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Magnetic Schrödinger operators with discrete spectra on non-compact Kähler manifolds

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Abstract

We identify a class of magnetic Schrödinger operators on Kähler manifolds which exhibit pure point spectrum. To this end we embed the Schrödinger problem into a Dirac-type problem via a parallel spinor and use a Bochner-Weitzenböck argument to prove our spectral discreteness criterion.

1. Introduction

Let (M, g) be a complete non-compact oriented Riemannian manifold of dimension $n \geq 2$, with Riemannian metric g, and let a be a *real* 1-form on M, of class C^∞. Then a induces a metric connection ∇^a on the trivial Hermitian bundle $M \times \mathbf{C}$, identifiable to the first order differential operator

$$C^\infty(M, \mathbf{C}) \ni \phi \longmapsto d^a\phi := d\phi + i\phi a \in C^\infty(M, T^*M \otimes \mathbf{C}),$$

where d represents ordinary exterior differentiation and $i = \sqrt{-1}$. As usual, the Riemannian metric allows one to consider pointwise Hermitian products $\langle \cdot, \cdot \rangle_x$, $x \in M$, in the complexified cotangent bundle $T^*M \otimes \mathbf{C}$ and, via the volume form, global (integrated) Hermitian products $(\cdot, \cdot)$, in the spaces $C^\infty_{\rm cpt}(M, \mathbf{C})$ and $C^\infty_{\rm cpt}(M, \mathbf{C} \otimes T^*M)$. With respect to these products the formal adjoint $(d^a)^*$ of d^a can be defined as a first order differential operator,

$$(d^a)^* : C^\infty(M, \mathbf{C} \otimes T^*M) \longrightarrow C^\infty(M, \mathbf{C}),$$

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and then the magnetic Schrödinger operator (magnetic bottle) with magnetic potential a is the second order differential operator $H_a := (d^a)^* d^a$, viewed as an unbounded operator in $L^2(M, \mathbf{C})$. (see Section 2 for more details). It is known that regardless of a, H_a with domain $C^\infty_{\mathrm{cpt}}(M, \mathbf{C})$ is an essentially self-adjoint operator in $L^2(M, \mathbf{C})$ [S1].

There is a great deal of work, especially on Euclidean spaces $M = \mathbf{R}^n$, dedicated to deciding which magnetic Schrödinger operators H_a have discrete spectrum, that is a spectrum consisting only in isolated eigenvalues of finite multiplicity [AHS, I, KS, A1]. Typically, these works provide sufficient conditions for spectral discreteness, in terms of the magnetic field B associated to a, $B := da$.

The purpose of this note is to provide one more result along these lines, in the case M is a Kähler manifold with Kähler form ω and Riemannian metric g naturally induced by ω. This result can easily be seen to generalize that of [A1], when n is even.

Theorem. *Let M be a non-compact Kähler manifold with Kähler form ω and Riemannian metric induced by ω. Assume that H_a is a magnetic Schrödinger operator on M associated to a real 1-form a of class C^∞. Then H_a has discrete spectrum if the real-valued function $\langle B(x), \omega(x)\rangle$ on M, where $\langle \cdot, \cdot \rangle$ denotes the natural pointwise inner product on 2-forms, satisfies the condition*

$$\lim_{x\to\infty} \langle B(x), \omega(x)\rangle = -\infty. \tag{1}$$

2. Magnetic Schrödinger operators on manifolds

Let (M, g) be a complete non-compact oriented Riemannian (C^∞) manifold of dimension n, equipped with the metric g. On the usual real C^∞-bundles of p-forms on M, $\Lambda^p(T^*M)$, $0 \le p \le n$, consider the standard inner products $\langle \cdot, \cdot \rangle_x$, $x \in M$. Specifically, if $(e_1, e_2, \ldots, e_n)$ is an oriented local orthonormal frame in the tangent bundle TM, with local dual frame of 1-forms in the cotangent bundle T^*M, $(e_1^*, e_2^*, \ldots, e_n^*)$, then a local orthonormal basis of $\Lambda^p(T^*M)$ is $\{e_J^*\}_J$, $e_J^* := e_{j_1}^* \wedge e_{j_2}^* \wedge \cdots \wedge e_{j_p}^*$, where J runs through the set of all multi-indices $1 \le j_1 < j_2 < \cdots < j_p \le n$.

