



Fixed points of Kannan mappings in metric spaces endowed with a graph

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Abstract

Let (X, d) be a metric space endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X . We define the notion of G -Kannan maps and obtain a fixed point theorem for such mappings.

1 Introduction

Let T be a selfmap of a metric space (X, d) . Following Petrușel and Rus [10], we say that T is a Picard operator (abbr., PO) if T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$ and is a weakly Picard operator (abbr. WPO) if the sequence $(T^n x)_{n \in \mathbb{N}}$ converges, for all $x \in X$ and the limit (which may depend on x) is a fixed point of T .

Let (X, d) be a metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X , and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [[6], p. 309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G , i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \mid (y, x) \in G\}.$$

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The letter \tilde{G} denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat \tilde{G} as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}) \quad (1.1)$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for any edge $(x, y) \in E'$, $x, y \in V'$.

Now we recall a few basic notions concerning the connectivity of graphs. All of them can be found, e.g., in [6]. If x and y are vertices in a graph G , then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^N$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if \tilde{G} is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G , then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x . In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected.

Recently, two results appeared giving sufficient conditions for f to be a PO if (X, d) is endowed with a graph. The first result in this direction was given by J. Jakhymski [5] who also presented its applications to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0, 1]$.

Definition 1 ([5], Def. 2.1). *We say that a mapping $f : X \rightarrow X$ is a Banach G -contraction or simply G -contraction if f preserves edges of G , i.e.,*

$$\forall x, y \in X ((x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)) \quad (1.2)$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1), \forall x, y \in X ((x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y)) \quad (1.3)$$

Theorem 1 ([5], Th 3.2). *Let (X, d) be complete, and let the triple (X, d, G) have the following property:*

for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $f : X \rightarrow X$ be a G -contraction, and $X_f = \{x \in X | (x, fx) \in E(G)\}$. Then the following statements hold.

1. $\text{card} \text{Fix } f = \text{card} \{ [x]_{\tilde{G}} | x \in X_f \}.$

2. $\text{Fix } f \neq \emptyset$ iff $X_f \neq \emptyset$.
3. f has a unique fixed point iff there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\tilde{G}}$.
4. For any $x \in X_f$, $f|_{[x]_{\tilde{G}}}$ is a PO.
5. If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.
6. If $X' := \cup \{[x]_{\tilde{G}} \mid x \in G\}$ then $f|_{X'}$ is a WPO.
7. If $f \subseteq E(G)$, then f is a WPO.

Subsequently, Bega, Butt and Radojević extended Theorem 1 for set valued mappings.

Definition 2 ([1], Def. 2.6). Let $F : X \rightsquigarrow X$ be a set valued mapping with nonempty closed and bounded values. The mapping F is said to be a G -contraction if there exists a $\alpha \in (0, 1)$ such that

$$D(Fx, Fy) \leq \alpha d(x, y) \text{ for all } x, y \in E(G)$$

and if $u \in Fx$ and $v \in Fy$ are such that

$$d(u, v) \leq \alpha d(x, y) + k, \text{ for each } k > 0$$

then $(u, v) \in E(G)$.

Theorem 2 ([1], Th. 3.1). Let (X, d) be a complete metric space and suppose that the triple (X, d, G) has the property:

for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $F : X \rightsquigarrow X$ be a G -contraction and

$$X_f = \{x \in X : (x, u) \in E(G) \text{ for some } u \in Fx\}.$$

Then the following statements hold:

1. For any $x \in X_F$, $F|_{[x]_{\tilde{G}}}$ has a fixed point.
2. If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X .
3. If $X' := \cup \{[x]_{\tilde{G}} : x \in X_F\}$, then $F|_{X'}$ has a fixed point.
4. If $F \subseteq E(G)$ then F has a fixed point.
5. $\text{Fix } F \neq \emptyset$ if and only if $X_F \neq \emptyset$.

Definition 3. Let (X, d) be a metric space. $T : X \rightarrow X$ is called a Kannan operator if there exists $a \in [0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq a [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$.

Kannan [7] proved that if X is complete, then every Kannan mapping has a fixed point. A number of interesting results have been obtained by different authors for Kannan mappings, see for example [3, 4, 14, 15].

The aim of this paper is to study the existence of fixed points for Kannan mappings in metric spaces endowed with a graph G by introducing the concept of G -Kannan mappings.

2 Main Results

Throughout this section we assume that (X, d) is a metric space, and G is a directed graph such that $V(G) = X$, $E(G) \supseteq \Delta$ and the graph G has no parallel edges. The set of all fixed points of a mapping T is denoted by $FixT$.

