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Several properties on quasi-class A operators

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Abstract

In this paper, we shall show a similar results corresponding the results of M. Ito [6] for quasi-class A introduced in [7] as a class of operators including class A and p-quasihyponormal. Moreover, we shall show several properties on quasi-class A which corresponding to the properties on class A and p-quasihyponormal.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$, we shall write $\ker(T)$, $ran(T)$ for the null space and range of T, respectively. An operator T is said to be *positive* (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and also T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

Recall ([1, 8, 9]) that an operator T is called *p-quasihyponormal* if $T^*((T^*T)^p - (TT^*)^pT) \geq 0$ for $p \in (0, 1]$, and T is called *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for all unit vector $x \in \mathcal{H}$. Following [5, 6, 10] we say that $T \in \mathbf{B}(\mathcal{H})$ belongs to *class A* if $|T^2| \geq |T|^2$ and T is called *normaloid* if $\|T^n\| = \|T\|^n$, for $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T). Recall [2], an operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T^*}|$. We remark that w-hyponormal operator is defined by using Aluthge transformation $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. An operator T is said to be *quasi-class A* if

$$T^* \left|T^2\right| T \geq T^* \left|T\right|^2 T.$$

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The *quasi-class A* operators were introduced , and their properties were studied in [7]. (see also [4]). In particular, it was shown in [7] that the class of *quasi-class A* operators contains properly classes of *class A* and *p-quasihyponormal* operators.
Quasi-class A operators were independently introduced by Jeon and Kim [7]. They gave an example of a *quasi-class A* operator which is not *paranormal* nor *normaloid*. Jeon and Kim example show that neither the class *paranormal* operators nor the class of *quasi-class A* contains the other. we shall denote classes of p-quasihyponormal operators, paranormal operators, normaloid operators, class A operators, and quasi-class A operators by $\mathcal{QH}(p), \mathcal{PN}, \mathcal{N}, \mathcal{A}$, and $\mathcal{QA}$, respectively. It is well known that

$$\mathcal{A} \subset \mathcal{PN} \subset \mathcal{N} \qquad \text{and} \qquad \mathcal{QH}(p) \subset \mathcal{PN} \subset \mathcal{N},$$

also, the following inclusions holds;

$$\mathcal{A} \subset \mathcal{QA} \qquad \text{and} \qquad \mathcal{QH}(p) \subset \mathcal{QA}.$$

Recently, M. Ito [6] showed the following results on powers of class A operators.

Theorem 1.1. *Let T be an invertible and class A operator. Then the following assertions holds;*

1. $|T^n|^{\frac{1}{2n}} \geq \left(T^*|T^{n-1}|^{\frac{2}{n-1}}T\right)^{\frac{1}{2}} \geq |T|^2$ *for* $n = 2, 3, \cdots$.
2. $\left|T^{n+1}\right|^{\frac{2n}{n+1}} \geq |T^n|^2$ *for all positive integer n.*
3. $\left|T^{2n}\right| \geq |T^n|^2$ *for all positive integer n.*
4. $|T|^2 \leq \left|T^2\right| \leq \ldots \leq |T^n|^{\frac{2}{n}}$ *for all positive integer n.*
5. $\left|T^{-2}\right| \geq \left|T^{-1}\right|^2$.

Theorem 1.2. *Let T be an invertible and class A. Then the following assertions holds;*

1. $|T^*|^2 \geq \left(T|T^{(n-1)*}|^{\frac{2}{n-1}}T^*\right)^{\frac{1}{2}} \geq |T^{*n}|^{\frac{2}{n}}$ *for* $n = 2, 3, \cdots$.
2. $|T^{n*}|^2 \geq \left|T^{(n+1)*}\right|^{\frac{2n}{n+1}}$ *for all integer* $n = 2, 3, \cdots$.
3. $|T^{n*}|^2 \geq \left|T^{2n*}\right|$ *for all integer* $n = 2, 3, \cdots$.

4. $|T^*|^2 \geq |T^{2*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$.

In this paper, we shall show similar results corresponding to Theorem 1.1 and Theorem 1.2 for a quasi-class A operators. Moreover, we shall show several properties on quasi-class A operators.

2 Results

We begin this section by introducing the following famous inequality which is quite useful for the study of quasi-class A operators.