There is a Levi-Cività metric connection ∇^{LC} on $\Lambda^p(T^*M)$, extending naturally the Levi-Cività connection on T^*M, the exterior product connection; For a local vector field e in TM and local forms v^* in T^*M and ϕ in $\Lambda^p(T^*M)$,

$$\nabla_e^{\mathrm{LC}}(v^* \wedge \phi) = \nabla_e^{\mathrm{LC}} v^* \wedge \phi + v^* \wedge \nabla_e^{\mathrm{LC}} \phi. \tag{2}$$

Denote now by $\Omega^p(M, \mathbf{C}) := C^\infty(M, \Lambda^p(T^*M) \otimes \mathbf{C})$ the Hermitian vector space of C^∞ complex global p-forms and by

$$d : \Omega^p(M, \mathbf{C}) \longrightarrow \Omega^{p+1}(M, \mathbf{C})$$

the usual exterior differential. In terms of the complexified Levi-Cività metric connection ∇^{LC} on $\Lambda^p(T^*M)\otimes\mathbf{C}$, d can be written locally as

$$d=\sum_{j=1}^n e_j^*\wedge\nabla^{\mathrm{LC}}_{e_j}.$$

Fix now $a\in\Omega^1(M,\mathbf{R})$ a real global 1-form. Then the twisted differential $d^a:=d+ia\wedge$, defined on $\Omega^p(M,\mathbf{C})$ by

$$\Omega^p(M,\mathbf{C})\ni\phi\longmapsto d^a\phi=d\phi+ia\wedge\phi\in\Omega^{p+1}(M,\mathbf{C}),$$

has the local frame counterpart

$$d^a=\sum_{j=1}^n e_j^*\wedge\nabla^{\mathrm{LC},a}_{e_j},$$

where $\nabla^{\mathrm{LC},a}$ is the twisted metric connection on $\Lambda^p(T^*M)\otimes\mathbf{C}$ defined by

$$\nabla^{\mathrm{LC},a}_v\phi=\nabla^{\mathrm{LC}}_v\phi+ia(v)\phi,\quad v \text{ global vector field in } TM,\ \phi\in\Omega^p(M,\mathbf{C}). \tag{3}$$

For $\phi\in\Omega^p(M,\mathbf{C})$ and $\psi\in\Omega^p_{\mathrm{cpt}}(M,\mathbf{C})$ the global Hermitian product $(\phi,\psi):=\int_M\langle\phi,\psi\rangle dvol$ induces the formal adjoint $(d^a)^*$ of d^a,

$$(d^a)^*:\Omega^{p+1}(M,\mathbf{C})\longrightarrow\Omega^p(M,\mathbf{C}),$$

subject to

$$((d^a)^*\phi,\psi)=(\phi,d^a\psi),\quad \phi\in\Omega^{p+1}(M,\mathbf{C}),\ \psi\in\Omega^p_{\mathrm{cpt}}(M,\mathbf{C}).$$

It follows that locally

$$(d^a)^*=-\sum_{j=1}^n e_j\lrcorner\,\nabla^{\mathrm{LC},a}_{e_j},$$

where $e_j\lrcorner$ denotes interior multiplication (contraction) by the local vector field e_j.

Making in the above discussion $p=0$ we get a second order differential operator

$$H_a:=(d^a)^*d^a:C^\infty(M,\mathbf{C})\longrightarrow C^\infty(M,\mathbf{C}).$$

Seen as an unbounded operator in $L^2(M,\mathbf{C})$, the completion of $C^\infty_{\mathrm{cpt}}(M,\mathbf{C})$ with respect to $(\cdot,\cdot)$, H_a is called the (scalar) magnetic Schrödinger operator generated by the potential a. It is then a nice exercise to see that in a local frame,

$$H_a=-\sum_{j=1}^n\left(e_j+ia(e_j)\right)^2+\sum_{j=1}^n\left(\nabla^{LC}_{e_j}e_j+ia(\nabla^{LC}_{e_j}e_j)\right).$$

H_a with domain $C^\infty_{\text{cpt}}(M,\mathbf{C})$ can be closed in only one way in $L^2(M,\mathbf{C})$, i.e., H_a is an essentially self-adjoint operator [S1].