In this section, by using the idea of Jakhymski [5], we will consider the following concept:

Definition 4. Let (X, d) be a metric space and G a graph. The mapping $T : X \rightarrow X$ is said to be a G -Kannan mapping if:

1. $\forall x, y \in X$ (If $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$).
2. there exists $a \in [0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq a [d(x, Tx) + d(y, Ty)]$$

for all $(x, y) \in E(G)$.

Remark 1. If T is a G -Kannan mapping, then T is both a G^{-1} -Kannan mapping and a \tilde{G} -Kannan mapping.

Example 1. Any Kannan mapping is a G_0 -Kannan mapping, where the graph G_0 is defined by $E(G_0) = X \times X$.

Example 2. Let $X = \{0, 1, 3\}$ and the euclidean metric $d(x, y) = |x - y|$, $\forall x, y \in X$. The mapping $T : X \rightarrow X$, $Tx = 0$, for $x \in \{0, 1\}$ and $Tx = 1$, for $x = 3$ is a G -Kannan mapping with constant $a = \frac{1}{3}$, where $G = \{(0, 1); (1, 3); (0, 0); (1, 1); (3, 3)\}$, but is not a Kannan mapping because $d(T0, T3) = 1$ and $d(0, T0) + d(3, T3) = 2$.

Definition 5. Let (X, d) be a metric space endowed with a graph G and $T : X \rightarrow X$ be a mapping. We say that the graph G is T -connected if for all vertices x, y of G with $(x, y) \notin E(G)$, there exists a path in $G, (x_i)_{i=0}^N$ from x to y such that $x_0 = x, x_N = y$ and $(x_i, Tx_i) \in E(G)$ for all $i = 1, \dots, N-1$. A graph G is weakly T -connected if \tilde{G} is T -connected.

Lemma 1. Let (X, d) be a metric space endowed with a graph G and $T : X \rightarrow X$ be a G -Kannan mapping with constant a . If the graph G is weakly T -connected then, given $x, y \in X$, there is $r(x, y) \geq 0$ such that

$$d(T^n x, T^n y) \leq ad(T^{n-1}x, T^n x) + \left(\frac{a}{1-a}\right)^n r(x, y) + ad(T^{n-1}y, T^n y) \quad (2.1)$$

for all $n \in \mathbb{N}^*$.

Proof. Let $x, y \in X$. If $(x, y) \in E(\tilde{G})$ then by induction $(T^n x, T^n y) \in E(\tilde{G})$ so (2.1) is true, with $r(x, y) = 0$ for all $n \in \mathbb{N}$. If $(x, y) \notin E(\tilde{G})$ then there is a path $(x_i)_{i=0}^N$ in \tilde{G} from x to y , i.e., $x_0 = x, x_N = y$ with $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, \dots, N$ and $(x_i, Tx_i) \in E(\tilde{G})$ for $i = 1, \dots, N-1$. By Remark 1, T is a \tilde{G} -Kannan mapping. An easy induction shows $(T^n x_{i-1}, T^n x_i) \in E(\tilde{G})$ for $i = 1, \dots, N$, $(T^{n-1}x_i, T^n x_i) \in E(\tilde{G})$ and

$$d(T^{n-1}x_i, T^n x_i) \leq \left(\frac{a}{1-a}\right)^{n-1} d(x_i, Tx_i)$$

for all $n \in \mathbb{N}^*$ and $i = 1, \dots, N-1$. Hence by the triangle inequality, we get

$$\begin{aligned} d(T^n x, T^n y) &\leq \sum_{i=1}^N d(T^n x_{i-1}, T^n x_i) \\ &\leq a \left[d(T^{n-1}x, T^n x) + 2 \sum_{i=1}^{N-1} d(T^{n-1}x_i, T^n x_i) + d(T^{n-1}y, T^n y) \right] \\ &\leq a \left[d(T^{n-1}x, T^n x) + 2 \left(\frac{a}{1-a}\right)^{n-1} \sum_{i=1}^{N-1} d(x_i, Tx_i) + d(T^{n-1}y, T^n y) \right], \end{aligned}$$

so it suffices to set

$$r(x, y) = 2(1-a) \sum_{i=1}^{N-1} d(x_i, Tx_i).$$

□

The main result of this paper is given by the following theorem.

Theorem 3. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \rightarrow X$ be a G -Kannan mapping. We suppose that:*

- (i.) *G is weakly T -connected;*
- (ii.) *for any $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.*

Then T is a PO.

