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Reidemeister type moves for knots and links in lens spaces

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Abstract

We introduce the concept of regular diagrams for knots and links in lens spaces, proving that two diagrams represent equivalent links if and only if they are related by a finite sequence of seven Reidemester type moves. As a particular case, we obtain diagrams and moves for links in $\mathbb{RP}^3$ previously introduced by Y.V. Drobothukina.

1 Preliminaries

In this paper we work in the *Diff* category (of smooth manifolds and maps). Every result also holds in the *PL* category.

Let X and Y be two smooth manifolds. A smooth map $f : X \to Y$ is an *embedding* if the differential $d_x f$ is injective for all $x \in X$ and if X and $f(X)$ are homeomorphic. As a consequence, X and $f(X)$ are diffeomorphic and $f(X)$ is a submanifold of Y. An *ambient isotopy* between two embeddings l_0 and l_1 from X to Y is a smooth map $H : Y \times [0, 1] \to Y$ such that, if we define $h_t(y) = H(y, t)$ for each $t \in [0, 1]$, then $h_t : Y \to Y$ is a diffeomorphism, $h_0 = \mathrm{Id}_Y$ and $l_1 = h_1 \circ l_0$.

A *link* in a closed 3-manifold M^3 is an embedding of ν copies of $\mathbf{S}^1$ into M^3, namely it is $l : \mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1 \to M^3$. A link can also be denoted by L, where $L = l(\mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1) \subset M^3$. A *knot* is a link with $\nu = 1$. Two links L_0 and L_1 are *equivalent* if there exists an ambient isotopy between the two related embeddings l_0 and l_1 (i.e., $h_1(L_0) = L_1$).

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Let p and q be two integer numbers such that $\gcd(p,q) = 1$ and $0 \leqslant q < p$. Consider $B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leqslant 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3. Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with B^3. Label with N and S the "north pole" $(0,0,1)$ and the "south pole" $(0,0,-1)$ of B^3, respectively. Let $g_{p,q} : E_+ \to E_+$ be the rotation of $2\pi q/p$ around the x_3 axis as in Figure 1, and let $f_3 : E_+ \to E_-$ be the reflection with respect to the plane $x_3 = 0$. The *lens space* $L(p,q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identify $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by $F : B^3 \to B^3/\sim$ the quotient map. Notice that on the equator $\partial B_0^2 = E_+ \cap E_-$ there are p points in each class of equivalence.

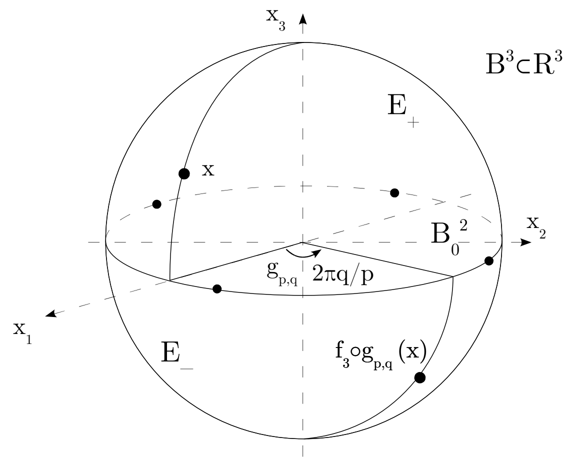

Figure 1: Model for $L(p,q)$.

It is easy to see that $L(1,0) \cong \mathbf{S}^3$ since $g_{1,0} = \mathrm{Id}_{E_+}$. Furthermore, $L(2,1)$ is $\mathbb{RP}^3$, since we obtain the usual model of the projective space where opposite points of the boundary of the ball are identified.

2 Links in $\mathbf{S}^3$

2.1 Diagrams

One of the first tools used to study links in $\mathbf{S}^3$ are diagrams, which are suitable projections of the links on a plane.

If L is a link in $\mathbf{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, we can always suppose, up to equivalence, that L belongs to $\mathrm{int}B^3$. Let $\mathbf{p} : B^3 \smallsetminus \{N, S\} \to B_0^2$ be the projection defined by $\mathbf{p}(x) = c(x) \cap B_0^2$, where $c(x)$ is the circle (possibly a line) through N, x and S.