In this note we will be interested in reasonably simple conditions on M and a which would ensure that H_a has pure point spectrum. We therefore conclude this section with a general criterion for spectral discreteness.

Proposition 1. *H_a being defined as above, if there is a function $f \in C^0(M,\mathbf{R})$, $\lim_{x\to\infty} f(x) = \infty$, such that*

$$(H_a\phi,\phi) \geq (f\phi,\phi), \quad \phi \in C^\infty_{cpt}(M,\mathbf{C}), \tag{4}$$

then H_a has discrete spectrum.

Proof. We will supply a somewhat less traditional proof to this proposition. To this end, let $W^2(M,a)$ be the domain of the unique closed extension of H_a from $C^\infty_{\text{cpt}}(M,\mathbf{C})$ into $L^2(M,\mathbf{C})$. $W^2(M,a)$ is the completion of $C^\infty_{\text{cpt}}(M,\mathbf{C})$ with respect to the Sobolev inner product $(\cdot,\cdot)_2 := (\cdot,\cdot) + (H_a\cdot, H_a\cdot)$. Since $H_a : W^2(M,a) \longrightarrow L^2(M,\mathbf{C})$ is self-adjoint, its spectrum is contained in the real line.

To prove the proposition it suffices to show that for every $\lambda \in \mathbf{R}$ the operator $H_a - \lambda$ with domain $W^2(M,a)$ is Fredholm, since for any Fredholm operator 0 is an isolated point of its spectrum, and in fact an eigenvalue with finite multiplicity.

Fix now a number $\lambda \in \mathbf{R}$. The assumption on the function f provides a compact subset K of M such that $f(x) \geq \lambda + 1$, if $x \in M \setminus K$. The hypothesis (4) and the density of $C^\infty_{\text{cpt}}(M,\mathbf{C})$ in $W^2(M,a)$ imply that

$$((H_a - \lambda)\phi,\phi) - ((f-\lambda)\phi,\phi)_K \geq (\phi,\phi)_{M\setminus K}, \quad \phi \in W^2(M,a), \tag{5}$$

where for a subset U of M, $(\cdot,\cdot)_U$ indicates integration is carried out only over U.

As in [A2], $H_a - \lambda$ will be a Fredholm operator if we can show that any sequence $\{\phi_n\}_n$ from $W^2(M,a)$, which is L^2-bounded and for which $\{(H_a - \lambda)\phi_n\}_n$ is L^2-convergent, admits a L^2-convergent subsequence.

Since $\{\phi_n\}_n$ is bounded in the Sobolev norm $||\cdot||_2$, by Rellich's lemma [S2] the sequence $\{\phi_n|_K\}_n$ has a convergent subsequence in $L^2(K,\mathbf{C})$ (assumed to be the sequence itself).

The property (5) applied now to the differences $\{\phi_m - \phi_n\}_{m,n}$ shows that $\{\phi_n|_{M\setminus K}\}_n$ is a Cauchy sequence in $L^2(M \setminus K,\mathbf{C})$. We conclude that $\{\phi_n\}_n$ converges in the L^2-norm, since its restrictions to K and $M \setminus K$ do so. $\square$

3. Generalized Dirac operators

As mentioned in the introduction, our spectral discreteness analysis will come about by embedding the magnetic Schrödinger operator formalism into a Dirac-type framework. It is then desirable to briefly review here the concept of generalized Dirac bundle with its associated Dirac operator [GL].

If (M, g) is, as before, a complete non-compact oriented Riemannian manifold of dimension n, let $Cl(M)$ be the real Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric g. There is a canonical embedding $TM \subset Cl(M)$, and then the Riemannian metric and Levi-Cività connection extend from TM to $Cl(M)$ in such a way that the connection ∇^{LC} of $Cl(M)$ preserves the metric and acts as a derivation.