Theorem 2.1. *(**Löwner-Heinz Theorem**) If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Theorem 2.2. *Let T be an invertible operator such that*

$$\left(T^*|T^{n-1}|^{2k}T\right)^{\frac{1}{(n-1)k+1}} \geq |T|^2$$

for some $k > 0$ and $n = 2, 3, \cdots$. Then for any fixed $\delta \geq -1$,

$$f_{n,\delta}(\ell) = {T^*}^{n-1}\left(T^*|T^{n-1}|^{2\ell}T\right)^{\frac{\delta+1}{(n-1)\ell+1}} T^{n-1} \tag{2.1}$$

is increasing for $\ell \geq \max\left\{k, \frac{\delta}{n-1}\right\}$.

We need the following Lemma in order to give a proof of Theorem 2.2.

Lemma 2.3. *[6, Theorem C] Let A and B be positive invertible operators such that*

$$\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B$$

holds for fixed $\alpha_0 \geq 0$ and $\beta_0 \geq 0$ with $\alpha_0 + \beta_0 > 0$. Then for any fixed $\delta \geq -\beta_0$,

$$g(\lambda,\mu) = B^{\frac{-\mu}{2}}\left(B^{\frac{\mu}{2}}A^{\lambda}B^{\frac{\mu}{2}}\right)^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} B^{\frac{-\mu}{2}}$$

is an increasing function of both λ and μ for $\lambda \geq 1$ and $\mu \geq 1$ such that $\alpha_0\lambda \geq \delta$.

Proof of Theorem 2.2. Let $T = U|T|$ be the polar decomposition of T. We remark that U is unitary since T is invertible. Suppose that

$$\left(T^*|T^{n-1}|^{2k}T\right)^{\frac{1}{(n-1)k+1}} \geq |T|^2. \tag{2.2}$$

Since

$$\left(T^*|T^{n-1}|^{2k}T\right)^{\frac{1}{(n-1)k+1}} = \left(U^*|T^*||T^{n-1}|^{2k}|T^*|U\right)^{\frac{1}{(n-1)k+1}}$$
$$= U^*\left(|T^*||T^{n-1}|^{2k}|T^*|\right)^{\frac{1}{(n-1)k+1}}U$$

(2.2) holds if and only if

$$\left(|T^*||T^{n-1}|^{2k}|T^*|\right)^{\frac{1}{(n-1)k+1}} \geq U|T|^2U^*$$

if and only if

$$\left(|T^*||T^{n-1}|^{2k}|T^*|\right)^{\frac{1}{(n-1)k+1}} \geq |T^*|^2 \tag{2.3}$$

Let $A = |T^{n-1}|^{2k}$ and $B = |T^*|^2$. Then (2.3) is equivalent to the following:

$$(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{(n-1)k+1}} \geq B. \tag{2.4}$$

By applying Lemma 2.3 to (2.4), for any fixed $\delta \geq -1$,

$$g(\lambda) = B^{\frac{-1}{2}}(B^{\frac{1}{2}}A^{\lambda}B^{\frac{1}{2}})^{\frac{\delta+1}{(n-1)k\lambda+1}}B^{\frac{-1}{2}}$$
$$= |T^*|^{-1}(|T^*||T^{n-1}|^{2k\lambda}|T^*|)^{\frac{\delta+1}{(n-1)k\lambda+1}}|T^*|^{-1}$$

is increasing for $\lambda \geq 1$ such that $(n-1)k\lambda \geq \delta$. Hence

$$g(\lambda) = C^{*(n-1)}g(\lambda)C^{n-1}$$
$$= C^{*(n-1)}B^{\frac{-1}{2}}(B^{\frac{1}{2}}A^{\lambda}B^{\frac{1}{2}})^{\frac{\delta+1}{(n-1)k\lambda+1}}B^{\frac{-1}{2}}C^{n-1}$$
$$= (UT^*U^*)^{n-1}|T^*|^{-1}(|T^*||T^{n-1}|^{2k\lambda}|T^*|)^{\frac{\delta+1}{(n-1)k\lambda+1}}|T^*|^{-1}(UTU^*)^{n-1}$$

is increasing for $\lambda \geq 1$ such that $(n-1)k\lambda \geq \delta$, and we have