Now project $L \subset \mathrm{int}B^3$ using $\mathbf{p}_{|L} : L \to B_0^2$. For any $P \in \mathbf{p}(L)$, $\mathbf{p}_{|L}^{-1}(P)$ may contains more than one point; in this case, we say that P is a multiple point. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, up to a "small" isotopy, that the projection $\mathbf{p}_{|L} : L \to B_0^2$ of L is *regular*, namely:

1. the arcs of the projection contain no cusps;
2. the arcs of the projection are not tangent to each other;
3. the set of multiple points is finite, and all of them are actually double points.

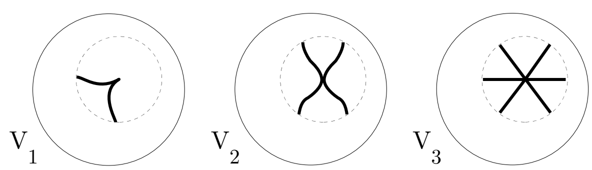

Figure 2:

These conditions correspond to forbidden configurations depicted in Figure 2.

Now let Q be a double point and consider $\mathbf{p}_{|L}^{-1}(Q) = \{P_1, P_2\}$.

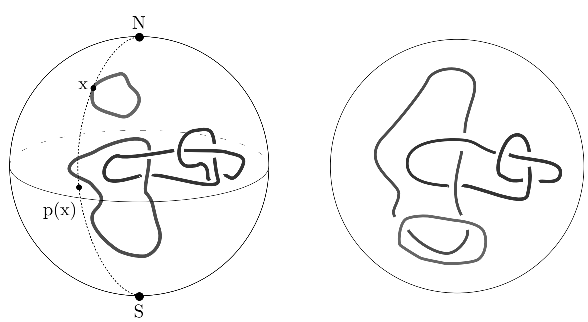

Figure 3: A link in S^3 and the corresponding diagram.

We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L such that $\mathbf{p}(\overline{U})$ does not contain other double points. We call U *underpass*. Take the complementary set in L of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called *overpass*. The overpass/underpass rule in the double points is visualized by removing U from L' before projecting the link (see Figure 3). Observe that some components of the link might have no underpasses.

A *diagram* of a link L in $\mathbf{S}^3$ is a regular projection of L on the equatorial disk B_0^2, with specified overpasses and underpasses.

2.2 Reidemeister moves

There are three (local) moves that allow us to determine when two links in $\mathbf{S}^3$ are equivalent, directly from their diagrams. Reidemeister proved this theorem for *PL* links. For the *Diff* case a possible reference is [3], where the result concerns links in arbitrary dimensions, so the proof is rather complicated.

The *Reidemeister moves* for a diagram of a link $L \subset \mathbf{S}^3$ are the moves R_1, R_2, R_3 of Figure 4.

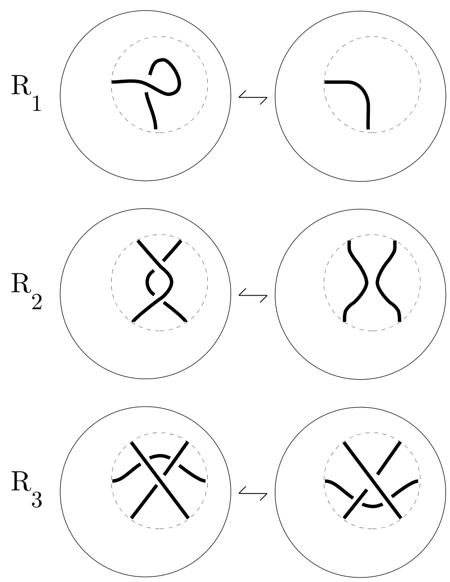

Figure 4: Reidemeister moves.

Theorem 2.1. [3] *Two links L_0 and L_1 in $\mathbf{S}^3$ are equivalent if and only if their diagrams can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : \mathbf{S}^3 \times [0,1] \to \mathbf{S}^3$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. From general position theory (see [3] for details), we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times only one condition is violated. From each type of violation a transformation of the diagram appears, that is to say, a Reidemeister move, as it follows (see Figure 5):

- from V_1 we obtain the move R_1;
- from V_2 we obtain the move R_2;
- from V_3 we obtain the move R_3.

