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On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary Δ_m^n-statistical convergence

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Abstract

In this article, we introduce the lacunary difference sequence spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$, $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ using a sequence $\boldsymbol{M} = (M_k)$ of Orlicz functions and investigate some relevant properties of these spaces. Then, we define and study the notion of q-lacunary Δ_m^n-statistical convergent sequences. Further, we study the relationship between q-lacunary Δ_m^n-statistical convergent sequences and the spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

1 Introduction

The notion of difference sequence space was introduced by Kizmaz [10], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [4] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [23], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

Tripathy, Esi and Tripathy [24] generalized the above notions and unified these as follows:

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Let m, n be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$, for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

The notion of difference sequences was investigated from different aspects by Tripathy [16], Tripathy, Altin and Et [17], Tripathy and Baruah [18], Tripathy and Borgogain [20], Tripathy, Choudhary and Sarma [21], Tripathy and Dutta [22], Tripathy and Mahanta [27] are a few to be named.

The notion of statistical convergence was studied at the initial stage by Fast [5] and Schoenberg [13] independently. Later on, it was further investigated by Fridy [6], Rath and Tripathy [12], Šalàt [14], Tripathy ([15], [16]), Tripathy and Baruah [19], Tripathy and Sarma [28], Tripathy and Sen [32] and many others.

A subset E of N is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. For $L = 0$, we say (x_k) is statistically null.

By a lacunary sequence $\theta = (k_r)$; $r = 1, 2, 3, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ and $\eta_r = \frac{k_r}{k_{r-1}}$, for $r = 1, 2, 3, \ldots$. The space of lacunary strongly convergent sequence N_θ was defined by Freedman, Sember and Raphael [7] as follows:

$$N_\theta = \{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L\}.$$

The space N_θ is a BK-space with the norm

$$\|x\|_\theta = \sup_r \frac{1}{h_r} \sum_{k \in I_r} |x_k|.$$

N_θ^0 denotes the subset of those sequences in N_θ for which $L = 0$. $(N_\theta^0, \|.\|_\theta)$ is also a BK-space. Freedman, Sember and Raphael [7] also defined the space $|\sigma_1|$ of strongly Cesàro summable sequences as follows:

$$|\sigma_1| = \{x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ for some } L\}.$$

In the special case when $\theta = (2^r)$, $N_\theta = |\sigma_1|$.

The notion of lacunary convergence has been investigated by Colak, Tripathy and Et [2], Tripathy and Baruah [19], Tripathy and Mahanta [27] and many others.

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [11] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

In the recent past the notion of Orlicz function was investigated from different aspects and sequence spaces have been studied by Altin, Et and Tripathy [1], Et, Altin, Choudhary and Tripathy [3], Hudzik, Kamińska and Mastylo [8], Isik, Et and Tripathy [9], Tripathy, Altin and Et [17], Tripathy and Borgogain [20], Tripathy and Dutta [22], Tripathy and Hazarika [26], Tripathy and Mahanta [27], Tripathy and Sarma ([29], [30], [31]) and many others.

Remark 1.1. An Orlicz function M satisfies the inequality $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D \left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}.$$

The notion of paranormed sequences has been investigated from sequence space point of view and linked with summability theory by Rath and Tripathy [12], Tripathy [16], Tripathy and Dutta [22], Tripathy and Hazarika [25], Tripathy and Sen ([32], [33]) and many others.

Definition 1.1. Two non-negative functions f, g are called equivalent, whenever $C_1 f \leq g \leq C_2 f$, for some $C_j > 0$, $j = 1, 2$ and in this case we write $f \approx g$.