$$g(\frac{\ell}{k}) = (UT^*U^*)^{n-1}|T^*|^{-1}(|T^*||T^{n-1}|^{2\ell}|T^*|)^{\frac{\delta+1}{(n-1)\ell+1}}|T^*|^{-1}(UTU^*)^{n-1}$$
$$= (UT^*U^*)^{n-1}|T^*|^{-1}(|T^*||T^{n-1}|^{2\ell}|T^*|)^{\frac{\delta+1}{(n-1)\ell+1}}|T^*|^{-1}(UTU^*)^{n-1}$$
$$= (UT^*U^*)^{n-1}|T^*|^{-1}U(T^*||T^{n-1}|^{2\ell}T)^{\frac{\delta+1}{(n-1)\ell+1}}U^*|T^*|^{-1}T^{n-1} \qquad \text{(Since } U \text{ is unitary)}$$
$$= (UT^*U^*)^{n-1}T^{-n^*}T^{n-1^*}(T^*||T^{n-1}|^{2\ell}T)^{\frac{\delta+1}{(n-1)\ell+1}}T^{n-1}T^{-n}(UTU^*)^{n-1}$$
$$= (UT^*U^*)^{n-1}T^{-n^*}f_{n,\delta}(\ell)T^{-n}(UTU^*)^{n-1}$$

is increasing for $\ell \geq k$ such that $(n-1)\ell \geq \delta$. Hence $f_{n,\delta}(\ell)$ is increasing for $\ell \geq \max\left\{k, \frac{\delta}{n-1}\right\}$, that is, the proof of Theorem 2.2 is achieved. □

By using Theorem 2.2, we obtain the following results.

Theorem 2.4. *Let T be an invertible and quasi-class A operator. Then the following assertions hold;*
(a) $T^{*^{n-1}}|T^n|^{\frac{2}{n}}T^{n-1} \geq T^{*^{n-1}}(T^*|T^{n-1}|^{\frac{2}{n-1}}T)^{\frac{1}{2}}T^{n-1} \geq T^*|T|^2T$ *for* $n = 2, 3, \cdots$.
(b) $T^{n*}|T^{n+1}|^{\frac{2n}{n+1}}T^n \geq T^{*n}|T^n|^2T^n$ *for all positive integer n.*
(c) $T^{n*}|T^{2n}|T^n \geq T^{n*}|T^n|^2T^n$ *for all positive integer n.*
(d) $T^*|T|^2T \leq T^*|T^2|T \leq \cdots \leq T^{*n}|T^n|^{\frac{2}{n}}T^n$ *for all positive integer n.*
(e) $T^{*^{-1}}|T^{-2}|T^{-1} \geq T^{*^{-1}}|T^{-1}|^2T^{-1}$.

Proof. Define $f_{n,\delta}(\ell)$ as (2.1) in Theorem 2.2.
(a). We will use induction to establish the inequality

$$\begin{aligned} T^{*^{n-1}}|T^n|^{\frac{2}{n}}T^{n-1} &\geq T^{*^{n-1}}(T^*|T^{n-1}|^{\frac{2}{n-1}}T)^{\frac{1}{2}}T^{n-1} \\ &\geq T^*|T|^2T \quad \text{for} \quad n = 2, 3, \cdots . \end{aligned} \tag{2.5}$$

In case $n = 2$,
$$T^*|T^2|T = T^*(T^*|T|^2T)^{\frac{1}{2}}T \geq T^*|T|^2T$$
hold since T is a quasi-class A operator.
Assume that (2.5) holds for some $n \geq 2$. Then

$$\begin{aligned} T^*|T|^2T &\leq T^{n^*}(T^*|T|^2T)^{\frac{1}{2}}T^n \qquad \text{(by Inequality (2.5))} \\ &\leq T^{n^*}(T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}}T^n \qquad \text{(by Inequality (2.5) and Löwner-Heinz Theorem).} \end{aligned} \tag{2.6}$$

Then (2.6) and Theorem 2.2 ensure that

$$f_{n+1,0}(\ell) = T^{n^*}(T^*|T^n|^{2\ell}T)^{\frac{1}{n\ell+1}}T^n \quad \text{is increasing for} \ell \geq \max\left\{\frac{1}{n}, 0\right\} = \frac{1}{n}, \tag{2.7}$$

and we have

$$\begin{aligned} T^{n^*}(T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}}T^n &= f_{n+1,0}(\frac{1}{n}) \\ &\leq f_{n+1,0}(1) \text{ by (2)} \\ &= T^{n^*}(T^*|T^n|^{\frac{1}{2}}T)^{\frac{1}{n+1}}T^n \\ &= T^{n^*}|T^{n+1}|^{\frac{2}{n+1}}T^n. \end{aligned} \tag{2.8}$$