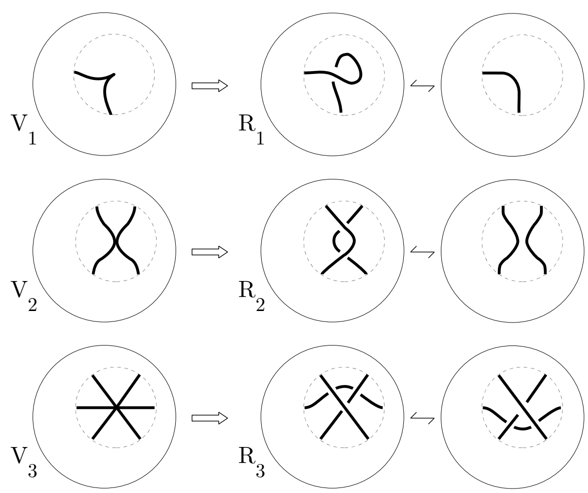

Figure 5:

So diagrams of two equivalent links can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies. □

3 Links in $\mathbb{RP}^3$

3.1 Diagrams

The definition of diagrams for links in the projective space, given by Drobothukina in [1], makes use of the model of the projective space $\mathbb{RP}^3$ explained in Section 1, as a particular case of $L(p,q)$ with $p = 2$ and $q = 1$. Namely, consider B^3 and let $\sim$ the equivalence relation which identify diametrically opposite points on ∂B^3, then $\mathbb{RP}^3 \cong B^3/\sim$.

Let L be a link in $\mathbb{RP}^3$ and consider $L' = F^{-1}(L)$, where F is the quotient map. By moving L with a small isotopy in $\mathbb{RP}^3$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\text{int}B^3$ and arcs properly embedded* in B^3.

Let $\mathbf{p} : B^3 \smallsetminus \{N, S\} \to B_0^2$ be the projection defined in the previous section. Take L' and project it using $\mathbf{p}_{|L'} : L' \to B_0^2$. A point $P \in \mathbf{p}(L')$ such that $\mathbf{p}_{|L'}^{-1}(P)$ contains more than one point is called a multiple point. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, by moving L via small isotopies, that the projection $\mathbf{p}(L')$ is *regular*,

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B_0^2;

5) no double point is on ∂B_0^2.

namely:

These conditions correspond to forbidden configurations $V_1, \ldots, V_5$ depicted in Figures 2 and 6.

Now let Q be a double point and consider $\mathbf{p}_{|L'}^{-1}(Q) = \{P_1, P_2\}$. We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L' such that $\mathbf{p}(\overline{U})$ does not contain other double points and does not meet ∂B_0^2. We call U *underpass*. Take the complementary set in L' of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called

*An arc is properly embedded in a compact 3-manifold M^3 if only the endpoints belong to ∂M^3.

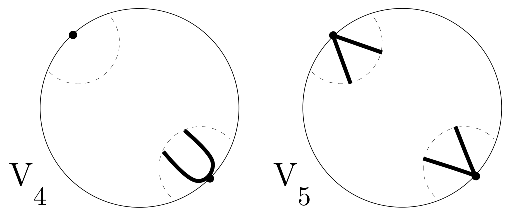

Figure 6:

overpass. Again the overpass/underpass rule in the double points is visualized by removing U from L' before projecting the link (see Figure 7).

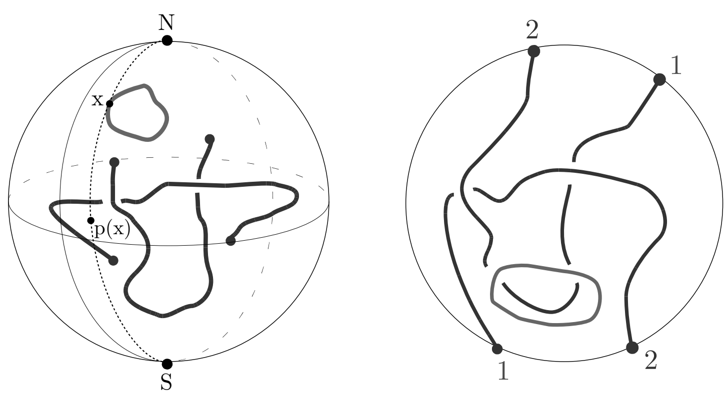

Figure 7: A link in $\mathbb{RP}^3$ and the corresponding diagram.