2 Definition and Preliminaries

Lemma 2.1. *(Işik, Et and Tripathy [9], Lemma1.1) Let p and q be seminorms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$.*

Let $\boldsymbol{M}=(M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and X be a seminormed space over the field C of complex numbers with the seminorm q. $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$, for all $k \in N$. We define the following sequence spaces:

$$w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = \{x \in w(X) : \lim_{r\to\infty} \tfrac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(q\left(\tfrac{\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} = 0, \text{ for some } \rho > 0\},$$

$$w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = \{x \in w(X) : \lim_{r\to\infty} \tfrac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(q\left(\tfrac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in X\},$$

$$w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = \{x \in w(X) : \sup_r \tfrac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(q\left(\tfrac{\Delta_m^n x_k}{\rho}\right)\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

If $M_k(x) = x$, for all $x \in [0,\infty)$, for all $k \in N$, $p_k = 1$, for all $k \in N$, $X = C$, $q(x) = |x|$, for all $x \in X$ and $n = 0$ so that $\Delta_m^0 x_k = x_k$, for all $k \in N$, then $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = N_\theta$ and $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = N_\theta^0$. If in addition, we take $\theta = (2^r)$, then $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q) = |\sigma_1|$.

3 Main Results

In this section, we investigate the results of this paper involving the spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$, $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

Theorem 3.1. *Let $\boldsymbol{M} = (M_k)$ be a sequence of Orlicz functions. Then*

$$w_0(\mathbf{M}, \theta, \Delta_m^n, p, q) \subset w_1(\mathbf{M}, \theta, \Delta_m^n, p, q) \subset w_\infty(\mathbf{M}, \theta, \Delta_m^n, p, q).$$

Proof. It is obvious that $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q) \subseteq w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$. We shall prove that $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q) \subseteq w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

Let $(x_k) \in w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$. Then there exist some $\rho > 0$ and $L \in X$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k \left(q \left(\frac{\Delta_m^n x_k - L}{\rho} \right) \right) \right]^{p_k} = 0.$$

On taking $\rho_1 = 2\rho$, we have

$$\begin{aligned} &\tfrac{1}{h_r} \sum_{k\in I_r} \left[M_k \left(q \left(\tfrac{\Delta_m^n x_k}{\rho_1} \right) \right) \right]^{p_k} \\ &\qquad \le \tfrac{D}{h_r} \sum_{k\in I_r} \left[\tfrac{1}{2} M_k \left(q \left(\tfrac{\Delta_m^n x_k - L}{\rho} \right) \right) \right]^{p_k} + \tfrac{D}{h_r} \sum_{k\in I_r} \left[\tfrac{1}{2} M_k \left(q \left(\tfrac{L}{\rho} \right) \right) \right]^{p_k} \\ &\qquad \le \tfrac{D}{h_r} \sum_{k\in I_r} \left[\tfrac{1}{2} M_k \left(q \left(\tfrac{\Delta_m^n x_k - L}{\rho} \right) \right) \right]^{p_k} + D \max \left(1, \sup \left[\tfrac{1}{2} M_k \left(q \left(\tfrac{L}{\rho} \right) \right) \right]^{H} \right), \end{aligned}$$

where $\sup\limits_k p_k = G$, $H = \max(1, G)$ and $D = \max(1, 2^{G-1})$.

Thus we get $(x_k) \in w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

The inclusions are strict follows from the following examples.

Example 3.1. Let $m = n = 2$, $\theta = (3^r)$, $p_k = 1$, for all $k \in N$, $X = C^2$, $q(x) = \max(|x^1|, |x^2|)$, for $x = (x^1, x^2) \in C^2$ and $M_k(x) = x^2$, for all $x \in [0, \infty)$ and $k \in N$. Consider the sequence (x_k) defined by $x_k = (k^2, k^2)$ for each fixed $k \in N$. Then $(x_k) \in w_1(\mathbf{M}, \theta, \Delta_m^n, p, q)$, but $(x_k) \notin w_0(\mathbf{M}, \theta, \Delta_m^n, p, q)$.

Example 3.2. Let $m = n = 2$, $\theta = (2^r)$, $p_k = 2$, for all k odd and $p_k = 3$, for all k even, $X = C^3$, $q(x) = \max(|x^1|, |x^2|, |x^3|)$, for $x = (x^1, x^2, x^3) \in C^3$ and $M_k(x) = x^4$, for all $x \in [0, \infty)$ and $k \in N$. Consider the sequence (x_k) defined by $x_k = (k, k, k)$ for each fixed $k \in N$. Then $(x_k) \in w_\infty(\mathbf{M}, \theta, \Delta_m^n, p, q)$, but $(x_k) \notin w_1(\mathbf{M}, \theta, \Delta_m^n, p, q)$.