Hence (2.6) and (2.8) ensure

$$T^{n^*}|T^{n+1}|^{\frac{2}{n+1}}T^n \geq T^{n^*}(T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}} \geq T^*|T|^2T,$$

so that (2.5) hold for $n = 2, 3, \cdots$ by induction, that is, the proof of (a) is achieved.
Proof of (b). We will use induction to establish the inequality

$$T^{n*}|T^{n+1}|^{\frac{2n}{n+1}}T^n \geq T^{*n}|T^n|^2T^n \text{ for all positive integer } n. \tag{2.9}$$

In case $n = 1$, $T^*|T^2|T \geq T^*|T|^2T$ holds since T is a quasi-class A operator.

Assume (2.9) holds for some n. We remark the following:
since $T^{n^*}|T^{n+1}|^{\frac{2}{n+1}}T^n \geq T^*|T|^2T$ holds by part(a), Theorem 2.2 ensures that

$$f_{n+2,n}(\ell) = T^{n+1^*}(T^*|T^{n+1}|^{2\ell}T)^{\frac{n+1}{(n+1)\ell+1}}T^{n+1} \tag{2.10}$$

is increasing for $\ell \geq \max\left\{\frac{1}{n+1}, \frac{n}{n+1}\right\} = \frac{n}{n+1}$.
Then we have

$$\begin{aligned}
T^{n^*}|T^{n+1}|^2T^n &= T^{n+1^*}|T^n|^2T^{n+1} \\
&\leq T^{n+1^*}|T^{n+1}|^{\frac{2n}{n+1}}T^{n+1} \qquad \text{(by Inequality (2.9))} \\
&= f_{n+2,n}(\frac{n}{n+1}) \\
&\leq f_{n+2,n}(1) \qquad \text{(by (2.10))} \\
&= T^{n+1^*}(T^*|T^{n+1}|^2T)^{\frac{n+1}{n+2}}T^{n+1} \\
&= T^{n+1^*}|T^{n+2}|)^{\frac{2(n+1)}{n+2}}T^{n+1}.
\end{aligned} \tag{2.11}$$

Hence (2.9) holds for all positive integer n by induction, that is, the proof of (b) is achieved.
Proof of (c). By part (b) and Löwner-Heinz Theorem, we obtain

$$\begin{aligned}
T^{n^*}|T^n|^2T^n &\leq T^{n^*}|T^{n+1}|^{\frac{2n}{n+1}}T^n = T^{n^*}|T^{n+1}|^{2\cdot\frac{n}{n+1}}T^n \\
&\leq \cdots \\
&\leq T^{n^*}|T^{2n}|^{\frac{2(2n-1)}{2n}\times\frac{n}{2n-1}}T^n = T^{n^*}|T^{2n}|^{2\times\frac{n}{2n}}T^n \\
&= T^{n^*}|T^{2n}|T^n,
\end{aligned}$$

so that we have (c).
Proof of (d). Applying Löwner-Heinz Theorem to (b),

$$T^{n*}|T^{n+1}|^{\frac{2n}{n+1}}T^n \geq T^{*n}|T^n|^2T^n$$

holds for all positive integer n. Therefore we obtain

$$T^*|T|^2T \leq T^*|T^2|T \leq \cdots \leq T^{*n}|T^n|^{\frac{2}{n}}T^n$$

for all positive integer n.
Proof of (e). We cite the following obvious result (see [3]): Let S be an invertible operator. Then

$$(S^*S)^\lambda = S^*(SS^*)^{\lambda-1}S \quad \text{holds for any real number } \lambda. \tag{2.12}$$