A *diagram* of a link L in $\mathbb{RP}^3$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2, with specified overpasses and underpasses.

The boundary points of the link projection are labelled in order to make clear the identifications. Assume that the number of boundary points is $2t$ and orient the equator counterclockwise (looking at it from N). Choose a point of $\mathbf{p}(L')$ on the equator and label it by 1, as well as its antipodal point; then following the orientation of ∂B_0^2, label the points of $\mathbf{p}(L')$ on the equatorial circle, as well as the antipodal ones, by $2, \ldots, t$ respectively (see Figure 7).

3.2 Generalized Reidemeister moves

The *generalized Reidemeister moves* on a diagram of a link $L \subset \mathbb{RP}^3$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5 of Figure 8.

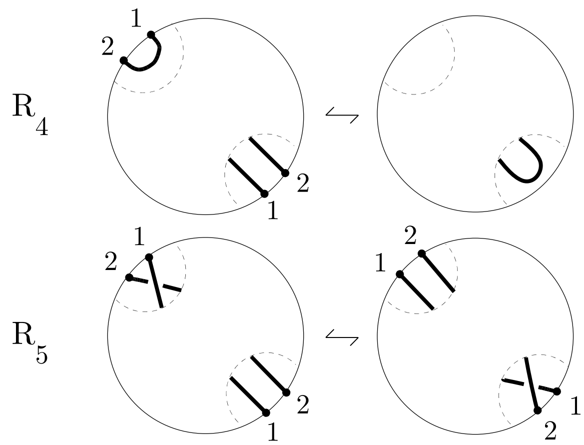

Figure 8: Generalized Reidemeister moves for links in $\mathbb{RP}^3$.

An analogue of the Reidemeister theorem can be proved in this contest:

Theorem 3.1. [1] *Two links L_0 and L_1 in $\mathbb{RP}^3$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_5$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : \mathbb{RP}^3 \times [0,1] \to \mathbb{RP}^3$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. As for links in $\mathbf{S}^3$, using general position theory we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times only one condition is violated. From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as it follows (see Figures 5 and 9):

– from V_1, V_2 and V_3 we obtain the classic Reidemeister moves R_1, R_2 and R_3;

– from V_4 we obtain the move R_4;
– from V_5 we obtain the move R_5.

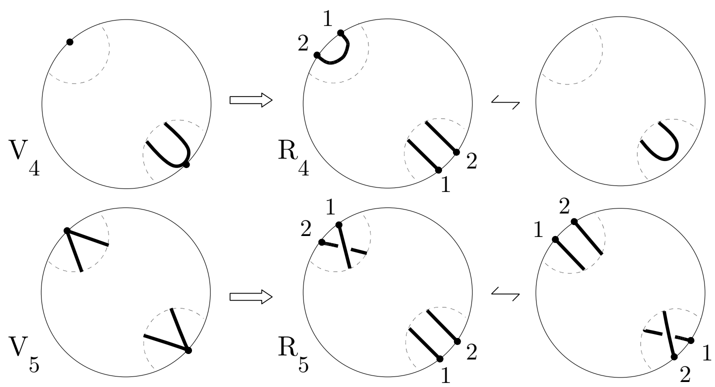

Figure 9:

So diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_5$ and diagram isotopies. $\square$

4 Links in $L(p,q)$

4.1 Diagrams

In this section we improve the definition of diagram for links in lens spaces given in [2]. We can assume $p > 2$, since the particular cases $L(1,0) \cong \mathbf{S}^3$ and $L(2,1) \cong \mathbb{RP}^3$ have been already considered in previous sections.

