.4}$$

Since Φ is continuous, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. From (3.4), we have

$$\mu_j \Phi\big(\|x_{n_j} - p\|\big) = \Phi\big(\|q - p\|\big) = 0$$

and so $p = q$. Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point p of T.

Next, we prove that $p \in F(T)$ is a solution to the variational inequality (3.1). For any $w \in F(T)$, we see that

$$\begin{aligned}
\|x_n - w\|\varphi\big(\|x_n - w\|\big) &= \big\langle x_n - w, j_\varphi(x_n - w)\big\rangle \\
&= t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\big\langle p - x_n, j_\varphi(x_n - w)\big\rangle \\
&\quad + t_n\big\langle x_n - w, j_\varphi(x_n - w)\big\rangle \\
&\quad + (1 - t_n)\big\langle Tx_n - w, j_\varphi(x_n - w)\big\rangle \\
&\le t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\|x_n - p\|\varphi\big(\|x_n - w\|\big) \\
&\quad + t_n\|x_n - w\|\varphi(\|x_n - w\|) \\
&\quad + (1 - t_n)\|x_n - w\|\varphi\big(\|x_n - w\|\big) \\
&= t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\|x_n - p\|\varphi\big(\|x_n - w\|\big) \\
&\quad + \|x_n - w\|\varphi\big(\|x_n - w\|\big).
\end{aligned}$$

This implies that

$$\big\langle u - p, j_\varphi(w - x_n)\big\rangle \le \|x_n - p\|\varphi\big(\|x_n - w\|\big). \tag{3.5}$$

Since j_φ is norm-weak* uniformly continuous on bounded subsets of E, we have

$$\big\langle u - p, j_\varphi(w - x_n)\big\rangle \to \big\langle u - p, j_\varphi(w - p)\big\rangle \quad (n \to \infty).$$

Thus, taking the limit as $n \to \infty$ in both sides of (3.5), we get

$$\big\langle u - p, j_\varphi(w - p)\big\rangle \le 0, \quad \forall w \in F(T).$$

Finally, we prove that $x_t \to p$ as $t \to 0$. To this end, let $\{x_{s_n}\}$ be another subsequence of $\{x_t\}$ such that $x_{s_n} \to p'$ as $n \to \infty$. We have to show that $p = p'$. For any $w \in F(T)$, we have

$$\begin{aligned}\big\langle Tx_t - x_t, j_\varphi(x_t - w)\big\rangle &= \big\langle Tx_t - w, j_\varphi(x_t - w)\big\rangle + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &\leq \|x_t - w\|\varphi\big(\|x_t - w\|\big) + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &= \big\langle x_t - w, j_\varphi(x_t - w)\big\rangle + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &= 0.\end{aligned}$$

On the other hand, since

$$x_t - Tx_t = \frac{t}{1-t}(u - x_t),$$

we have

$$\big\langle x_t - u, j_\varphi(x_t - w)\big\rangle \leq 0, \quad \forall w \in F(T).$$

In particular, we have

$$\big\langle x_{t_n} - u, j_\varphi(x_{t_n} - p')\big\rangle \leq 0$$

and

$$\big\langle x_{s_n} - u, j_\varphi(x_{s_n} - p)\big\rangle \leq 0$$

or, equivalently,

$$\|x_{t_n} - p'\|\varphi\big(\|x_{t_n} - p'\|\big) + \big\langle p' - u, j_\varphi(x_{t_n} - p')\big\rangle \leq 0$$

and

$$\|x_{s_n} - p\|\varphi\big(\|x_{s_n} - p\|\big) + \big\langle p - u, j_\varphi(x_{s_n} - p)\big\rangle \leq 0.$$

Taking the limit as $n \to \infty$, since φ is continuous and j_φ is norm-to-weak* uniformly continuous on bounded subsets of E, we obtain

$$\|p - p'\|\varphi\big(\|p - p'\|\big) + \big\langle p' - u, j_\varphi(p - p')\big\rangle \leq 0$$

and

$$\|p' - p\|\varphi\big(\|p' - p\|\big) + \big\langle p - u, j_\varphi(p' - p)\big\rangle \leq 0.$$

Summing the above inequalities, we also have

$$2\|p - p'\|\varphi\big(\|p - p'\|\big) + \big\langle p' - p, j_\varphi(p - p')\big\rangle \leq 0.$$

This implies that

$$\big\langle p - p', j_\varphi(p - p')\big\rangle \leq 0$$

and hence $p = p'$. Therefore, $\{x_t\}$ converges strongly to a fixed point of T. This completes the proof. $\square$

4 Strong convergence theorems

In this section, using Theorem 3.3, we prove a strong convergence theorem in a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function on $[0, \infty)$.