Proof. Let $x \in X$. By Lemma 1, there exists $r(x, Tx)$ such that:

$$d(T^n x, T^{n+1} x) \leq ad(T^{n-1} x, T^n x) + \left(\frac{a}{1-a}\right)^n r(x, Tx) + ad(T^n x, T^{n+1} x).$$

for all $n \in \mathbb{N}^*$. Hence

$$d(T^n x, T^{n+1} x) \leq \frac{a}{1-a} d(T^{n-1} x, T^n x) + \frac{a^n}{(1-a)^{n+1}} r(x, Tx). \quad (2.2)$$

Using the relation (2.2) and an elementary calculus we get

$$d(T^n x, T^{n+1} x) \leq \left(\frac{a}{1-a}\right)^n d(x, Tx) + n \frac{a^n}{(1-a)^{n+1}} r(x, Tx) \quad (2.3)$$

for all $n \in \mathbb{N}$.

Let $b := \frac{a}{1-a}$, because $a \in [0, \frac{1}{2})$ then $b \in [0, 1)$. Using the relation (2.3) we have

$$\begin{aligned} \sum_{i=0}^n d(T^i x, T^{i+1} x) &\leq d(x, Tx) \sum_{i=0}^n b^i + \frac{r(x, Tx)}{1-a} \sum_{i=0}^n i \cdot b^i \\ &= d(x, Tx) \frac{1-b^{n+1}}{1-b} + \frac{r(x, Tx)}{1-a} \cdot \frac{nb^{n+2} - (n+1)b^{n+1} + b}{(1-b)^2} \end{aligned}$$

Hence $\sum_{n=0}^{\infty} d(T^n x, T^{n+1} x) < \infty$ and because

$$d(T^n x, T^{n+p} x) \leq \sum_{i=0}^{n+p} d(T^i x, T^{i+1} x) - \sum_{i=0}^{n-1} d(T^i x, T^{i+1} x) \rightarrow 0$$

as $n \rightarrow \infty$ and for all $p \in \mathbb{N}$, we get $(T^n x)_{n \geq 0}$ is a Cauchy sequence. But (X, d) is a complete metric space, therefore $(T^n x)_{n \geq 0}$ converges to some $x^* \in X$.

Let $x, y \in X$ then $(T^n x)_{n \geq 0} \rightarrow x^*$ and $(T^n y)_{n \geq 0} \rightarrow y^*$, as $n \rightarrow \infty$.
By Lemma 1, for all $n \in \mathbb{N}^*$, we get

$$d(T^n x, T^n y) \leq ad(T^{n-1}x, T^{n-1}y) + \left(\frac{a}{1-a}\right)^n r(x, y) + ad(T^{n-1}y, T^n y)$$

Letting $n \rightarrow \infty$ obtain that $d(x^*, y^*) \leq 0$, hence $x^* = y^*$ and for all $x \in X$ there exists a unique x^* such that

$$\lim_{n \rightarrow \infty} T^n x = x^*.$$

Now we will prove that $x^* \in \text{Fix}T$. Because the graph G is weakly T -connected, there is at least $x_0 \in X$ such that $(x_0, Tx_0) \in E(\tilde{G})$ so $(T^n x_0, T^{n+1} x_0) \in E(\tilde{G})$ for all $n \in \mathbb{N}$. But $\lim_{n \rightarrow \infty} T^n x_0 = x^*$, then by (ii.) there is a subsequence $(T^{k_n} x_0)_{n \in \mathbb{N}}$ with $(T^{k_n} x_0, x^*) \in E(G)$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, T^{k_n+1} x_0) + d(T^{k_n+1} x_0, Tx^*) \\ &\leq d(x^*, T^{k_n+1} x_0) + a [d(T^{k_n} x_0, T^{k_n+1} x_0) + d(x^*, Tx^*)]. \end{aligned}$$

Hence

$$d(x^*, Tx^*) \leq \frac{1}{1-a} d(x^*, T^{k_n+1} x_0) + \frac{a}{1-a} d(T^{k_n} x_0, T^{k_n+1} x_0)$$

Now, letting $n \rightarrow \infty$, we obtain

$$d(x^*, Tx^*) = 0 \Leftrightarrow x^* = Tx^*, \text{ that is, } x^* \in \text{Fix}T.$$

If we have $Ty = y$ for some $y \in X$, then from above, we must have $T^n y \rightarrow x^*$, so $y = x^*$ and therefore, T is a PO. □

The next example shows that the condition (ii.) is a necessary condition for G -Kannan mapping to be a PO.