The model for the lens space $L(p,q) = B^3/\sim$ is the one given in Section 1. Let L be a link in $L(p,q)$ and let $L' = F^{-1}(L)$, where F is the quotient map. By moving L via a small isotopy in $L(p,q)$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\mathrm{int} B^3$ and arcs properly embedded in B^3.

Let $\mathbf{p} : B^3 \smallsetminus \{N, S\} \to B^2_0$ be the projection defined in Section 2. Take L' and project it using $\mathbf{p}_{|L'} : L' \to B^2_0$. As before, a point $P \in \mathbf{p}(L')$ such that $\mathbf{p}^{-1}_{|L'}(P)$ contains more than one point is called a multiple point. In particular, if it contains exactly two points, P is called a *double point*. We can assume, by moving L via a small isotopy, that the projection $\mathbf{p}_{|L'} : L' \to B_0^2$ of L is *regular*, namely:

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B_0^2;

5) no double point is on ∂B_0^2;

6) $L' \cap \partial B_0^2 = \emptyset$.

Of course, overpasses and underpasses are defined as in the previous section. A *diagram* of a link L in $L(p,q)$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2, with specified overpasses and underpasses.

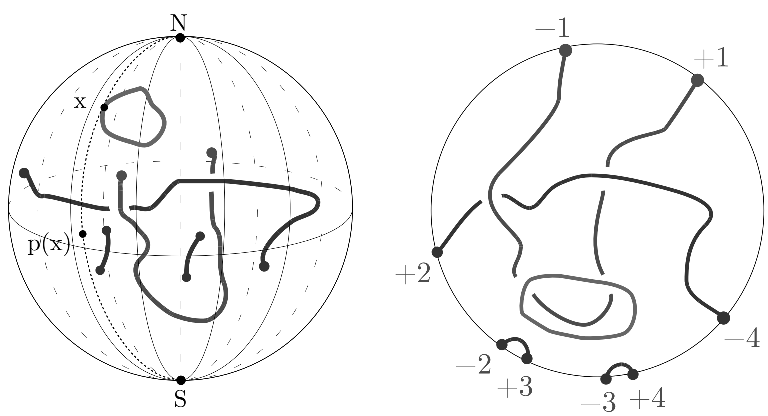

Figure 10: A link in $L(9,1)$ and the corresponding diagram.

We label the boundary points of the link projection, in order to make clear the identification. Assume the equator is oriented counterclockwise, and consider the t endpoints of the overpasses belonging to the upper hemisphere. Label their projecton by $+1, \ldots, +t$, according to the orientation of ∂B_0^2. Then label the other t boundary points by $-1, \ldots, -t$, where for each $i = 1, \ldots, t$, according to the identifications. An example is shown in Figure 10.

Now we explain which diagram violations arise from condition 1)-6). As usual, conditions 1), 2) and 3) correspond to forbidden configurations V_1, V_2 and V_3 of Figure 2.

Condition 4), as in the projective case, corresponds to V_4. On the contrary, condition 5) does not behave as in the projective case. Indeed two forbidden configurations arise from it (V_5 and V_6), as Figure 11 shows. The difference between them is that V_5 involves two arcs of L' that end in the same hemisphere of ∂B^3, on the contrary V_6 involves arcs that end in different hemispheres.

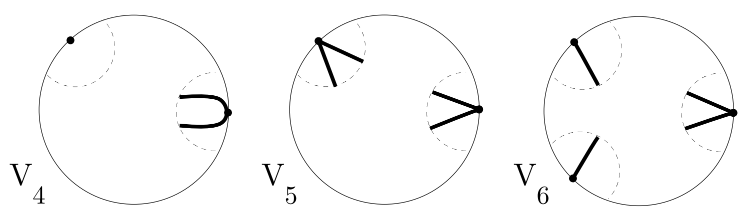

Figure 11:

Finally, condition 6) produces a family of forbidden configurations, called $V_{7,1}, \dots, V_{7,p-1}$ (see Figure 12). The difference between them is that $V_{7,1}$ has the arcs of the projection identified directly by $g_{p,q}$, while $V_{7,k}$ has the arcs identified by $g^k_{p,q}$, for $k = 2, \dots, p-1$.