Theorem 4.1. *Let K be a nonempty closed and convex subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\{T_n\}_{n=1}^\infty : K \to K$ be a sequence of nonexpansive mappings such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let $u \in K$ be fixed. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^\infty \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.6) converge strongly to $p \in F$ which also solves the variational inequality (3.1).

Proof. First, we see that the sequences $\{x_n\}$ and $\{y_n\}$ is bounded. In fact, for any $w \in F$, we have

$$\|y_n - w\| \leq \beta_n \|x_n - w\| + (1-\beta_n)\|T_n x_n - w\| \leq \|x_n - w\|$$

and so

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|u - w\| + (1-\alpha_n)\|y_n - w\| \\ &\leq \alpha_n \|u - w\| + (1-\alpha_n)\|x_n - w\| \\ &\leq \max\Big\{\|x_n - w\|, \|u - w\|\Big\}. \end{aligned}$$

Hence the sequence $\{x_n\}$ is bounded by induction and so is $\{y_n\}$.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Putting $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we get

$$x_{n+1} = (1-\beta_n) l_n + \beta_n x_n, \quad \forall n \geq 1.$$

Thus we have

$$\begin{aligned}
& l_{n+1} - l_n \\
= \quad & \frac{\alpha_{n+1}u + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\
= \quad & \frac{\alpha_{n+1}(u-y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(u-y_n)}{1-\beta_n} + T_{n+1}x_{n+1} - T_n x_n,
\end{aligned}$$

which implies

$$\begin{aligned}
& \|l_{n+1} - l_n\| \\
\leq \quad & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-y_n\| + \|x_{n+1}-x_n\| + \|T_{n+1}x_n - T_n x_n\| \\
\leq \quad & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-y_n\| + \|x_{n+1}-x_n\| + \sup_{z\in\{x_n\}}\|T_{n+1}z - T_n z\|.
\end{aligned}$$

Since $\{T_n\}$ satisfies the AKTT-condition, it follows from the conditions (a) and (c) that

$$\limsup_{n\to\infty}\big(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|\big) \leq 0.$$

By Lemma 2.7, we also obtain

$$\lim_{n\to\infty}\|l_n - x_n\| = 0.$$

Since

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n),$$

we have

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|l_n - x_n\| \to 0 \quad (n\to\infty). \tag{4.1}$$

On the other hand, we see that

$$\|x_{n+1} - y_n\| = \alpha_n\|u - y_n\| \to 0 \quad (n\to\infty). \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{4.3}$$

Noting that

$$\begin{aligned}
\|x_n - T_n x_n\| \quad & \leq \quad \|x_n - y_n\| + \|y_n - T_n x_n\| \\
& = \quad \|x_n - y_n\| + \beta_n\|x_n - T_n x_n\|,
\end{aligned}$$

from (4.3) and the condition (c), we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{4.4}$$

Further, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\leq \|x_n - T_n x_n\| + \sup_{z\in\{x_n\}} \|T_n z - Tz\|. \end{aligned}$$

Thus, by Lemma 2.9 and (4.4), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.5}$$

Since T is nonexpansive, by Theorem 3.3, we know that the net $\{x_t\}$ generated by (1.2) converges strongly to a fixed point $p \in F(T) = F$ which also solves the variational inequality (3.1).

