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Iterative methods for zero points of accretive operators in Banach spaces

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Abstract

The purpose of this paper is to consider the problem of approximating zero points of accretive operators. We introduce and analysis Mann-type iterative algorithm with errors and Halpern-type iterative algorithms with errors. Weak and strong convergence theorems are established in a real Banach space. As applications, we consider the problem of approximating a minimizer of a proper lower semicontinuous convex function in a real Hilbert space.

1 Introduction-Preliminaries

Let C be a nonempty closed and convex subset of a Banach space E and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to be have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

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exists for each $x, y \in U$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for all $x \in U$. E is said to be uniformly smooth or said to be have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then

$$\Big\|\frac{x+y}{2}\Big\| \leq r\Big(1 - \delta(\frac{\epsilon}{r})\Big)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \epsilon$.

In this paper, $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

A Banach space E is said to satisfy Opial's condition [13] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies that

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$.

Recall that a mapping $T : C \to C$ is said to be nonexpanisve if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(T)$ to denote the set of fixed points of T. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D.

A mapping P of C into itself is called a retraction if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0.$$

An accretive operator A is said to satisfy the range condition if

$$\overline{D(A)} \subset \cap_{r>0} R(I + rA),$$

where $\overline{D(A)}$ denote the closure of $D(A)$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by

$$J_r = (I + rA)^{-1}$$

for each $r > 0$, which is called the resolvent of A. We also define the Yosida approximation A_r by

$$A_r = \frac{1}{r}(I - J_r).$$

It is known that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \le \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$.

One of classical methods of studying the problem $0 \in Ax$, where $A \subset E \times E$ is an accretive operator, is the following:

$$x_0 \in E, \quad x_{n+1} = J_{r_n} x_n, \quad n \ge 0, \tag{Δ}$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers.

The convergence of (Δ) has been studied by many authors; see, for example, Benavides, Acedo and Xu [1], Brézis and Lions [2], Bruck [3], Bruck and Passty [4], Bruck and Reich [5], Cho, Zhou and Kim [7], Ceng, Wu and Yao [8], Kamimur and Takahashi [10,11], Pazy [14], Qin, Kang and Cho [15], Qin and Su [16], Rockafellar [17], Reich [19-22], Takahashi and Ueda [23], Takahashi [24], Xu [26] and Zhou [27].

In this paper, motivated by the research work going on in this direction, we introduce and analysis Mann-type iterative algorithms with errors and Halpern-type iterative algorithms with errors. Weak and strong convergence theorems are established in a real Banach space.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([21],[23]). *Let E be a real reflexive Banach space whose norm is uniformly Gâteaux differentiable and $A \subset E \times E$ be an accretive operator. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty, closed and convex subset of E such that $\overline{D(A)} \subset C \subset \cap_{t>0} R(I + tA)$. If $A^{-1}(0) \neq \emptyset$, then the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$ for all $x \in C$, where $J_t = (I + tA)^{-1}$ is the resolvent of A for all $t > 0$.*

Lemma 1.2 ([12]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{t_n\}$ is a sequence in $[0,1]$. Assume that the following conditions are satisfied

(a) $\sum_{n=0}^{\infty} t_n = \infty$ *and* $b_n = o(t_n)$*;*

(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3 ([6]). *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$, then $Tz = z$.*

Lemma 1.4 ([25]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then the limit of $\{a_n\}$ exists.

Lemma 1.5 ([9]). In a Banach space E, there holds the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

2 Main results

Theorem 2.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed and convex subset of E. Let P be a nonexpansive retraction of E onto C and $A \subset E \times E$ an accretive operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in E, \quad x_{n+1} = \alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0, \qquad (\Upsilon)$$

where $u \in C$ is a fixed point, $\{f_n\} \subset E$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{e_n\}$ is a sequence in E, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ *generated by* (Υ) *converges strongly to a zero of* A.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we have

$$\begin{aligned} \|x_1 - p\| &= \|\alpha_0 u + \beta_0 J_{r_0}(x_0 + e_1) + \gamma_0 P f_0 - p\| \\ &\le \alpha_0\|u - p\| + \beta_0\|J_{r_0}(x_0 + e_1) - p\| + \gamma_0\|Pf_0 - p\| \\ &\le \alpha_0\|u - p\| + \beta_0\|(x_0 + e_1) - p\| + \gamma_0\|f_0 - p\| \\ &\le \alpha_0\|u - p\| + \beta_0(\|x_0 - p\| + \|e_1\|) + \gamma_0\|f_0 - p\| \\ &\le K, \end{aligned}$$