Corollary 3.2. *$w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ are nowhere dense subsets of $w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.*

Proof. Proof is a consequence of Theorem 3.1.

Proof of the following theorem is easy, so omitted.

Theorem 3.3. *The spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$, $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ are linear.*

Theorem 3.4. *The spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$, $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and*

$w_\infty(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ *are paranormed spaces paranormed by*

$$g(x) = \sum_{i=1}^{mn} q(x_i) + \inf\left\{\rho^{\frac{p_r}{H}} : \sup_k \left[M_k\left(q\left(\frac{\Delta_m^n x_k}{\rho}\right)\right)\right] \leq 1, \rho > 0, r \in N\right\},$$

where $H = \max(1, \sup_r p_r)$.

Proof. Clearly $g(x) = g(-x)$. Since $M_k(0) = 0$, for all $k \in N$, we get $\inf\left\{\rho^{\frac{p_r}{H}}\right\} = 0$ for $x = \theta$. Now let $x, y \in w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and choose $\rho_1, \rho_2 > 0$ such that

$$\sup_k k\left[M_k\left(q\left(\frac{\Delta_m^n x_k}{\rho_1}\right)\right)\right] \leq 1 \text{ and } \sup_k \left[M_k\left(q\left(\frac{\Delta_m^n y_k}{\rho_2}\right)\right)\right] \leq 1$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned} &\sup_k \left[M_k\left(q\left(\tfrac{\Delta_m^n (x_k + y_k)}{\rho}\right)\right)\right] \\ &\quad\leq \left(\tfrac{\rho_1}{\rho_1+\rho_2}\right) \sup_k \left[M_k\left(q\left(\tfrac{\Delta_m^n x_k}{\rho_1}\right)\right)\right] + \left(\tfrac{\rho_2}{\rho_1+\rho_2}\right) \sup_k \left[M_k\left(q\left(\tfrac{\Delta_m^n y_k}{\rho_2}\right)\right)\right] \\ &\quad\leq \left(\tfrac{\rho_1}{\rho_1+\rho_2}\right) + \left(\tfrac{\rho_2}{\rho_1+\rho_2}\right) = 1. \end{aligned}$$

Hence $g(x+y) \leq g(x) + g(y)$.

Finally let λ be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality

$$g(\lambda x) = \sum_{i=1}^{mn} q(\lambda x_i) + \inf\left\{\rho^{\frac{p_r}{H}} : \sup_k \left[M_k\left(q\left(\tfrac{\Delta_m^n(\lambda x_k)}{\rho}\right)\right)\right] \leq 1\right\}$$

$$= |\lambda| \sum_{i=1}^{mn} q(x_i) + \inf\left\{(|\lambda| s)^{\frac{p_r}{H}} : \sup_k \left[M_k\left(q\left(\tfrac{\Delta_m^n(x_k)}{s}\right)\right)\right] \leq 1\right\}, \text{ where } s = \tfrac{\rho}{|\lambda|}.$$

This completes the proof.

Proof of the following result is easy, so omitted.

Theorem 3.5. *Let* $\boldsymbol{M} = (M_k)$ *and* $\boldsymbol{T} = (T_k)$ *be sequences of Orlicz functions and* $Z = w_0$, w_1 *and* w_∞. *Then for any two sequences* $p = (p_k)$ *and* $t = (t_k)$ *of bounded positive real numbers and for any two seminorms* q_1 *and* q_2, *we have*
(i) If q_1 *is stronger than* q_2, *then* $Z(\boldsymbol{M}, \theta, \Delta_m^n, p, q_1) \subset Z(\boldsymbol{M}, \theta, \Delta_m^n, p, q_2)$,
(ii) $Z(\boldsymbol{M}, \theta, \Delta_m^n, p, q_1) \bigcap Z(\boldsymbol{M}, \theta, \Delta_m^n, p, q_2) \subset Z(\boldsymbol{M}, \theta, \Delta_m^n, p, q_1 + q_2)$,