Next, we prove that

$$\limsup_{n\to\infty} \big\langle u - p, j_\varphi(x_n - p)\big\rangle \leq 0.$$

Observe that

$$\begin{aligned} & \|x_t - x_n\|\varphi\big(\|x_t - x_n\|\big) \\ = \;& t\big\langle u - x_n, j_\varphi(x_t - x_n)\big\rangle + (1-t)\big\langle Tx_t - x_n, j_\varphi(x_t - x_n)\big\rangle \\ = \;& t\big\langle p - x_t, j_\varphi(x_t - x_n)\big\rangle + t\big\langle u - p, j_\varphi(x_t - x_n)\big\rangle \\ & + t\big\langle x_t - x_n, j_\varphi(x_t - x_n)\big\rangle + (1-t)\big\langle Tx_t - Tx_n, j_\varphi(x_t - x_n)\big\rangle \\ & + (1-t)\big\langle Tx_n - x_n, j_\varphi(x_t - x_n)\big\rangle \\ \leq \;& t\|p - x_t\|\varphi(\|x_t - x_n\|) + t\big\langle u - p, j_\varphi(x_t - x_n)\big\rangle \\ & + \|x_t - x_n\|\varphi(\|x_t - x_n\|) + \|Tx_n - x_n\|\varphi(\|x_t - x_n\|). \end{aligned}$$

Therefore, it follows that

$$\big\langle u - p, j_\varphi(x_n - x_t)\big\rangle \leq \frac{\|Tx_n - x_n\|\varphi(\|x_t - x_n\|)}{t} + \|x_t - p\|\varphi(\|x_t - x_n\|). \tag{4.6}$$

Using (4.5) and taking the limit as $n \to \infty$ first and then, as $t \to 0$, the inequality (4.6) becomes

$$\limsup_{t\to 0}\limsup_{n\to\infty} \big\langle u - p, j_\varphi(x_n - x_t)\big\rangle \leq 0. \tag{4.7}$$

Since j_φ is norm-weak* uniformly continuous on bounded sets,

$$\big\langle u-p, j_\varphi(x_n-x_t)\big\rangle \to \big\langle u-p, j_\varphi(x_n-p)\big\rangle \quad (t\to 0).$$

We see that

$$\big\langle u-p, j_\varphi(x_n-p)\big\rangle = \big\langle u-p, j_\varphi(x_n-x_t)\big\rangle + \big\langle u-p, j_\varphi(x_n-p)-j_\varphi(x_n-x_t)\big\rangle.$$

By the uniform continuity of j_φ, we can interchange the two limits above and deduce that

$$\limsup_{n\to\infty} \big\langle u-p, j_\varphi(x_n-p)\big\rangle \le 0. \tag{4.8}$$

Finally, we prove that $x_n \to p$ as $n\to\infty$. Observe that

$$\begin{aligned}
\Phi\big(\|y_n-p\|\big) &= \Phi\big(\|\beta_n(x_n-p)+(1-\beta_n)(T_nx_n-p)\|\big)\\
&\le \beta_n\Phi\big(\|x_n-p\|\big)+(1-\beta_n)\Phi\big(\|T_nx_n-p\|\big)\\
&\le \Phi\big(\|x_n-p\|\big).
\end{aligned}$$

Form (2.2), it follows that

$$\begin{aligned}
\Phi\big(\|x_{n+1}-p\|\big) &= \Phi\big(\|\alpha_n(u-p)+(1-\alpha_n)(y_n-p)\|\big)\\
&\le \Phi\big((1-\alpha_n)\|y_n-p\|\big)+\alpha_n\big\langle u-p, j_\varphi(x_{n+1}-p)\big\rangle\\
&\le (1-\alpha_n)\Phi\big(\|x_n-p\|\big)+\alpha_n\big\langle u-p, j_\varphi(x_{n+1}-p)\big\rangle.
\end{aligned}$$

Applying Lemma 2.8, we have $\Phi\big(\|x_n-p\|\big)\to 0$ as $n\to\infty$ by the condition (b) and (4.8). Hence $x_n\to p$ as $n\to\infty$ since Φ is continuous. Moreover, the sequence $\{y_n\}$ also strongly converges to p. This completes the proof. □

Remark 4.2. From Examples 2.10 and 2.11, the ordered pair $\big(\{T_n\},T\big)$ in Theorem 4.1 can be replaced by $\big(\{W_n\},W\big)$ and $\big(\{V_n\},V\big)$.

Remark 4.3. Theorem 4.1 mainly improves and extends the results of Kim-Xu [10] in the following aspects:

(1) we relax the restrictions imposed on the parameters in Theorem 1 of [10];

(2) we extend Theorem 1 of [10] from a single nonexpansive mapping to an infinite family of nonexpansive mappings;

(3) we extend Theorem 1 of [10] from a uniformly smooth Banach space to a much more general setting.