where $K = \|u - p\| + \|x_0 - p\| + \|e_1\| + \|f_0 - p\| < \infty$. Putting

$$M = \max\{K, \sup_{n\ge 0} \|f_n - p\|\},$$

we prove that

$$\|x_n - p\| \le M + \sum_{i=1}^{n} \|e_i\|, \quad \forall n \ge 1. \tag{2.1}$$

It is easy to see that the result holds for $n = 1$. We assume that the result holds for some n. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - p\| \\ &\le \alpha_n\|u - p\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - p\| + \gamma_n\|Pf_n - p\| \\ &\le \alpha_n\|u - p\| + \beta_n\|(x_n + e_{n+1}) - p\| + \gamma_n\|f_n - p\| \\ &\le \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \|e_{n+1}\| + \gamma_n\|f_n - p\| \\ &\le \alpha_n M + \beta_n(M + \textstyle\sum_{i=0}^{n} \|e_i\|) + \|e_{n+1}\| + \gamma_n M \\ &= M + \textstyle\sum_{i=1}^{n+1} \|e_i\|. \end{aligned}$$

This shows that (2.1) holds. From the condition $\sum_{i=1}^{\infty} \|e_i\| < \infty$, we see that the sequence $\{x_n\}$ is bounded.

Next, we show that $\limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z)\rangle \le 0$, where $z = \lim_{t\to\infty} J_t u$, which is guaranteed by Lemma 1.1. Note that $\frac{u - J_t u}{t} \in AJ_t u$, $A_{r_n}x_n \in AJ_{r_n}x_n$ and A is accretive. It follows that

$$\langle A_{r_n}x_n - \frac{u - J_t u}{t}, J(J_{r_n}x_n - J_t u)\rangle \ge 0.$$

This implies that

$$\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle \le \langle t A_{r_n} x_n, J(J_{r_n} x_n - J_t u)\rangle. \tag{2.2}$$

On the other hand, we have

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \|\frac{x_n - J_{r_n} x_n}{r_n}\| = 0.$$

In view of (2.2), we arrive at

$$\limsup_{n\to\infty} \langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle \le 0, \quad \forall t \ge 0. \tag{2.3}$$

Since $z = \lim_{t\to\infty} J_t u$ and the norm of E is uniformly Gâteaux differentiable, for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|\langle z - J_t u, J(J_{r_n} x_n - J_t u)\rangle| \le \frac{\epsilon}{2}$$

and

$$|\langle u - z, J(J_{r_n} x_n - J_t u) - J(J_{r_n} x_n - z)\rangle| \le \frac{\epsilon}{2}$$

for all $t \ge t_0$ and $n \ge 0$. It follows that

$$\begin{aligned} &|\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - z)\rangle| \\ &\le |\langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - J_t u)\rangle| \\ &\quad + |\langle u - z, J(J_{r_n} x_n - J_t u)\rangle - \langle u - z, J(J_{r_n} x_n - z)\rangle| \\ &= |\langle z - J_t u, J(J_{r_n} x_n - J_t u)\rangle| + |\langle u - z, J(J_{r_n} x_n - J_t u) - J(J_{r_n} x_n - z)\rangle| \\ &\le \epsilon \end{aligned} \tag{2.4}$$

for all $t \ge t_0$ and $n \ge 0$. It follows from (2.3) and (2.4) that

$$\limsup_{n\to\infty} \langle u - z, J(J_{r_n} x_n - z)\rangle \le \limsup_{n\to\infty} \langle u - J_t u, J(J_{r_n} x_n - J_t u)\rangle + \epsilon \le \epsilon.$$

Since ϵ is arbitrary, we see that

$$\limsup_{n\to\infty} \langle u - z, J(J_{r_n} x_n - z)\rangle \le 0. \tag{2.5}$$