(iii) $Z(\boldsymbol{M},\theta,\Delta_m^n,p,q_1) \bigcap Z(\boldsymbol{T},\theta,\Delta_m^n,p,q_1) \subset Z(\boldsymbol{M}+\boldsymbol{T},\theta,\Delta_m^n,p,q_1)$,
(iv) $Z(\boldsymbol{M},\theta,\Delta_m^n,p,q_1) \bigcap Z(\boldsymbol{M},\theta,\Delta_m^n,t,q_2) \neq \phi$,
(v) The inclusions $Z(\boldsymbol{M},\theta,\Delta_m^{n-1},p,q_1) \subset Z(\boldsymbol{M},\theta,\Delta_m^n,p,q_1)$ *are strict. In general* $Z(\boldsymbol{M},\theta,\Delta_m^i,p,q_1) \subset Z(\boldsymbol{M},\theta,\Delta_m^n,p,q_1)$ *for* $i = 1, 2, \ldots,$ n-1 *and the inclusion is strict.*

Theorem 3.6. *Let* $Z = w_0,\ w_1$ *and* w_∞. *Then we have the followings.*
(i) Let $0 < \inf p_k \le p_k \le 1$. *Then* $Z(\boldsymbol{M},\theta,\Delta_m^n,p,q) \subset Z(\boldsymbol{M},\theta,\Delta_m^n,q)$,
(ii) Let $1 \le p_k \sup p_k < \infty$. *Then* $Z(\boldsymbol{M},\theta,\Delta_m^n,q) \subset Z(\boldsymbol{M},\theta,\Delta_m^n,p,q)$,
(iii) Let $0 < p_k \le t_k$ *and* $\left(\frac{p_k}{t_k}\right)$ *be bounded. Then* $Z(\boldsymbol{M},\theta,\Delta_m^n,t,q) \subseteq Z(\boldsymbol{M},\theta,\Delta_m^n,p,q)$.

Proof. Proof of the parts *(i)* and *(ii)* is easy and so omitted. We prove the part *(iii)* for $Z = w_1$ and for $Z = w_0, w_\infty$, it will follow on applying similar technique.

We write $S_k = \left[M_k\left(q\left(\frac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]_k^t$ and $\mu_k = \frac{p_k}{t_k}$ so that $0 < \mu \le \mu_k \le 1$.

Define $S_k^{'} = S_k$ if $S_k \ge 1$ $\qquad$ $S_k^{''} = 0$ if $S_k \ge 1$
$\qquad = 0$ if $S_k < 1$, $\qquad\qquad = S_k$ if $S_k < 1$

Then $S_k = S_k^{'} + S_k^{''}$, $S_k^{\mu_k} = S_k^{'\mu_k} + S_k^{''\mu_k}$.

Now it follows that $S_k^{'\mu_k} \le S_k^{'} \le S_k$, $S_k^{''\mu_k} \le S_k^{''\mu}$.

We have the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r} S_k^{\mu_k} \le \frac{1}{h_r}\sum_{k\in I_r} S_k + \frac{1}{h_r}\sum_{k\in I_r} S_k^{''\mu}.$$

Therefore if $(x_k) \in w_1(\mathbf{M},\theta,\Delta_m^n,t,q)$, then $(x_k) \in w_1(\mathbf{M},\theta,\Delta_m^n,p,q)$.

The following Theorem is a direct consequence of Definition 1.1.

Theorem 3.7. *Let* $\boldsymbol{M} = (M_k)$ *and* $\boldsymbol{T} = (T_k)$ *be two sequences of Orlicz functions such that* $M_k \approx T_k$, *for each* $k \in N$. *Then for* $Z = w_0,\ w_1$ *and* w_∞, *we have* $Z(\boldsymbol{M},\theta,\Delta_m^n,p,q)$=$Z(\boldsymbol{T},\theta,\Delta_m^n,p,q)$.

Theorem 3.8. *Let* $\boldsymbol{M} = (M_k)$ *be a sequence of Orlicz functions and* $Z = w_0,\ w_1$ *and* w_∞. *Then* $Z(\boldsymbol{M},\theta,\Delta_m^n,p,q)$=$Z(\theta,\Delta_m^n,p,q)$, *if the following conditions hold*

$$\lim_{t\to 0}\frac{M_k(t)}{t} > 0 \text{ and } \lim_{t\to 0}\frac{M_k(t)}{t} < \infty, \text{ for each } k \in N.$$

Proof. If the given conditions are satisfied, we have $M_k(t) = t$, for each $k \in N$. Then the proof from using Theorem 3.7.