Remark 4.4. If $f: K\to K$ is a contraction and we replace u by $f(x_n)$ in the recursion formula (1.6), we can obtain the so-called viscosity iteration method (see [22]).

Remark 4.5. Theorem 3.3 and Theorem 4.1 can be applied to the spaces L^p, ℓ^p $(1 < p < \infty)$, the Sobolev spaces W_m^p $(1 < p < \infty)$ and Hilbert spaces. Moreover, our results hold for a Banach space which has the generalized duality mapping j_q $(q > 1)$ and the normalized the duality mapping j.

Acknowledgement. The first author was supported by the Thailand Research Fund, the Commission on Higher Education, and University of Phayao under Grant MRG5580016.

References

[1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer, New York (2009).

[2] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 2350-2360.

[3] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Natl. Acad. Sci. USA 53 (1965) 1272-1276.

[4] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, Math. Z. 100 (1967) 201-225.

[5] S.S. Chang, Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 323 (2006) 1402-1416.

[6] Y.J. Cho, S.M. Kang, X. Qin, Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces, Comput. Math. Appl. 56 (2008) 2058-2064.

[7] C.E. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, in: Springer Lecture Notes Series, 2009.

[8] P. Cholamjiak, S. Suantai, Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions, J. Glob. Optim. (2011), doi: 10.1007/s10898-011-9756-4.

[9] J.P. Gossez, D.E. Lami, Some geometric properties related to the fixed point theory for nonexpansive mappings, Pacific J. Math. 40 (1972) 565-573.

[10] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51-60.

[11] W.R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4 (1953) 506-510.

[12] R.E. Megginson, An Introduction to Banach Space Theory, Springer, New York (1998).

[13] A. Moudafi, Viscosity approximation methods for fixed point problems, J. Math. Anal. Appl. 241 (2000) 46-55.

[14] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591-597.

[15] J.W. Peng, J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings, Nonlinear Anal. 71 (2009) 6001-6010.

[16] X. Qin, Y.J. Cho, J.I. Kang, S.M. Kang, Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces, J. Comput. Appl. Math. 230 (2009) 121-127.

[17] T.-L. Radulescu, V. Radulescu, T. Andreescu, Problems in Real Analysis: Advanced Calculus on the Real Axis, Springer, New York, 2009.

[18] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287-292.

[19] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274-276.

[20] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwan. J. Math. 5 (2001) 387-404.

[21] T. Suzuki, Strong convergence of Krasnoselskii and Manns type sequences for one parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005) 227-239.

[22] T. Suzuki, Moudafi's viscosity approximations with Meir-Keeler contractions, J. Math. Anal. Appl. 325 (2007) 342-352.

[23] W. Takahashi, Nonlinear Function Analysis, Yokahama Publishers, Yokahama (2000).

[24] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276-286.

[25] W. Takahashi, Y. Ueda, On Reich's strong convergence for resolvents of accretive operators, J. Math. Anal. Appl. 104 (1984) 546-553.

[26] R. Wangkeeree, N. Petrot, R. Wangkeeree, The general iterative methods for nonexpansive mappings in Banach spaces, J. Glob. Optim., doi 10.1007/s10898-010-9617-6.

[27] H.K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, J. Math. Anal. Appl. 314 (2006) 631-643.

[28] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (2002) 240-256.

[29] Y. Yao, R. Chen, J.C. Yao, Strong convergence and certain control conditions for modified Mann iteration, Nonlinear Anal. 68 (2008) 1687-1693.

[30] Y. Yao, J.C. Yao, H. Zhou, Approximation methods for common fixed points of infinite countable family of nonexpansive mappings, Comput. Math. Appl. 53 (2007) 1380-1389.

Prasit Cholamjiak,
School of Science,
University of Phayao,
Phayao 56000, Thailand.
Email: prasitch2008@yahoo.com

Yeol Je Cho,
Department of Mathematics Education and the RINS,
Gyeongsang National University,
Jinju 660-701, Republic of Korea.
Email: yjcho@gnu.ac.kr

Suthep Suantai,
Department of Mathematics,
Faculty of Science,
Chiang Mai University,
Chiang Mai 50200, Thailand.
Email: scmti005@chiangmai.ac.th