Suppose that T is an invertible quasi-class A operator. Then

$$T^{2^*}T^2 = T^*|T|^2T \leq T^*|T^2|T = T^*(T^{2^*}T^2)^{\frac{1}{2}}T = T^{3^*}(T^2T^{2^*})^{\frac{-1}{2}}T^3 \tag{2.13}$$

holds by (2.12). (2.13) holds if and only if

$$T^{*^{-1}}T^{-1} \leq (T^{*^{-2}}T^{-2})^{\frac{1}{2}} \tag{2.14}$$

if and only if

$$T^{*^{-2}}T^{-2} \leq T^{*^{-1}}(T^{*^{-2}}T^{-2})^{\frac{1}{2}}T^{-1}$$

if and only if

$$T^{*^{-1}}|T^{-1}|^2T^{-1} \leq T^{*^{-1}}|T^{-2}|T^{-1},$$

so that the proof of (e) is complete. □

Corollary 2.5. (i) *If T is an invertible and quasi-class A operator, then T^n is also a quasi-class A operator.*
(ii)*If T is an invertible and quasi-class A operator, then T^{-1} is also a quasi-class A operator.*

Theorem 2.6. *Let T be an invertible and quasi-class A operator. Then the following assertions hold;*
(a) $T|T^*|^2T^* \geq T^{n-1}(T|T^{n-1^*}|^{\frac{2}{n-1}}T^*)^{\frac{1}{2}}T^{*^{n-1}} \geq T^{n-1}|T^{*n}|^{\frac{2}{n}}T^{*n-1}$ *for* $n = 2, 3, \cdots$.
(b) $T^n|T^{n+1^*}|^{\frac{2n}{n+1}}T^{n^*} \leq T^n|T^{n^*}|^2T^{n^*}$ *for all positive integer n.*
(c) $T^n|T^{2n^*}|T^{n^*} \leq T^n|T^{n^*}|^2T^{n^*}$ *for all positive integer n.*
(d) $T|T^*|^2T^* \geq T|T^{2^*}|T^* \geq \cdots \geq T^n|T^{n^*}|^{\frac{2}{n}}T^{n^*}$ *for all positive integer n.*

Proof. First of all, we remark that

$$|S^{-1}| = (S^{*^{-1}}S^{-1})^{\frac{1}{2}} = (SS^*)^{\frac{-1}{2}} = |S^*|^{-1} \quad \text{for any invertible operator } S. \tag{2.15}$$

Suppose that T is an invertible and quasi-class A operator. Then T^{-1} is also a quasi-class A operator by part (e) of Theorem 2.4.
Proof of (a). Since T^{-1} a quasi-class A operator, applying part (a) of Theorem 2.4, we have

$$T^{*^{-n+1}}|T^{-n}|^{\frac{2}{n}}T^{-n+1} \geq T^{*^{-n+1}}(T^{-1^*}|T^{-n+1}|^{\frac{2}{n-1}}T^{-1})^{\frac{1}{2}}T^{-n+1} \geq T^{-1^*}|T^{-1}|^2T^{-1}. \tag{2.16}$$

By (2.15), (2.16) hold if and only if

$$T^{*^{-n+1}}|T^{n^*}|^{\frac{-2}{n}}T^{-n+1} \geq T^{*^{-n+1}}(T^{-1^*}|T^{n-1^*}|^{\frac{-2}{n-1}}T^{-1})^{\frac{1}{2}}T^{-n+1} \geq T^{-1^*}|T^*|^{-2}T^{-1}.$$

if and only if

$$T^{n-1}|T^{n^*}|^{\frac{2}{n}}T^{n-1^*} \leq T^{n-1}(T|T^{n-1^*}|^{\frac{2}{n-1}}T^*)^{\frac{1}{2}}T^{n-1^*} \leq T|T^*|^2T^*.$$

Proof of (b). Since T^{-1} a quasi-class A operator, applying part (b) of Theorem 2.4, we have

$$T^{(-n)*}|T^{-(n+1)}|^{\frac{2n}{n+1}}T^{-n} \geq T^{(-n)^*}|T^{-n}|^2T^{-n}. \tag{2.17}$$

By (2.15), (2.17) hold if and only if

$$T^{(-n)*}|T^{(n+1)^*}|^{\frac{-2n}{n+1}}T^{-n} \geq T^{(-n)^*}|T^{n^*}|^{-2}T^{-n}.$$

if and only if

$$T^n|T^{(n+1)^*}|^{\frac{2n}{n+1}}T^{n^*} \leq T^n|T^{n^*}|^2T^{n^*}$$

Proof of (c). Since T^{-1} a quasi-class A operator, applying part (c) of Theorem 2.4, we have