4 q-Lacunary Δ_m^n-Statistical Convergence

In this section, we define the notion of q-lacunary Δ_m^n-statistical convergence and investigate some of its properties. Further, we establish some relations between q-lacunary Δ_m^n-statistical convergence and the spaces $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$ and $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

Definition 4.1. Let θ be a lacunary sequence, then the sequence $x = (x_k)$ is said to be q-lacunary Δ_m^n-statistical convergent to the number L provided that for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}.card\,\{k \in I_r : q\,(\Delta_m^n x_k - L) \geq \varepsilon\} = 0.$$

In this case, we write $x_k \to L\,(S_\theta^q(\Delta_m^n))$ or $S_\theta^q(\Delta_m^n)$-$\lim x_k = L$ and we define

$$S_\theta^q(\Delta_m^n) = \{x \in w(X) : S_\theta^q(\Delta_m^n) - \lim x_k = L, \text{ for some } L\}.$$

In the case $\theta = (2^r)$, we write $S^q(\Delta_m^n)$ instead of $S_\theta^q(\Delta_m^n)$.

If $X = C$, $q(x) = |x|$, we write $S_\theta(\Delta_m^n)$ instead of $S_\theta^q(\Delta_m^n)$ and if $\theta = (2^r)$ we write $S(\Delta_m^n)$ instead of $S_\theta(\Delta_m^n)$.

In the special case $L = 0$, we denote it by $S_{0\theta}^q(\Delta_m^n)$.

Theorem 4.1. *Let θ be a lacunary sequence and $0 < p < \infty$.*
(i) If $x_k \to L(w_\theta^q(\Delta_m^n))$, then $x_k \to L(S_\theta^q(\Delta_m^n))$,
(ii) If $x \in \ell_\infty(q, \Delta_m^n)$ and $x_k \to L(S_\theta^q(\Delta_m^n))$, then $x_k \to L(w_\theta^q(\Delta_m^n))$, where $\ell_\infty(q, \Delta_m^n)$=$x \in w(X) : \sup_k q(\Delta_m^n x_k) < \infty$ and

$$w_\theta^q(\Delta_m^n)=\left\{ x \in w(X) : \lim_r \frac{1}{h_r}\sum_{k\in I_r}(q\,(\Delta_m^n x_k - L))^p = 0, \text{ for some } L \right\}.$$

(iii) $\ell_\infty(q, \Delta_m^n) \cap S_\theta^q(\Delta_m^n)$=$\ell_\infty(q, \Delta_m^n) \cap w_\theta^q(\Delta_m^n)$.

Proof. (i) Let $x_k \to L(w_\theta^q(\Delta_m^n))$ and $\varepsilon > 0$. Then we have
$\sum_{k\in I_r}(q\,(\Delta_m^n x_k - L))^p \geq \varepsilon^p \text{card}\{k \in I_r : q\,(\Delta_m^n x_k - L) \geq \varepsilon\}$.

Hence $x_k \to L(S_\theta^q(\Delta_m^n))$.

(ii) Suppose $x \in \ell_\infty(q, \Delta_m^n)$ and $x_k \to L(S_\theta^q(\Delta_m^n))$. Let $\varepsilon > 0$ be given and $n_0(\varepsilon) \in N$ such that
$\frac{1}{h_r}\text{card}\left\{k \in I_r : q\left(\Delta_m^n x_k - L\right) \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\} < \frac{\varepsilon}{2K^p}$ for all $r > n_0(\varepsilon)$, where $K = \sup_k \left(q\left(\Delta_m^n x_k - L\right)\right)$ and we set $L_r = \left\{k \in I_r : q\left(\Delta_m^n x_k - L\right) \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}$.
Now for all $r > n_0$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left(q\left(\Delta_m^n x_k - L\right)\right)^p &= \frac{1}{h_r}\sum_{k\in I_r, k\in L_r}\left(q\left(\Delta_m^n x_k - L\right)\right)^p \\ &+ \frac{1}{h_r}\sum_{k\in I_r, k\notin L_r}\left(q\left(\Delta_m^n x_k - L\right)\right)^p \\ &\leq \frac{1}{h_r}\left(\frac{h_r\varepsilon}{2K^p}\right)K^p + \frac{1}{h_r}h_r\left(\frac{\varepsilon}{2}\right) = \varepsilon.\end{aligned}$$

Hence $x_k \to L(w_\theta^q(\Delta_m^n))$.