Note that

$$\|J_{r_n} x_n - J_{r_n}(x_n + e_{n+1})\| \le \|e_{n+1}\|.$$

This implies that

$$\lim_{n\to\infty} \|J_{r_n} x_n - J_{r_n}(x_n + e_{n+1})\| = 0.$$

Since E has a uniformly Gâteaux differentiable norm, we arrive at

$$\limsup_{n\to\infty}\langle u-z, J(J_{r_n}(x_n+e_{n+1})-z)\rangle \le 0. \tag{2.6}$$

On the other hand, , we see from the iterative (Υ) that

$$x_{n+1}-J_{r_n}(x_n+e_{n+1}) = \alpha_n[u-J_{r_n}(x_n+e_{n+1})]+\gamma_n[Pf_n-J_{r_n}(x_n+e_{n+1})].$$

That is,

$$\|x_{n+1}-J_{r_n}(x_n+e_{n+1})\| \le \alpha_n\|u-J_{r_n}(x_n+e_{n+1})\|+\gamma_n\|Pf_n-J_{r_n}(x_n+e_{n+1})\|.$$

From the conditions (b) and (c), we obtain that

$$\limsup_{n\to\infty}\|x_{n+1}-J_{r_n}(x_n+e_{n+1})\| = 0,$$

which combines with (2.6) yields that

$$\limsup_{n\to\infty}\langle u-z, J(x_{n+1}-z)\rangle \le 0. \tag{2.7}$$

From the algorithm (Υ), we see that

$$\begin{aligned} x_{n+1}-z &= \alpha_n(u-z)+\beta_n[J_{r_n}(x_n+e_{n+1})-z]+\gamma_n(Pf_n-z)\\ &= (1-\alpha_n)[J_{r_n}(x_n+e_{n+1})-z]+\alpha_n(u-z)+\gamma_n[Pf_n-J_{r_n}(x_n+e_{n+1})]. \end{aligned}$$

It follows from Lemma 1.5 that

$$\begin{aligned} &\|x_{n+1}-z\|^2\\ &\le (1-\alpha_n)^2\|J_{r_n}(x_n+e_{n+1})-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\langle Pf_n-J_{r_n}(x_n+e_{n+1}), J(x_{n+1}-z)\rangle\\ &\le (1-\alpha_n)\|(x_n+e_{n+1})-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|Pf_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)(\|x_n-z\|^2-2\langle e_{n+1}, J[(x_n+e_{n+1})-z]\rangle)+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)(\|x_n-z\|^2+2\|e_{n+1}\|\|(x_n+e_{n+1})-z\|)+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\\ &\le (1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle\\ &\quad +2\gamma_n\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|+2\|e_{n+1}\|\|(x_n+e_{n+1})-z\|\\ &\le (1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z, J(x_{n+1}-z)\rangle+(\gamma_n+\|e_{n+1}\|)B, \end{aligned}$$

where B is an appropriate constant such that

$$B \ge \max\{\sup_{n\ge 0}\{2\|f_n-J_{r_n}(x_n+e_{n+1})\|\|x_{n+1}-z\|\}, \sup_{n\ge 0}\{2\|(x_n+e_{n+1})-z\|\}\}$$

Let $\lambda_n = \max\{\langle u - z, J(x_{n+1} - z)\rangle, 0\}$. Next, we show that $\lim_{n\to\infty} \lambda_n = 0$. Indeed, from (2.7), for any give $\epsilon > 0$, there exists a positive integer n_1 such that

$$\langle u - z, J(x_{n+1} - z)\rangle < \epsilon, \quad \forall n \geq n_1.$$

This implies that $0 \leq \lambda_n < \epsilon$ $\forall n \geq n_1$. Since $\epsilon > 0$ is arbitrary, we see that $\lim_{n\to\infty} \lambda_n = 0$. Put $a_n = \|x_n - z\|$, $b_n = 2\alpha_n\lambda_n$, $c_n = (\gamma_n + \|e_{n+1}\|)B$ and $t_n = \alpha_n$. In view of Lemma 1.2, we can obtain the desired conclusion immediately. This completes the proof.

In a real Hilbert space, Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. Let P be a metric projection of H onto C and $A \subset H \times H$ a monotone operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in H, \quad x_{n+1} = \alpha_n u + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0,$$

where $u \in C$ is a fixed point, $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{e_n\}$ is a sequence in H, $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to a zero of A.