A complex bundle of left modules over the bundle of algebras $Cl(M)$, say $S \longrightarrow M$, will be called a (generalized) Dirac bundle if S is furnished with a Hermitian metric $\langle \cdot, \cdot \rangle$ and a metric connection ∇^S such that

i) The action on S by unit vectors in $TM \subset Cl(M)$ is a pointwise isometry.

ii) The connection ∇^S is compatible with the Clifford multiplication, in the sense that for local sections e in TM, ϕ in $Cl(M)$, and s in S, we have

$$\nabla_e^S(\phi \cdot s) = \left(\nabla_e^{LC}\phi\right) \cdot s + \phi \cdot \left(\nabla_e^S s\right).$$

Above, the "$\cdot$" indicates the action of $Cl(M)$ on S, while the multiplication in $Cl(M)$ will be simply represented by juxtaposition. Since TM generates $Cl(M)$, the action $\cdot$ of $Cl(M)$ on S is completely determined by its restriction to TM.

There are several fundamental examples and constructs of Dirac bundles associated to M, which are relevant to us:

a) $S = Cl(M) \otimes \mathbf{C}$. In this case $Cl(M)$ acts on S by left algebra multiplication and ∇^S is the complexification of ∇^{LC}.

b) $S = \Lambda(T^*M) \otimes \mathbf{C}$. This case, where $\Lambda(T^*M)$ represents the real bundle of exterior forms on M, is relevant to our concept of magnetic Schrödinger operator, in the sense that the scalar concept we work with admits an extension to a concept of exterior form magnetic Schrödinger operator.

If $(e_1, e_2, \ldots, e_n)$ is a local frame in TM then the action $\cdot$ of e_j on S is given by $e_j \cdot = e_j^* \wedge -e_j \lrcorner$. ∇^S is the exterior form extension of the Levi-Cività connection ∇^{LC} on T^*M, cf. (2). In fact case *b*) coincides with case *a*) under the canonical vector bundle linear isometry $\Lambda(T^*M) \simeq Cl(M)$, $e_J^* \longmapsto e_{j_1} e_{j_2} \ldots e_{j_p}$. This is a vector bundle isomorphism which also preserves the Levi-Cività connections, but of course not an algebra bundle isomorphism.

c) For a Kähler manifold M of complex dimension m [GH] let ω be the Kähler 2-form and let g be the Riemannian metric naturally induced on TM by ω. Then the integrable complex structure J in the tangent bundle TM makes

(TM, g) a Hermitian bundle, and there is a complex linear isometry between (TM, J) and the Hermitian bundle of $(0,1)$-forms $T^{*0,1}M \subset T^*M \otimes \mathbf{C}$. Since M is Kähler this isometry takes the Levi-Civitá connection of TM to the unique anti-holomorphic Hermitian connection $\nabla_{\overline{z}}$ on $T^{*0,1}M$. Then $S := \Lambda(T^{*0,1}M)$ is a Dirac bundle, when endowed with a Clifford multiplication similar to that of case *b*), via the above-said complex isometry, and with the exterior product connection induced by, and extending, $\nabla_{\overline{z}}$ [B].

d) If M is a *spin* manifold [LM] then S can be taken to be the spinor bundle $\Sigma(M)$ of M. To be more specific, for a spin manifold the principal $\mathrm{SO}(n)$-bundle $P_{\mathrm{SO}}(M)$ of oriented frames in TM lifts to a principal Spin-bundle $P_{\mathrm{Spin}}(M)$, equivariantly with respect to the 2-cover map $\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. The spinor bundle $\Sigma(M)$ is then the fiber product $\Sigma(M) := P_{\mathrm{Spin}}(M) \times_\mu \Delta$, where Δ is an irreducible representation of the Euclidean Clifford algebra on n generators $Cl_n \otimes \mathbf{C}$ and μ is the unitary representation $\mu : \mathrm{Spin}(n) \longrightarrow U(\Delta)$ induced by the left multiplication with elements of $\mathrm{Spin}(n) \subset Cl_n \otimes \mathbf{C}$. We get then the compatible connection ∇^{Spin} of $\Sigma(M)$ by lifting the Riemannian connection on $P_{\mathrm{SO}}(M)$ to $P_{\mathrm{Spin}}(M)$, via the Lie algebra isomorphism $\mathrm{so}(n) \simeq \mathrm{spin}(n)$.

e) If S is a Dirac bundle and F is any Hermitian bundle over M, equipped with a metric connection ∇^F, then the twisted bundle $S \otimes F$ is naturally a Dirac bundle, with Clifford multiplication induced by that of S and connection $\nabla^{S\otimes F} := \nabla^S \otimes Id + Id \otimes \nabla^E$.

