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A certain class of quadratures with even Tchebychev weights

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Abstract

We are considering the quadrature formulas of "practical type" (with five knots) for approximate computation of integral

$$\int_{-1}^{1} w(x)f(x)dx, \tag{1}$$

where $w(\cdot)$ denotes (even) Tchebychev weight function. We prove that algebraic degree of exactness of those formulas can not be greater than five. We also determined some admissible nodes and compared proposed formula with some other quadrature formulas.

1 Introduction and preliminaries

Very important place in approximation theory stands for the Tchebychev weight functions

$$w_1(x) = \frac{1}{\sqrt{1-x^2}} \text{ (the first Tchebychev weight)}$$

and

$$w_2(x) = \sqrt{1-x^2} \text{ (the second Tchebychev weight).}$$

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Except those weight functions, one can find the third $w_3(x) = \sqrt{(1-x) \,/\, (1+x)}$ and the fourth $w_4(x) = \sqrt{(1+x) \,/\, (1-x)}$ Tchebychev weight, which are not under our consideration. Therein, the problem of approximate computation of the integral (1) is very frequent. In this paper we are investigating a certain class of quadratures (the so called quadratures of "practical type") for approximate computation of the previous integral. The paper was motivated by results recently published in [1] and [2], where the same class of quadratures was considered, but without weight function. Similar problem (with linear and cubic B-spline as a weight function) was considered in [4] and [5]. We shall suppose that in integral (1) and in all following integrals we have $w(x) = w_1(x)$ or $w(x) = w_2(x)$.

We say that quadrature formula

$$\int_{-1}^{1} w(x)f(x)dx = \sum_{i=1}^{5} A_i f(x_i) + R[f] \tag{2}$$

is of "practical type" if the following conditions hold:

1. $A_1 = A_5$ and $A_2 = A_4$;
2. nodes $x_k, 1 \leq k \leq 5$ are symmetric and rational numbers from the interval $[-1, 1]$, i.e. $x_1 = -r_1, x_2 = -r_2, x_3 = 0, x_4 = r_2$ and $x_5 = r_1$, for some $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$ (as usual, $\mathbb{Q}$ denotes the set of rational numbers).

Hence, quadratures of "practical type" have the following form:

$$\begin{aligned} \int_{-1}^{1} w(x)f(x)dx \quad &= \quad A(f(-r_1) + f(r_1)) + \\ &\qquad B(f(-r_2) + f(r_2)) + Cf(0) + R[f], \end{aligned} \tag{3}$$

for some $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$.

Quadrature rule (2) has algebraic degree of exactness equal to $m, m \in \mathbb{N}$, if and only if $R[p] = 0$ whenever $p(\cdot)$ is a polynomial of degree not greater than m and there exists the polynomial $q(\cdot)$, of degree $m + 1$, such that $R[q] \neq 0$. Our aim is construction of the quadrature rules of "practical type" with maximal algebraic degree of exactness.

We continue this section with some well known facts from the theory of numerical integration. More details one can find in [3].

Lemma 1. *Quadrature rule* (2) *(i.e.* (3)*) integrates exactly all polynomials of degree* $m, m \in \mathbb{N}$, *if and only if* $R[x^k] = 0$ *for all* $k \in \{0, 1, \ldots, m\}$.

Lemma 2. *Quadrature rule* (3) *integrates exactly every odd function* $f(\cdot)$ *(i.e.* $R[f] = 0$ *for every odd function* $f(\cdot)$*).*

From the previous lemmas follows that algebraic degree of exactness of the formula (3) has to be odd.

Finally, with the choice

$$x_{1,2,4,5} = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}, x_3 = 0$$

and

$$A_k = \pi/5, 1 \leq k \leq 5,$$

for the first Tchebychev weight, or with the choice

$$x_{1,5} = \pm\frac{\sqrt{3}}{2}, x_{2,4} = \pm\frac{1}{2}, x_3 = 0$$

and

$$A_{1,5} = \frac{\pi}{24}, A_{2,4} = \frac{\pi}{8}, A_3 = \frac{\pi}{6}$$

for the second Tchebychev weight, formula (2) attains maximal algebraic degree of exactness (which is equal to nine), but those formulas obviously are not of "practical type". Hence, algebraic degree of exactness of the formula (3) can not be greater than seven.

Now, let us determine (by using standard procedure) the coefficients A, B and C such that formula (3) has maximal algebraic degree of exactness. Therein, we use notation $m_k = \int_{-1}^{1} w(x)x^k dx, \quad k \in \{0, 2, 4\}$.