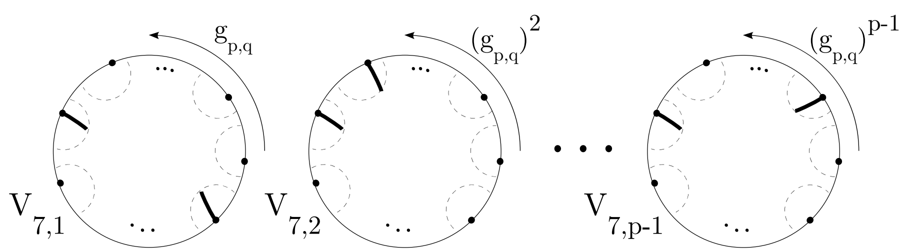

Figure 12:

It is easy to see what kind of small isotopy on L is necessary, in order to make the projection of the link regular when we deal with configurations $V_1, \dots, V_6$. Now we explain how the link can avoid to meet ∂B^2_0 up to isotopy, that is to say, avoid $V_{7,1}, \dots, V_{7,p-1}$.

We start with a link with two arcs ending on ∂B_0^2. Suppose that the endpoints B and C of the arcs are connected by $g_{p,q}$, (a $V_{7,1}$ violation), namely $C = g_{p,q}(B)$. In this case the required isotopy is the one that lift up a bit the arc ending in B and lower down the other one.

Now if we suppose that the endpoints B and C of the arcs are connected by a power of $g_{p,q}$, (a $V_{7,k}$ violation with $k > 1$), namely $C = g_{p,q}^k(B)$. In this case the required isotopy is similar to the one of the example in $L(9,1)$ of Figure 13. In lens spaces with $q \neq 1$, the new arcs end in the faces specified by the map $f_3 \circ g_{p,q}$.

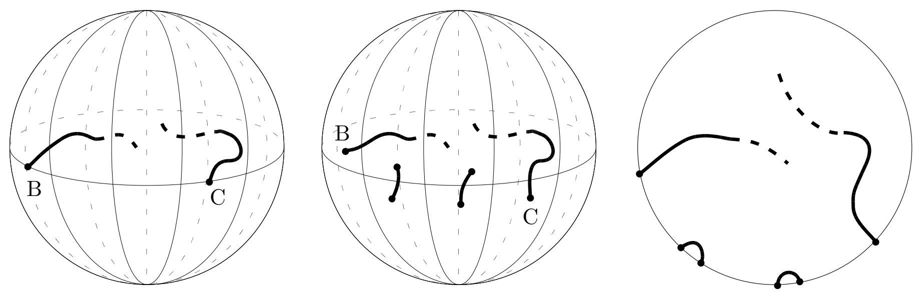

Figure 13: How to avoid ∂B_0^2 in $L(9,1)$.

4.2 Generalized Reidemeister moves

Again, with the aim of discovering when two diagrams represent equivalent links in $L(p,q)$, we generalize Reidemeister moves to this contest.

The *generalized Reidemeister moves* on a diagram of a link $L \subset L(p,q)$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5, R_6 and R_7 of Figure 14.

Theorem 4.1. *Two links L_0 and L_1 in $L(p,q)$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_7$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : L(p,q) \times [0,1] \to L(p,q)$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. As before, using general position