Theorem 2.3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of E. Let P be a nonexpansive retraction of E onto C and $A \subset E \times E$ an accretive operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \geq 0, \qquad (\Upsilon\Upsilon)$$

where $\{f_n\} \subset E$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0, 1)$, $\{e_n\}$ is a sequence in E, $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ *generated by* $(\Upsilon\Upsilon)$ *converges weakly to a zero of* A.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we have

$$\begin{aligned}
\|x_1 - p\| &= \|\alpha_0 x_0 + \beta_0 J_{r_0}(x_0 + e_1) + \gamma_0 P f_0 - p\| \\
&\le \alpha_0\|x_0 - p\| + \beta_0\|J_{r_0}(x_0 + e_1) - p\| + \gamma_0\|P f_0 - p\| \\
&\le \alpha_0\|x_0 - p\| + \beta_0\|(x_0 + e_1) - p\| + \gamma_0\|f_0 - p\| \\
&\le \alpha_0\|x_0 - p\| + \beta_0(\|x_0 - p\| + \|e_1\|) + \gamma_0\|f_0 - p\| \\
&\le K',
\end{aligned}$$

where $K' = \|x_0 - p\| + \|e_1\| + \|f_0 - p\| < \infty$. Putting

$$M' = \max\{K, \sup_{n\ge 0}\|f_n - p\|\},$$

we prove that

$$\|x_n - p\| \le M' + \sum_{i=1}^{n}\|e_i\|, \quad \forall n \ge 1. \tag{2.8}$$

It is easy to see that the result holds for $n = 1$. We assume that the result holds for some n. It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - p\| \\
&\le \alpha_n\|x_n - p\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - p\| + \gamma_n\|P f_n - p\| \\
&\le \alpha_n\|x_n - p\| + \beta_n\|(x_n + e_{n+1}) - p\| + \gamma_n\|f_n - p\| \\
&\le \alpha_n\|x_n - p\| + \beta_n\|x_n - p\| + \|e_{n+1}\| + \gamma_n\|f_n - p\| \\
&\le \alpha_n M + \beta_n(M + \textstyle\sum_{i=0}^{n}\|e_i\|) + \|e_{n+1}\| + \gamma_n M \\
&= M + \textstyle\sum_{i=1}^{n+1}\|e_i\|.
\end{aligned}$$

This shows that (2.8) holds. From the condition $\sum_{i=1}^{\infty}\|e_i\| < \infty$, we see that the sequence $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_n - x^*\|$ exists for any $x^* \in A^{-1}(0)$. In fact, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n - x^*\| \\
&\le \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - x^*\| + \gamma_n\|P f_n - x^*\| \\
&\le \alpha_n\|x_n - x^*\| + \beta_n\|(x_n + e_{n+1}) - x^*\| + \gamma_n\|f_n - x^*\| \\
&\le \|x_n - x^*\| + \lambda_n,
\end{aligned}$$

where $\lambda_n = \|e_{n+1}\| + \gamma_n\|f_n - x^*\|$ for each $n \geq 0$. From the assumption, we see that $\sum_{n=0}^{\infty} \lambda_n < \infty$. It follows from Lemma 1.4 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any $x^* \in A^{-1}(0)$. Put $d = \lim_{n\to\infty} \|x_n - x^*\|$ for any $x^* \in A^{-1}(0)$. We may, without loss of generality, assume that $d > 0$. Since A is accretive and E is uniformly convex, we have

$$\begin{aligned} \|J_{r_n}x_n - x^*\| &\leq \|J_{r_n}x_n - x^* + \tfrac{r_n}{2}(A_{r_n}x_n - 0)\| \\ &= \|J_{r_n}x_n - x^* + \tfrac{1}{2}(x_n - J_{r_n}x_n)\| \\ &= \|\tfrac{x_n + J_{r_n}x_n}{2} - x^*\| \\ &\leq \|x_n - x^*\|[1 - \delta(\tfrac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|})]. \end{aligned} \tag{2.9}$$

Note that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|\alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|J_{r_n}(x_n + e_{n+1}) - J_{r_n}x_n\| + \beta_n\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \beta_n\|e_{n+1}\| + \beta_n\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|. \end{aligned}$$