From the condition that formula (3) is exact for $f(x) = 1$ (i.e. exact for any polynomial of zero degree) simply follows

$$C = m_0 - 2A - 2B. \tag{4}$$

Furthermore, conditions that formula (3) is exact for $f(x) = x^2$ and $f(x) = x^4$ give the following system of linear equations

$$\begin{array}{rcl} 2r_1^2 A + 2r_2^2 B & = & m_2, \\ 2r_1^4 A + 2r_2^4 B & = & m_4, \end{array}$$

which has unique solution

$$A = \frac{m_2 r_2^2 - m_4}{2r_1^2(r_2^2 - r_1^2)} \quad \text{and} \quad B = \frac{m_4 - m_2 r_1^2}{2r_2^2(r_2^2 - r_1^2)}. \tag{5}$$

Hence, with this choice of the coefficients A, B and C formula (3) has algebraic degree of exactness equal to five. It is natural to ask is it possible to choose rational nodes r_1 and r_2 such that formula (3) has algebraic degree of exactness equal to six, i.e. seven. Negative answer to this question is given in the following section.

2 Main result

We are considering Tchebychev weights separately.

2.1 The first Tchebychev weight

For the first Tchebychev weight we have $m_0 = \pi, m_2 = \pi/2$ and $m_4 = 3\pi/8$. Putting those values in (5) and (4) we obtain the corresponding values for the coefficients A, B and C in (3). Therein, in formula (3) we have

$$R[x^6] = \frac{\pi}{8}\left[\frac{5}{2} - \left(3r_1^2 - 4r_1^2 r_2^2 + 3r_2^2\right)\right].$$

Lemma 3. *There are no numbers $r_1, r_2 \in (0,1] \cap \mathbb{Q}$ such that*

$$3r_1^2 - 4r_1^2 r_2^2 + 3r_2^2 = \frac{5}{2}. \tag{6}$$

Proof: Let us assume contrary, i.e. that $r_1 = a/b$ and $r_2 = c/d$, for some $a, b, c, d \in \mathbb{N}$, such that $(a, b) = 1$ and $(c, d) = 1$. Putting this in equality (6), after simplification, gives

$$6a^2d^2 - 3a^2c^2 + 6b^2c^2 = 5\left(a^2c^2 + b^2d^2\right), \tag{7}$$

from which follows that $a^2c^2 + b^2d^2 \equiv 0 (\text{mod } 3)$. Having in mind that the square of number divided by three gives reminder equal to zero or one, from the last relation follows that it has to be $a \equiv 0(\text{mod } 3)$ and $d \equiv 0(\text{mod } 3)$ or $b \equiv 0(\text{mod } 3)$ and $c \equiv 0(\text{mod } 3)$.

Let $a = 3\alpha$ and $d = 3\delta$, for some $\alpha, \delta \in \mathbb{N}$. Putting this in (7), after simplification, we obtain that

$$6\alpha^2\left(27\delta^2 - 4c^2\right) = b^2(15\delta^2 - 2c^2),$$

from which simply follows that it has to be $b \equiv 0(\text{mod } 3)$ or $c \equiv 0(\text{mod } 3)$, which is impossible since $a \equiv 0(\text{mod } 3)$ and $d \equiv 0(\text{mod } 3)$.

The case $b = 3\beta$ and $c = 3\gamma$, for some $\beta, \gamma \in \mathbb{N}$, because of symmetry of the relation (7) can be proved analogue. This completes the proof. ▶

Let us estimate the error of the formula (3) in the case of the first Tchebychev weight. Let $H_5(\cdot)$ be Hermite's interpolating polynomial which interpolates the function $f(\cdot)$ through the points $\pm r_1, \pm r_2$ and 0, where the node 0 has multiplicity two. Then (see for example [3], p. 55),

$$f(x) - H_5(x) = \frac{f^{(vi)}(\xi(x))}{6!} x^2(x^2 - r_1^2)(x^2 - r_2^2),$$

and the error of the formula (3) in this case is given by

$$\begin{aligned} R[f] &= \int_{-1}^{1} \frac{f^{(vi)}(\xi(x))}{\sqrt{1-x^2}} \cdot \frac{x^2(x^2-r_1^2)(x^2-r_2^2)}{6!} dx \\ &= \frac{\pi}{2 \cdot 6!} f^{(vi)}(\eta)(\eta^2-r_1^2)(\eta^2-r_2^2), \end{aligned}$$