(iii) The proof follows from (i) and (ii).

Theorem 4.2. *Let θ be a lacunary sequence.*
(i) If $\liminf_r \eta_r > 1$, then $S^q(\Delta_m^n) \subseteq S_\theta^q(\Delta_m^n)$,
(ii) If $\limsup_r \eta_r < \infty$, then $S_\theta^q(\Delta_m^n) \subseteq S^q(\Delta_m^n)$,
(iii) If $1 < \liminf_r \eta_r \leq \limsup_r \eta_r < \infty$, then $S_\theta^q(\Delta_m^n) = S^q(\Delta_m^n)$.

Proof. (i) If $\liminf_r \eta_r > 1$, then there exists a $\delta > 0$ such that $1 + \delta \leq \eta_r$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$. Let $(x_k) \in L\left(S^q(\Delta_m^n)\right)$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{k_r}\text{card}\{k \leq k_r : q\left(\Delta_m^n x_k - L\right) \geq \varepsilon\} &\geq \frac{1}{k_r}\text{card}\{k \in I_r : q\left(\Delta_m^n x_k - L\right) \geq \varepsilon\} \\ &\geq \left(\frac{\delta}{\delta+1}\right)\frac{1}{h_r}\text{card}\{k \in I_r : q\left(\Delta_m^n x_k - L\right) \geq \varepsilon\}.\end{aligned}$$

Thus $x_k \to L\left(S_\theta^q(\Delta_m^n)\right)$. Hence $S^q(\Delta_m^n) \subseteq S_\theta^q(\Delta_m^n)$.

(ii) Suppose $\limsup_r \eta_r < \infty$. Then there exists $M > 0$ such that $\eta_r < M$ for all $r \geq 1$.

Let $x_k \to L\left(S_\theta^q(\Delta_m^n)\right)$ and $\varepsilon > 0$. Suppose $E_r =$card$\{k \in I_r : q\left(\Delta_m^n x_k - L\right) \geq \varepsilon\}$, then there exists $n_0 \in N$ such that $\frac{1}{h_r}E_r < \varepsilon$ for all $r > n_0$. Let $K = \max_{1\leq r\leq n_0} E_r$ and choose n such that $k_{r-1} < n \leq K_r$, then we have

$$
\begin{aligned}
\tfrac{1}{n}\text{card}\{k \le n : q\left(\Delta_m^n x_k - L\right) \ge \varepsilon\} \le \tfrac{1}{k_{r-1}}&\text{card}\{k \le k_r : q\left(\Delta_m^n x_k - L\right) \ge \varepsilon\} \\
&\le \tfrac{1}{k_{r-1}}\left\{E_1 + \cdots + E_{n_0} + \cdots + E_r\right\} \\
&\le \tfrac{K}{k_{r-1}} n_0 + \tfrac{1}{k_{r-1}}\left\{\tfrac{E_{n_0+1}}{h_{n_0+1}} h_{n_0+1} + \cdots + \tfrac{E_r}{h_r} h_r\right\} \\
&\le \tfrac{K}{k_{r-1}} n_0 + \tfrac{1}{k_{r-1}}\left(\sup_{r>n_0} \tfrac{E_r}{h_r}\right)\left\{h_{n_0+1} + \cdots + h_r\right\} \\
&\le \tfrac{K}{k_{r-1}} n_0 + \varepsilon \tfrac{k_r - k_{n_0}}{k_{r-1}} \\
&\le \tfrac{K}{k_{r-1}} n_0 + \varepsilon \eta_r \\
&\le \tfrac{K}{k_{r-1}} n_0 + \varepsilon M.
\end{aligned}
$$

Since $k_{r-1} \to \infty$ as $n \to \infty$, it follows that $x_k \to L\,(S^q(\Delta_m^n))$. Hence $S_\theta^q(\Delta_m^n) \subseteq S^q(\Delta_m^n)$.