Example 3. Let $X := [0, 1]$ be endowed with the Euclidean metric d_E . Define the graph G by

$$E(G) = \{(x, y) \in (0, 1] \times (0, 1] \mid x \geq y\} \cup \{(0, 0), (0, 1)\}$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0, 1], \text{ and } T0 = 1$$

It is easy to verify (X, d) is a complete metric space, G is weakly T -connected and T is a G -Kannan mapping with $a = \frac{3}{7}$. Clearly, $T^n x \rightarrow 0$ for all $x \in X$, but T has no fixed points.

The next example shows that the graph G must be T -connected so the G -Kannan mapping T to be a PO.

Example 4. Let $X = \mathbb{N} \setminus \{0, 1\}$ be endowed with the Euclidean metric and define $T : X \rightarrow X$, $Tx = 2x$. Consider the graph G define by:

$$V(G) = X \text{ and } E(G) = \{(2^k n, 2^k(n+1)) : n \in X, k \in \mathbb{N}\} \cup \Delta$$

Then T is G -Kannan operator with $a = \frac{2}{5}$ because

$$\begin{aligned} d_E(T2^k n, T2^k(n+1)) &= 2^{k+1} \leq \frac{2}{5} 2^k(2n+1) \\ &= \frac{2}{5} [d_E(2^k n, T2^k n) + d_E(2^k(n+1), T2^k(n+1))] \end{aligned}$$

for all $n \in X$ and for all $k \in \mathbb{N}$.

Then (X, d) is a complete metric space, G is weakly connected but not weakly T -connected because $(2, 4) \notin E(\tilde{G})$ and the only path in \tilde{G} from 2 to 4 is

$$y_0 = 2, y_1 = 3, y_2 = 4 \text{ and } (3, T3) = (3, 6) \notin E(\tilde{G}).$$

Clearly, $T^n x$ not converge for all $x \in X$ and T has no fixed points.

From Theorem 3, we obtain the following corollary concerning the fixed point of Kannan operator in partially ordered metric spaces.

Corollary 1. Let (X, \leq) be a partially ordered set and d be a metric on X such that the metric space (X, d) is complete. Let $T : X \rightarrow X$ be an increasing operator such that the following three assertions hold:

- (i.) There exist $a \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$ for each $x, y \in X$ with $x \leq y$;
- (ii.) For each $x, y \in X$, incomparable elements of (X, \leq) , there exists $z \in X$ such that $x \leq z$, $y \leq z$ and $z \leq Tz$;
- (iii.) If an increasing sequence (x_n) converges to x in X , then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then T is a PO.

Proof. Consider the graph G with $V(G) = X$, and

$$E(G) = \{(x, y) \in X \times X \mid x \leq y\}.$$

Because the mapping T is increasing and (i.) holds we get the mapping T is a G -Kannan mapping. By (ii.) the graph G is weakly T -connected and the condition (iii.) implies the condition (ii.) from Theorem 3. The conclusion follows now from Theorem 3. \square

In the next we show the fixed point theorem for cyclic Kannan mapping proved in [9] by Petric is a consequence of the Theorem 3.

Let $p \leq 2$ and $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space X . A mapping $T : \cup_{i=1}^p A_i \rightarrow \cup_{i=1}^p A_i$ is called a *cyclical operator* if

$$T(A_i) \subseteq A_{i+1}, \quad \text{for all } i \in \{1, 2, \dots, p\} \quad (2.4)$$

where $A_{p+1} := A_1$.

Theorem 4. *Let $A_1, A_2, \dots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space (X, d) and suppose $T : \cup_{i=1}^p A_i \rightarrow \cup_{i=1}^p A_i$ is a cyclical operator, and there exists $a \in [0, \frac{1}{2})$ such that for each pair $(x, y) \in A_i \times A_{i+1}$, for $i \in \{1, 2, \dots, p\}$, we have*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)].$$

Then T is a PO.

Proof. Let $Y = \cup_{i=1}^p A_i$ then (Y, d) is a complete metric space.

Consider the graph G with $V(G) = Y$, and

$$E(G) = \{(x, y) \in Y \times Y \mid \exists i \in \{1, 2, \dots, p\} \text{ such that } x \in A_i \text{ and } y \in A_{i+1}\}$$

Because T is a cyclic operator we get

$$(Tx, Ty) \in E(G), \text{ for all } (x, y) \in E(G)$$

and via hypothesis the operator T is a G -Kannan operator and the graph G is weakly T -connected. Now let $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$. Then there is $i \in \{1, 2, \dots, p\}$ such that $x \in A_i$. However in view of (2.4) the sequence $\{x_n\}$ has an infinite number of terms in each A_i , for all $i \in \{1, 2, \dots, p\}$. The subsequence of the sequence $\{x_n\}$ formed by the terms which is in A_{i-1} satisfies the condition (ii.) from Theorem 3. In conclusion the operator T is PO. □

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