Any Dirac bundle S generates a distinguished differential operator $D_S : C^\infty(M, S) \longrightarrow C^\infty(M, S)$, the generalized Dirac operator, defined as follows: If $m : T^*M \otimes S \longrightarrow S$ denotes the restriction to T^*M (metrically identified with TM) of the Clifford action $\cdot$ of $Cl(M)$ on S, then $D_S = m \circ \nabla^S$. Locally, D_S admits the representation

$$D_S = \sum_{j=1}^{n} e_j \cdot \nabla^S_{e_j},$$

where as usual $(e_1, e_2, \ldots, e_n)$ is a local orthonormal frame in TM.

Since M is complete, D_S with domain $C^\infty_{\mathrm{cpt}}(M, S)$ is an essentially self-adjoint first order elliptic differential operator in $L^2(M, S)$ [GL].

Clearly, the Dirac operator associated to $S = \Lambda(T^*M) \otimes \mathbf{C}$ (case *b*) above) is $d + d^*$, where d is the exterior differential and d^* its formal adjoint, as in section 2.

In case *c*), when M is a Kähler manifold and $S = \Lambda(T^{*0,1}M)$ the Dirac operator becomes $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$, where $\overline{\partial}$ is the Dolbeault operator and $\overline{\partial}^*$ its formal adjoint [B].

On a spin manifold M the Dirac operator associated to the spinor bundle $\Sigma(M)$ of case *d*) is called the *classical* Dirac operator.

For the square of a generalized Dirac operator D_S the following Bochner-Witzenböck formula holds true [GL],

$$D_S^2 = \left(\nabla^S\right)^* \nabla^S + \mathcal{R}^S,$$

where $\mathcal{R}^S$ is the Hermitian curvature bundle morphism acting on S according to the formula

$$\mathcal{R}^S = \sum_{j<k} e_j \cdot e_k \cdot R^S_{e_j,e_k}, \quad R^S_{e_j,e_k} = [\nabla^S_{e_j}, \nabla^S_{e_k}] - \nabla^S_{[e_j,e_k]}.$$

In case b), $\mathcal{R}^{\Lambda(T^*M)\otimes \mathbf{C}}$ preserves $\Lambda^p(T^*M)\otimes \mathbf{C}$ and evidently, $\mathcal{R}^{\Lambda(T^*M)\otimes \mathbf{C}}|_{\Lambda^0(T^*M)\otimes \mathbf{C}} = 0$.

In case d), $\mathcal{R}^{\Sigma(M)} = k/4$, where k is the scalar curvature of the spin manifold M (Lichnerowicz's theorem [LM]).

In case e), $\mathcal{R}^{S\otimes F}$ can be written as

$$\mathcal{R}^{S\otimes F} = \mathcal{R}^S \otimes Id + \sum_{j<k} e_j \cdot e_k \cdot \otimes R^F_{e_j,e_k}. \tag{6}$$

If $F = \mathbf{C}_a$, the trivial bundle $M \times \mathbf{C}$ equipped with the metric connection ∇^a associated to some real 1-form $a \in \Omega^1(M, \mathbf{R})$, as in the introduction, then $S \otimes \mathbf{C}_a = S$, and so (6) becomes $\mathcal{R}^{S\otimes \mathbf{C}_a} = \mathcal{R}^S + i\rho^a\cdot$, where ρ^a is the global section of $Cl(M)$ given by

$$\rho^a = \sum_{j<k} R^a_{e_j,e_k} e_j e_k, \quad R^a_{e_j,e_k} = e_j(a(e_k)) - e_k(a(e_j)) - a([e_j, e_k]). \tag{7}$$

It is elementary to see that under the linear isometry $\Lambda(T^*M) \simeq Cl(M)$ explained at case b) above, $\rho^a \in C^\infty(M, Cl(M))$ is the image of the real 2-form $B = da \in \Omega^2(M, \mathbf{R})$.