(*iii*) The proof follows from (*i*) and (*ii*).

Theorem 4.3. (*i*) $w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q) \subseteq S_\theta^q(\Delta_m^n)$,
(*ii*) $w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q) \subseteq S_{0\theta}^q(\Delta_m^n)$.

Proof. (*i*) Let $(x_k) \in w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$. Then there exist some $\rho > 0$ and $L \in X$ such that

$$
\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} = 0.
$$

Let $\varepsilon > 0$ be given and $\sum_1$ denote the sum over $k \in I_r$ such that $q(\Delta_m^n - L) \ge \varepsilon$ and $\sum_2$ denote the sum over $k \in I_r$ such the $q(\Delta_m^n - L) < \varepsilon$. Then

$$
\begin{aligned}
\tfrac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(q\left(\tfrac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} &= \tfrac{1}{h_r}\sum_1\left[M_k\left(q\left(\tfrac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} \\
&\quad + \tfrac{1}{h_r}\sum_2\left[M_k\left(q\left(\tfrac{\Delta_m^n x_k - L}{\rho}\right)\right)\right]^{p_k} \\
&\ge \tfrac{1}{h_r}\sum_1 [M_k(\varepsilon_1)]^{p_k}, \text{ where } \tfrac{\varepsilon}{\rho} = \varepsilon_1 \\
&\ge \tfrac{1}{h_r}\sum_1 \min\left\{[M_k(\varepsilon_1)]^{\inf p_k}, [M_k(\varepsilon_1)]^G\right\} \\
&\ge \tfrac{1}{h_r}\text{card}\{k \in I_r : q(\Delta_m^n x_k - L) \ge \varepsilon\} \min\left\{[M_k(\varepsilon_1)]^{\inf p_k}, [M_k(\varepsilon_1)]^G\right\}.
\end{aligned}
$$

Hence $(x_k) \in S_\theta^q(\Delta_m^n)$.

(*ii*) Proof is similar to that of part (*i*).

Theorem 4.4. (i) $\ell_\infty(q, \Delta_m^n) \cap S_\theta^q(\Delta_m^n) = \ell_\infty(q, \Delta_m^n) \cap w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$,
(ii) $\ell_\infty(q, \Delta_m^n) \cap S_{0\theta}^q(\Delta_m^n) = \ell_\infty(q, \Delta_m^n) \cap w_0(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

Proof. (i) Using Theorem 4.3, it is enough to show that $\ell_\infty(q, \Delta_m^n) \cap S_\theta^q(\Delta_m^n) \subseteq \ell_\infty(q, \Delta_m^n) \cap w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$. Let $(x_k) \in \ell_\infty(q, \Delta_m^n) \cap S_\theta^q(\Delta_m^n)$ and $t_k = (\Delta_m^n x_k - L) \to 0\,(S_\theta^q(\Delta_m^n))$. Let $\sum_1$ and $\sum_2$ be the same as in the proof of the previous Theorem. Since $(x_k) \in \ell_\infty(q, \Delta_m^n)$, there exists $K > 0$ such that $M_k\left(q\left(\frac{t_k}{\rho}\right)\right) \leq K$ for all $k \in N$. Then given $\varepsilon > 0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} M_k\left(q\left(\frac{t_k}{\rho}\right)\right) = \frac{1}{h_r}\sum_1 M_k\left(q\left(\frac{t_k}{\rho}\right)\right) + \frac{1}{h_r}\sum_2 M_k\left(q\left(\frac{t_k}{\rho}\right)\right)$$

$$\leq \tfrac{K}{h_r}\mathrm{card}\{k \in I_r : q(t_k) \geq \varepsilon\rho\} + \tfrac{1}{h_r}\sum_{k\in I_r} M_k\left(\tfrac{\varepsilon}{\rho}\right).$$

Hence $(x_k) \in \ell_\infty(q, \Delta_m^n) \cap w_1(\boldsymbol{M}, \theta, \Delta_m^n, p, q)$.

(ii) Proof is similar to that of part (i).

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