for some $\eta \in [-1,1]$, assuming $f(\cdot) \in C^6[-1,1]$. Let

$$\Phi(\eta) = (\eta^2 - r_1^2)(\eta^2 - r_2^2).$$

It is easy to check that

$$\begin{aligned} \max_{\eta \in [-1,1]} |\Phi(\eta)| &= \max \left\{ |\Phi(0)|, \left| \Phi(\sqrt{\frac{r_1^2+r_2^2}{2}}) \right|, |\Phi(1)| \right\} \\ &= \max \left\{ r_1^2 r_2^2, \frac{(r_1^2-r_2^2)^2}{4}, (1-r_1^2)(1-r_2^2) \right\}, \end{aligned}$$

so the error of the formula (3) can be estimated in the following way

$$\left| R[f] \right| \le \frac{M_6 \cdot \pi}{2 \cdot 6!} \max \left\{ r_1^2 r_2^2, \frac{(r_1^2-r_2^2)^2}{4}, (1-r_1^2)(1-r_2^2) \right\}, \tag{8}$$

where $M_6 = \max_{x \in [-1,1]} \left| f^{(vi)}(x) \right|$.

2.2 The second Tchebychev weight

In this case we have $m_0 = \pi/2, m_2 = \pi/8$ and $m_4 = \pi/16$, and after determining the coefficients A, B and C (by using (5) and (4)), elementary calculation gives

$$R[x^6] = \frac{\pi}{16} \left[\frac{5}{8} - \left(r_1^2 - 2r_1^2 r_2^2 + r_2^2 \right) \right]$$

in formula (3).

Lemma 4. *There are no numbers $r_1, r_2 \in (0,1] \cap \mathbb{Q}$ such that*

$$r_1^2 - 2r_1^2 r_2^2 + r_2^2 = \frac{5}{8}. \tag{9}$$

Proof: Let us assume contrary again, i.e. that $r_1 = a/b$ and $r_2 = c/d$, for some $a, b, c, d \in \mathbb{N}$, such that $(a,b) = 1$ and $(c,d) = 1$. Putting this in equality (9), after simplification, we have

$$8\left(a^2 d^2 - 2a^2 c^2 + b^2 c^2 \right) = 5b^2 d^2, \tag{10}$$

from which follows that $b^2d^2 \equiv 0 (\mathrm{mod}\ 8)$, i.e. that $b \equiv 0(\mathrm{mod}\ 2)$ or $d \equiv 0(\mathrm{mod}\ 2)$.

Let $b = 2k$, for some $k \in \mathbb{N}$, and put this in (10). After simplification, we obtain

$$2\left(a^2d^2 - 2a^2c^2 + 4k^2c^2\right) = 5k^2d^2, \tag{11}$$

and conclude that $k^2d^2 \equiv 0(\mathrm{mod}\ 2)$, i.e. $k^2d^2 \equiv 0(\mathrm{mod}\ 2^2)$. Hence, it has to be $a^2d^2 - 2a^2c^2 + 4k^2c^2 \equiv 0(\mathrm{mod}\ 2)$, from which follows that $a^2d^2 \equiv 0(\mathrm{mod}\ 2)$, i.e. $d \equiv 0(\mathrm{mod}\ 2)$. Let $d = 2l$, for some $l \in \mathbb{N}$. Now, equation (11) becomes

$$2a^2l^2 - a^2c^2 + 2k^2c^2 = 5k^2l^2.$$

From the last equality we obtain (since the numbers a and c are odd) that k and l have to be odd too. Finally, since the square of odd number divided by four gives reminder equal to one, we conclude that the left hand side of the last equation, divided by four gives reminder equal to three, while the right hand side of the same equation, divided by four gives reminder equal to one. Contradiction!

Because of symmetry, the case $d = 2m$, for some $m \in \mathbb{N}$, can be proved analogue. The proof is complete. ►

By using the same technics as in the case of the first Tchebychev weight, we obtain that the error of the formula (3) can be estimated in the following way

$$\left|R[f]\right| \leq \frac{M_6 \cdot \pi}{8 \cdot 6!} \max\left\{r_1^2r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2)\right\}. \tag{12}$$

3 Determination of some admissible nodes

Estimations (8) and (12) naturally impose the following problem

$$F(r_1, r_2) = \max\left\{r_1^2r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2)\right\} \to \min.,$$

where $r_1, r_2 \in (0, 1] \cap \mathbb{Q}, r_2 < r_1$. It is obvious that, for fixed $r_1 \in (0, 1] \cap \mathbb{Q}$, the function $F(\cdot, \cdot)$ attains its minimum in one of the intersection points among three curves $r_1^2r_2^2$, $\left(r_1^2 - r_2^2\right)^2/4$ and $(1 - r_1^2)(1 - r_2^2)$.