Finally, if $S = \Lambda(T^*M) \otimes \mathbf{C}$ and $F = \mathbf{C}_a$, then $\nabla^{(\Lambda(T^*M)\otimes \mathbf{C})\otimes \mathbf{C}_a} = \nabla^{\mathrm{LC},a}$, in the notation of section 2, cf. (3). The connection Laplacian $(\nabla^{\mathrm{LC},a})^*\nabla^{\mathrm{LC},a}$ can then be called an exterior form magnetic Schrödinger operator, since it restricts to H_a on $\Omega^0(M, \mathbf{C})$.

4. Our results

We are now ready to state and prove an abstract discreteness criterion for certain H_a's and, as an application, supply a proof to the theorem given in the introduction.

Proposition 2. *Suppose that are given a non-compact Riemannian manifold (M, g), a real 1-form $a \in \Omega^1(M, \mathbf{R})$ with associated scalar Schrödinger operator H_a, and a generalized Dirac bundle S over M with Clifford multiplication $\cdot$, compatible connection ∇^S, and Dirac operator D_S.*

In addition, suppose that there exists a ∇^S-parallel global section $\sigma \in C^\infty(M,S)$ such that

$$\lim_{x\to\infty} \langle i\rho^a \cdot \sigma, \sigma\rangle = -\infty, \tag{8}$$

where ρ^a is the global section of $Cl(M)$ given by (7). Then the magnetic Schrödinger operator H_a has discrete spectrum.

Proof. Consider the twisted Dirac bundle $S \otimes \mathbf{C}_a$ and its Dirac operator $D_{S\otimes\mathbf{C}_a}$. We have the Bochner-Weitzenböck formula

$$D^2_{S\otimes\mathbf{C}_a} = \left(\nabla^{S\otimes\mathbf{C}_a}\right)^* \nabla^{S\otimes\mathbf{C}_a} + \mathcal{R}^S + i\rho^a\cdot,$$

which will be applied to sections of type $\phi\sigma = \sigma\otimes\phi \in C^\infty_{\text{cpt}}(M, S\otimes\mathbf{C}_a)$, for arbitrary $\phi \in C^\infty_{\text{cpt}}(M,\mathbf{C})$.

Therefore,

$$\left(D^2_{S\otimes\mathbf{C}_a}(\phi\sigma), \phi\sigma\right) = \left(\nabla^{S\otimes\mathbf{C}_a}\sigma\otimes\phi, \nabla^{S\otimes\mathbf{C}_a}\sigma\otimes\phi\right) + \left(\phi\mathcal{R}^S\sigma, \rho\sigma\right) + \left(i\phi\rho^a\cdot\sigma, \phi\sigma\right). \tag{9}$$

However, $\nabla^{S\otimes\mathbf{C}_a}\sigma\otimes\phi = \nabla^S\sigma\otimes\phi + \sigma\otimes d^a\phi = \sigma\otimes d^a\phi$, since σ is ∇^S-parallel. For the same reason, $\mathcal{R}^S\sigma = 0$. By the hypothesis (8), σ is non-trivial, and since ∇^S is a metric connection, $\langle\sigma,\sigma\rangle$ is a (positive) constant function on M. By scaling σ appropriately we can assume that $\langle\sigma,\sigma\rangle = 1$.

Consequently, $\left(\nabla^{S\otimes\mathbf{C}_a}\sigma\otimes\phi, \nabla^{S\otimes\mathbf{C}_a}\sigma\otimes\phi\right) = (\sigma\otimes d^a\phi, \sigma\otimes d^a\phi) = \int_M \langle\sigma,\sigma\rangle\langle d^a\phi, d^a\phi\rangle dvol = \int_M \langle d^a\phi, d^a\phi\rangle dvol = (H_a\phi,\phi)$.