This is,

$$-(\alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|) \leq -\|x_{n+1} - x^*\|. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned} &(1 - \alpha_n)\|x_n - x^*\|\delta(\tfrac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|}) \\ &\leq (1 - \alpha_n)(\|x_n - x^*\| - \|J_{r_n}x_n - x^*\|) \\ &= \|x_n - x^*\| - (\alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\|) \\ &= \|x_n - x^*\| - (\alpha_n\|x_n - x^*\| + \|e_{n+1}\| + (1 - \alpha_n)\|J_{r_n}x_n - x^*\| + \gamma_n\|Pf_n - x^*\|) \\ &\quad + \|e_{n+1}\| + \gamma_n\|Pf_n - x^*\| \\ &\leq \|x_n - x^*\| - \|x_{n+1} - x^*\| + \|e_{n+1}\| + \gamma_n\|Pf_n - x^*\|. \end{aligned}$$

From the conditions (b), (c) and $\lim_{n\to\infty} \|x_n - x^*\| = d > 0$, we arrive at

$$\delta(\frac{\|x_n - J_{r_n}x_n\|}{\|x_n - x^*\|}) \to 0$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}\|J_{r_n}x_n - J_1J_{r_n}x_n\| &= \|(I-J_1)J_{r_n}x_n\| \\ &= \|A_1J_{r_n}x_n\| \\ &\le \inf\{\|u\| : u \in AJ_{r_n}x_n\} \\ &\le \|A_{r_n}x_n\| \\ &= \|\tfrac{x_n - J_{r_n}x_n}{r_n}\|.\end{aligned}$$

From (2.11) and the condition (d), we obtain that

$$\lim_{n\to\infty} \|J_{r_n}x_n - J_1J_{r_n}x_n\| = 0. \tag{2.12}$$

Letting $v \in C$ be a weak subsequential limit of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. From (2.11), we see that $J_{r_{n_i}}x_{n_i} \rightharpoonup v$. In view of Lemma 1.3, we obtain that $v \in F(J_1) = A^{-1}(0)$. Since the space satisfies Opial's condition (see [18]), we see that the desired conclusion holds. This completes the proof.

In a real Hilbert space, Theorem 2.3 is reduced to the following.

Corollary 2.4. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of E. Let P be a metric projection of E onto C and $A \subset H \times H$ a monotone operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0}R(I + rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n J_{r_n}(x_n + e_{n+1}) + \gamma_n P f_n, \quad n \ge 0,$$

where $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$, $\{e_n\}$ is a sequence in H, $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_nA)^{-1}$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to a zero of A.

3 Applications

In this section, as applications of main Theorems 2.1 and 2.3, we consider the problem of finding a minimizer of a convex function f.

Let H be a Hilbert space and $h : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂h of h is defined as follows:

$$\partial h(x) = \{y \in H : h(z) \geq h(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

Theorem 3.1. *Let H be a real Hilbert space and $h : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial h(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = \arg\min_{x \in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}, \\ x_{n+1} = \alpha_n u + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $u \in H$ is a fixed point, $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{e_n\}$ is a sequence in H and $\{r_n\} \subset (0, \infty)$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of h.

Proof. Since $h : H \to (-\infty, +\infty]$ is a proper convex lower semi-continuous function, we have that the subdifferential ∂h of h is maximal monotone by Rockafellar [18]. Notice that

$$y_n = \arg\min_{x \in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}$$

is equivalent to the following

$$0 \in \partial h(y_n) + \frac{1}{r_n}(y_n - x_n - e_{n+1}).$$

It follows that

$$x_n + e_{n+1} \in y_n + r_n \partial h(y_n), \quad \forall n \geq 0.$$

By Theorem 2.1, we can obtain the desired conclusion immediately.

Theorem 3.2. *Let H be a real Hilbert space and $h : H \to (-\infty, +\infty]$ a proper convex lower semi-continuous function such that $\partial h(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = \arg\min_{x \in H}\{h(x) + \frac{1}{2r_n}\|x - x_n - e_{n+1}\|^2\}, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $\{f_n\} \subset H$ is a bounded sequence, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$, $\{e_n\}$ is a sequence in H and $\{r_n\} \subset (0, \infty)$. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges weakly to a minimizer of h.

Proof. We can easily obtain from the proof of Theorem 2.3 and Theorem 3.1 the desired conclusion.

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