1. Curves $r_1^2r_2^2$ and $\left(r_1^2 - r_2^2\right)^2/4$ (r_1 is fixed) intersect each other at $r_2 = \pm(1 \pm \sqrt{2})r_1$, and since $r_2 \notin \mathbb{Q}$ we will not consider this case.

2. Similarly, curves $\left(r_1^2 - r_2^2\right)^2/4$ and $(1 - r_1^2)(1 - r_2^2)$ (r_1 is fixed) intersect each other at $r_2 = \pm\sqrt{3r_1^2 - 2 \pm 2\sqrt{2}(r_1^2 - 1)}$, and again because of $r_2 \notin \mathbb{Q}$ we will not consider this case.

3. Finally, curves $r_1^2 r_2^2$ and $(1-r_1^2)(1-r_2^2)$ (r_1 is still fixed) intersect each other at $r_2 = \sqrt{1-r_1^2}$, and we will look for the nodes r_1 and r_2 among "rational points" from the unit circle.

In the Table 1 we give some admissible values of the nodes r_1 and r_2 for which the function $F(\cdot,\cdot)$ attains its local minimums. The corresponding rational numbers are round off to the six decimal places.

r_1			r_2			$F(r_1,r_2)$
$\frac{4}{5}$	$=$	0.8	$\frac{3}{5}$	$=$	0.6	0.230400
$\frac{21}{29}$	$=$	0.724138	$\frac{20}{29}$	$=$	0.689655	0.249406
$\frac{55}{73}$	$=$	0.753425	$\frac{48}{73}$	$=$	0.657534	0.245424
$\frac{72}{97}$	$=$	0.742268	$\frac{65}{97}$	$=$	0.670103	0.247403
$\frac{377}{505}$	$=$	0.746535	$\frac{336}{505}$	$=$	0.665347	0.246715
$\frac{987}{1325}$	$=$	0.744906	$\frac{884}{1325}$	$=$	0.667170	0.246988
$\frac{1292}{1733}$	$=$	0.745528	$\frac{1155}{1733}$	$=$	0.666474	0.246885

Table 1: Some admissible nodes

Let us also say that, by using any of the given choices for the nodes r_1 and r_2, the errors (8) and (12) can be roughly estimated by

$$\left|R[f]\right| \le 0.6 \cdot 10^{-3} \cdot M_6,$$

i.e. by

$$\left|R[f]\right| \le 0.2 \cdot 10^{-3} \cdot M_6$$

respectively.

4 Comparation with some other formulas and conclusions

In the last section we shall compare formula (3), with rational nodes $(r_1, r_2) = (4/5, 3/5)$, with a several quadrature formulas of algebraic degree of exactness equal to five. Therein, we shall bound the error of each formulas by

$$\frac{M_6 \cdot \pi}{2 \cdot 6!} \cdot C_1$$

in the case of the first Tchebychev weight, i.e. by

$$\frac{M_6 \cdot \pi}{8 \cdot 6!} \cdot C_2$$

in the case of the second Tchebychev weight. Of course, by the choice $(r_1, r_2) = (4/5, 3/5)$ we have

$$C_1 = C_2 = \max\left\{r_1^2 r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2)\right\} = \frac{144}{625} = 0.2304.$$

The first two formulas we are going to compare with the proposed one are obtained from (3) by the choice $(r_1, r_2) = (2/3, 1/3)$, i.e. $(r_1, r_2) = (1, 1/2)$. Obviously, in this case we have open and closed formula with equidistant nodes, respectively. In the case of open formula we have

$$\begin{aligned} C_1 &= C_2 \\ &= \max\left\{r_1^2 r_2^2, \frac{(r_1^2 - r_2^2)^2}{4}, (1 - r_1^2)(1 - r_2^2)\right\} = \frac{40}{81} = 0.4938\ldots, \end{aligned}$$

while the constants C_1 and C_2, in the case of closed formula, we calculated by using standard procedure and obtained

$$C_1 = \frac{3}{16} = 0.1875$$

and

$$C_2 = \frac{3}{8} = 0.3750.$$

The following two formulas require a more detailed explanation. Namely, it is well known that

$$\min_{r_1, r_2 \in [-1,1]} \left(\max_{\eta \in [-1,1]} \left|(\eta^2 - r_1^2)(\eta^2 - r_2^2)\right| \right) = \frac{1}{8}$$