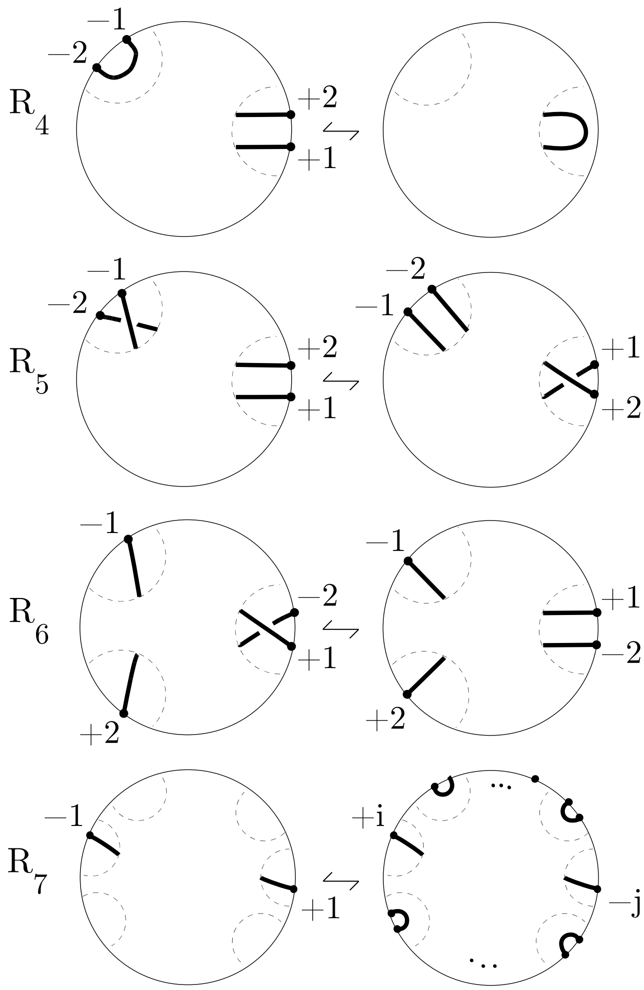

Figure 14: Generalized Reidemeister moves for links in $L(p,q)$.

arguments we can assume that the projection $\mathbf{p}(L'_t)$ is not regular only a finite number of times, and that at each of these times only one condition is violated.

From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as follows (see Figures 5 and 15):

– from V_1, V_2 and V_3 we obtain the classical Reidemeister moves R_1, R_2

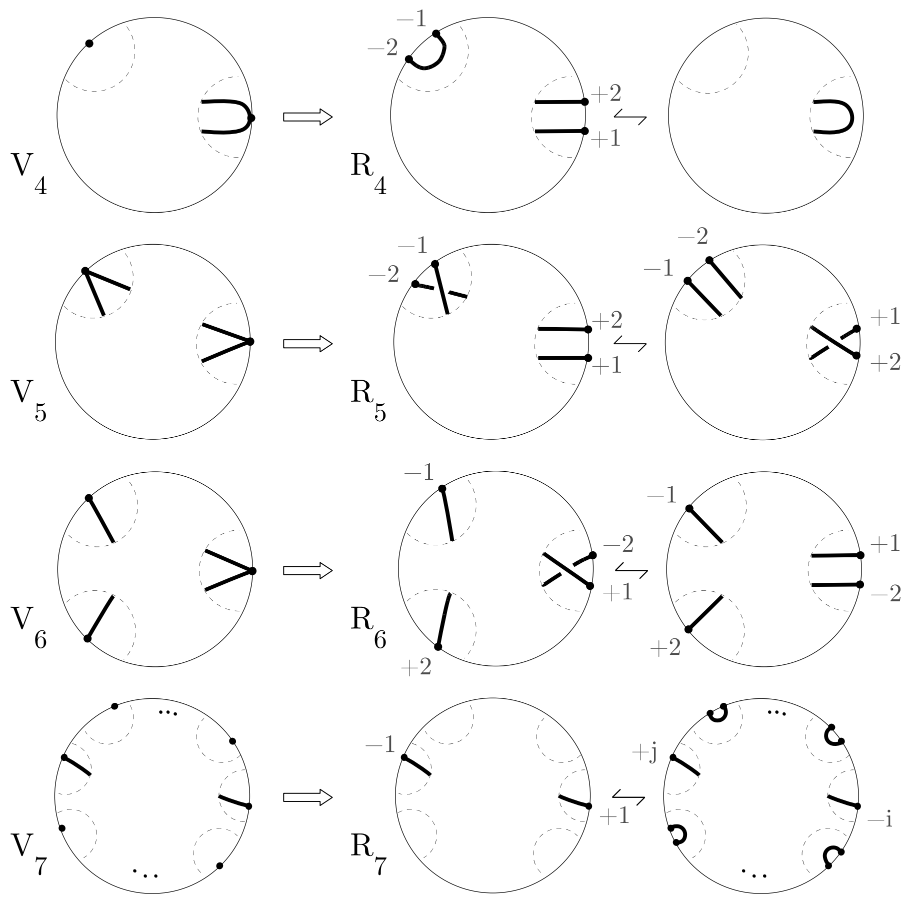

Figure 15:

and R_3;

– from V_4 we obtain the move R_4;
– from V_5, we obtain two different moves: if the endpoints of the arcs corresponding to the double point belong in the same hemisphere, then we obtain R_5; on the contrary we obtain R_6;
– from condition 6 we have a family of forbidden configurations $V_{7,1}, \ldots, V_{7,p-1}$, from which we obtain the moves $R_{7,1}, \ldots, R_{7,p-1}$.