Equation (9) now becomes

$$||D_{S\otimes\mathbf{C}_a}(\phi\sigma)||^2 = (H_a\phi,\phi) + \left(\langle i\rho^a\cdot\sigma,\sigma\rangle\phi,\phi\right),$$

which implies

$$(H_a\phi,\phi) \geq \left(-\langle i\rho^a\cdot\sigma,\sigma\rangle\phi,\phi\right).$$

The result follows by applying Proposition 1 to the function $f = -i\langle\rho^a\cdot\sigma,\sigma\rangle$, in the presence of the hypothesis (8). $\square$

A successful application of the above proposition rests obviously on the ability of finding Dirac bundles with non-trivial parallel sections σ for which $\langle\rho^a\cdot\sigma,\sigma\rangle$ can be effectively computed. This is indeed the case with the theorem stated in the introduction.

Proof of the Theorem. For a Kähler manifold of complex dimension m, $n = 2m$. If ω is the Kähler form inducing the Riemannian metric g and if J is the integrable complex structure on TM then there is a local orthonormal frame $(e_1, \mathrm{J}e_1, e_2, \mathrm{J}e_2, \ldots, e_m, \mathrm{J}e_m)$ in TM such that $\omega = e_1^*\wedge(\mathrm{J}e_1)^* + e_2^*\wedge(\mathrm{J}e_2)^* + \cdots + e_m^*\wedge(\mathrm{J}e_m)^*$. Expanding on the discussion on Kähler manifolds initiated in section 3, case c), $T^{*0,1}M$ is the space dual to $T^{0,1}M := \{v \in T^*M \otimes \mathbf{C} \mid \mathrm{J}v = -iv\}$. Since a local orthonormal basis of $T^{0,1}M$ is $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \ldots, \overline{\epsilon}_m\}$, $\overline{\epsilon}_j := \frac{1}{\sqrt{2}}(e_j + i\mathrm{J}e_j)$, a local orthonormal basis of $T^{*0,1}M$ will be $\{\overline{\epsilon}_1{}^*, \overline{\epsilon}_2{}^*, \ldots, \overline{\epsilon}_m{}^*\}$, with $\overline{\epsilon}_j{}^* := \frac{1}{\sqrt{2}}\left(e_j^* - i(\mathrm{J}e_j)^*\right)$. So, for the Dirac bundle $\Lambda(T^{*0,1}M)$ a local orthonormal basis for $\Lambda^p(T^{*0,1}M)$ is $\{\overline{\epsilon}_J{}^*\}_J$, $\overline{\epsilon}_J{}^* = \overline{\epsilon}_{j_1}{}^* \wedge \overline{\epsilon}_{j_2}{}^* \wedge \ldots \overline{\epsilon}_{j_p}{}^*$, $J = (j_1, j_2, \ldots, j_p)$ p-multi-index.

The Clifford multiplication in $\Lambda(T^{*0,1}M)$ is then implemented by setting

$$e_j \cdot = \overline{\epsilon}_j{}^* \wedge - \,\overline{\epsilon}_j \lrcorner\,, \quad (\mathrm{J}e_j) \cdot = i\,(\overline{\epsilon}_j{}^* \wedge + \,\overline{\epsilon}_j \lrcorner)\,, \quad j = 1, 2, \ldots, m. \tag{10}$$

In preparation for applying proposition 2 notice that $\sigma := 1 \in C^\infty(M, \Lambda^0(T^{*0,1}M))$ is a parallel section of $\Lambda(T^{*0,1}M))$. An elementary calculation based on (10) and (7) shows now that

$$\langle i\rho^a \cdot \sigma, \sigma \rangle = \sum_{j=1}^m R^a_{e_j, \mathrm{J}e_j}.$$

The theorem follows from proposition 2 and the hypothesis (1), since $a = \sum_{j=1}^m a(e_j)e_j^* + \sum_{j=1}^m a(\mathrm{J}e_j)(\mathrm{J}e_j)^*$ implies $\langle da, \omega \rangle = \sum_{j=1}^m R^a_{e_j, \mathrm{J}e_j} = \langle i\rho^a \cdot \sigma, \sigma \rangle$. $\square$

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