and that this value will be obtained in the case when $(\eta^2-r_1^2)(\eta^2-r_2^2)$ is a monic Tchebychev polynomial of the first kind and fourth degree, i.e. in the case $(\eta^2-r_1^2)(\eta^2-r_2^2)=\eta^4-\eta^2+1/8$. In this case, nodes of the quadrature formula (3) are given by $r_{1,2}=\sqrt{2\pm\sqrt{2}}\Big/2$ (those nodes also lie on the unit circle). Of course, with those nodes formula (3) is not of "practical type", but the previous nodes can be approximated by the rational with an arbitrary precision. Hence, we shall compare proposed formula with the formulas obtained from (3) by the choice $(r_1,r_2)=(\sqrt{2+\sqrt{2}}/2,\sqrt{2-\sqrt{2}}/2)$, i.e. by the choice $(r_1,r_2)=(924/1000,383/1000)$. Therein we have

$$C_1=C_2=\frac{1}{8}=0.1250$$

for the formula which is not "practical type" and

$$\begin{aligned} C_1 &= C_2 \\ &= \max\left\{r_1^2r_2^2,\frac{(r_1^2-r_2^2)^2}{4},(1-r_1^2)(1-r_2^2)\right\}=0.1252\ldots \end{aligned}$$

in the case of rational approximation of the corresponding nodes.

Finally, we shall also compare proposed formula with the Gaussian quadrature formulas of algebraic degree of exactness equal to five, i.e. by the formulas

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx\approx\frac{\pi}{3}\left(f\left(-\frac{\sqrt{3}}{2}\right)+f\left(0\right)+f\left(\frac{\sqrt{3}}{2}\right)\right)$$

and

$$\int_{-1}^{1}\sqrt{1-x^2}f(x)dx\approx\frac{\pi}{8}\left(f\left(-\frac{\sqrt{2}}{2}\right)+2f\left(0\right)+f\left(\frac{\sqrt{2}}{2}\right)\right).$$

For these formulas (see for example [3], p. 168) we have

$$C_1=C_2=\frac{1}{16}=0.0625.$$

Hence, error estimation of the proposed formula is weaker than error estimations of the other formulas, except in the case of open formula with equidistant nodes and in the case of closed formula with the second Tchebychev weight. So, it is natural to expect weaker results of the proposed formula. The following numerical examples justify expected results just in the case of the first Tchebychev weight. On the other hand, in the case of the second Tchebychev weight, numerical examples indicate quite reasonable application of the proposed formula.

r_1	r_2	$\left\|R\left[\sqrt{x^2-4x+13}\right]\right\|$	$\left\|R\left[\cos x^2\right]\right\|$
$\frac{4}{5}$	$\frac{3}{5}$	0.1498[−4]	0.1244[−1]
$\frac{2}{3}$	$\frac{1}{3}$	0.3694[−4]	0.2397[−1]
1	$\frac{1}{2}$	0.8862[−5]	0.7721[−2]
$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	0.4627[−7]	0.8863[−3]
$\frac{924}{1000}$	$\frac{383}{1000}$	0.6175[−7]	0.8727[−3]
-	-	0.8862[−5]	0.7725[−2]

Table 2: The first Tchebychev weight

Example 1. *In the first example we approximately calculated*

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx,$$

where we put $f(x) = \sqrt{x^2-4x+13}$, *i.e.* $f(x) = \cos x^2$. *Results of computation are given in the* Table 2. *Knots of the quadrature formulas are given in the first and in the second column, while the absolute errors are given in the last two columns. The last row of the table corresponds to the Gaussian quadrature rule. As usually, the numbers in parentheses indicate the decimal exponent.*

Hence, in the case of the first Tchebychev weight, results obtained by using the proposed formula are better only than the results obtained by using the open formula with equidistant knots.

Example 2. *In this example we approximately compute*

$$\int_{-1}^{1} \sqrt{1-x^2} f(x) dx,$$

with the same choices of the function $f(\cdot)$ *as in the previous one. Results of*

r_1	r_2	$\left\|R\left[\sqrt{x^2-4x+13}\right]\right\|$	$\left\|R\left[\cos x^2\right]\right\|$
$\frac{4}{5}$	$\frac{3}{5}$	$0.1536[-5]$	$0.1014[-2]$
$\frac{2}{3}$	$\frac{1}{3}$	$0.3019[-5]$	$0.1849[-2]$
1	$\frac{1}{2}$	$0.2216[-5]$	$0.1936[-2]$
$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$0.2238[-5]$	$0.1481[-2]$
$\frac{924}{1000}$	$\frac{383}{1000}$	$0.2238[-5]$	$0.1482[-2]$
-	-	$0.2238[-5]$	$0.1481[-2]$

Table 3: The second Tchebychev weight

computation are given in the Table 3 *which hase the same form as the* Table 2*.*

So, in this case, the best results are obtained just by using proposed formula.

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