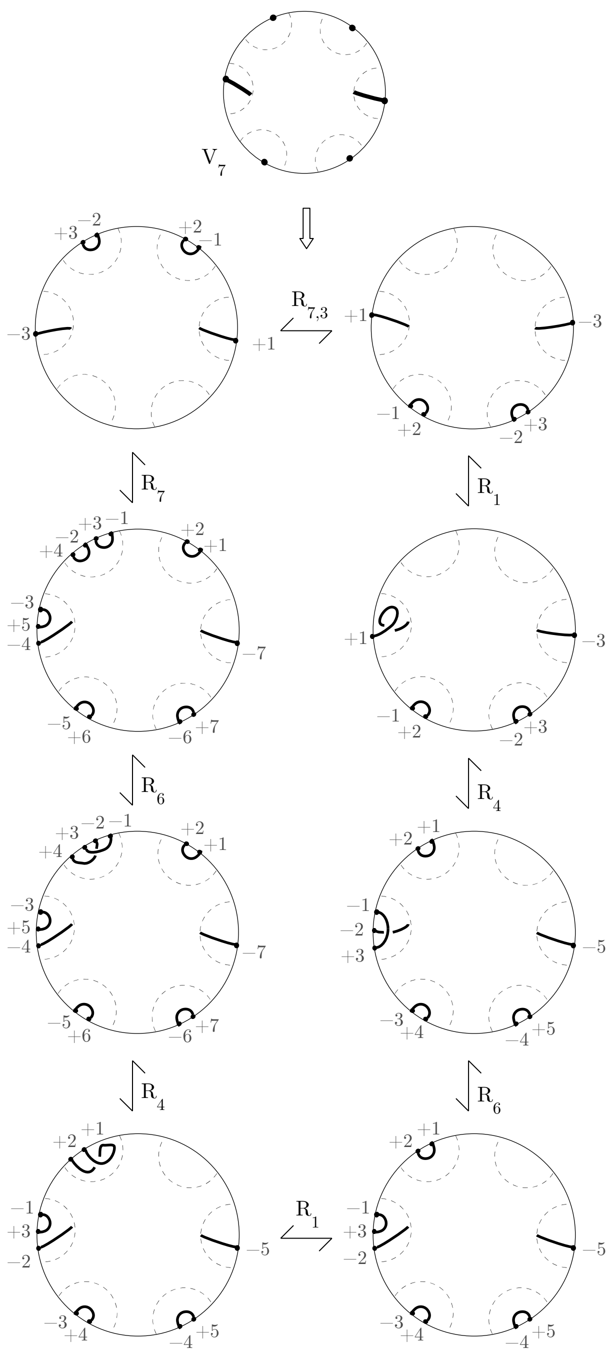

Figure 16: How to reduce a composite move.

Indeed, if an arc cross the equator during the isotopy, then we have a class of moves, $R_{7,1} = R_7, R_{7,2}, \ldots, R_{7,p-1}$. All these moves can be seen as the composition of R_7, R_6, R_4 and R_1 moves. More precisely, the move $R_{7,k}$ with $k = 2, \ldots, p-1$, can be obtained by the following sequence of moves: first we perform an R_7 move on one overpass that end on the equator and the corresponding point in a small arc, then we repeat for $k-1$ times the three moves R_6-R_4-R_1 necessary to retract the small arc with ending points having the same sign (see an example in Figure 16).

So we can drop $R_{7,2}, \ldots, R_{7,p-1}$ from the set of moves, and keep only $R_{7,1} = R_7$. As a consequence, any pair of diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_7$ and diagram isotopies. □

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