$$T^{(-n)^*}|T^{-2n}|T^{-n} \geq T^{(-n)^*}|T^{-n}|^2T^{-n}. \tag{2.18}$$

By (2.15), (2.18) hold if and only if

$$T^{(-n)^*}|T^{(2n)^*}|^{-1}T^{-n} \geq T^{(-n)^*}|T^{n^*}|^{-2}T^{-n}.$$

if and only if

$$T^n|T^{(2n)^*}|T^{n^*} \leq T^n|T^{n^*}|^2T^{n^*}.$$

Proof of (d). Since T^{-1} a quasi-class A operator, applying part (d) of Theorem 2.4, we have

$$T^{*^{-1}}|T^{-1}|^2T^{-1} \leq T^{-1*}|T^{-2}|T^{-1} \leq \cdots \leq T^{(-n)^*}|T^{-n}|^{\frac{2}{n}}T^{-n}. \tag{2.19}$$

By (2.15), (2.19) hold if and only if

$$T^{*^{-1}}|T^*|^{-2}T^{-1} \leq T^{*^{-1}}|T^{2^*}|^{-1}T^{-1} \leq \cdots \leq T^{(-n)^*}|T^{n^*}|^{\frac{-2}{n}}T^{-n}.$$

if and only if

$$T|T^*|^2T^* \geq T|T^{2^*}|T^* \geq \cdots \geq T^n|T^{n^*}|^{\frac{2}{n}}T^{n^*}.$$

Hence the proof of the theorem is achieved. □

Hölder-McCarthy Inequality. Let T be a positive operator. Then the following inequalities hold for all $x \in \mathcal{H}$:
(i) $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r \le 1$.
(ii) $\langle T^r x, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \ge 1$.

Theorem 2.7. *Let T be a quasi-class A. Then the following assertions hold.*

(i) $\left\|T^{k+1}x\right\|^2 \le \left\|T^k x\right\| \left\|T^{k+2}x\right\|$ *for all unit vectors $x \in \mathcal{H}$ and all positive integer k.*

(ii) $\left\|T^{k+1}\right\|^{k+1} \le r(T^{k+1}) \left\|T^k\right\|^{k+1}$ *for all positive integer k, where $r(T^k)$ denote the spectral radius of T^k.*

Proof. (i) Suppose that T is a quasi-class A. Then for every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
\left\|T^{k+1}x\right\|^2 &= \left\langle T^{*k}|T|^2 T^k x, x\right\rangle \\
&\le \left\langle T^{*k}|T^2| T^k x, x\right\rangle \\
&\le \left\langle (T^{*2}T^2)^{1/2} T^k x, T^k x\right\rangle \\
&\le \left\langle (T^{*2}T^2) T^k x, T^k x\right\rangle^{1/2} \left\|T^k x\right\| && \text{(by Hölder-McCarthy Inequality)} \\
&\le \left\|T^{k+2}x\right\| \left\|T^k x\right\|.
\end{aligned}$$

(ii) If $T^k = 0$ for some $k > 1$, then $r(T^k) = 0$. Hence (ii) is obvious. Hence we may assume $T^k \neq 0$ for all $k \ge 1$. Then

$$\frac{\left\|T^{k+1}\right\|}{\left\|T^k\right\|} \le \frac{\left\|T^{k+2}\right\|}{\left\|T^{k+1}\right\|} \le \cdots \le \frac{\left\|T^{m(k+1)}\right\|}{\left\|T^{m(k+1)-1}\right\|}$$

by (i), and we have

$$\left(\frac{\left\|T^{k+1}\right\|}{\left\|T^k\right\|}\right)^{m(k+1)-k} \le \frac{\left\|T^{k+1}\right\|}{\left\|T^k\right\|} \times \cdots \times \frac{\left\|T^{m(k+1)}\right\|}{\left\|T^{m(k+1)-1}\right\|} = \frac{\left\|T^{m(k+1)}\right\|}{\left\|T^k\right\|}.$$

Hence

$$\left(\frac{\left\|T^{k+1}\right\|}{\left\|T^k\right\|}\right)^{(k+1)-\frac{k}{m}} \le \frac{\left\|T^{m(k+1)}\right\|^{\frac{1}{m}}}{\left\|T^k\right\|^{\frac{1}{m}}},$$

letting $m \to \infty$, we have

$$\left\|T^{k+1}\right\|^{k+1} \le r(T^{k+1}) \left\|T^k\right\|^{k+1}.$$

□

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