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Two sufficient conditions for fractional k-deleted graphs

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Abstract

Let G be a graph, and k a positive integer. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. A graph G is a fractional k-deleted graph if $G - e$ has a fractional k-factor for each $e \in E(G)$. In this paper, we obtain some sufficient conditions for graphs to be fractional k-deleted graphs in terms of their minimum degree and independence number. Furthermore, we show the results are best possible in some sense.

1 Introduction

The graphs considered here will be finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. Let S and T be two disjoint subsets of $V(G)$, we denote by $e_G(S, T)$ the number of edges with one end in S and the other end in T. A subset S of $V(G)$ is called an independent set of G if every edge of G is incident with at most one

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vertex of S. We use $\alpha(G)$ and $\delta(G)$ to denote the independence number and minimum degree of G, respectively.

Let k be a positive integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. If $k = 1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. If $k = 1$, then a fractional k-factor is a fractional 1-factor. A graph G is a fractional k-deleted graph if $G - e$ has a fractional k-factor for each $e \in E(G)$. If $k = 1$, then a fractional k-deleted graph is a fractional 1-deleted graph. If G_1 and G_2 are disjoint graphs, then the union is denoted by $G_1 \cup G_2$ and the join by $G_1 \bigvee G_2$. The other terminologies and notations not given here can be found in [1].

Many authors have investigated graph factors [6,7,11,12]. Many authors have investigated fractional k-factors [2,5,8,13] and fractional k-deleted graphs [3,9,10]. The following results on k-factors, fractional k-factors and fractional k-deleted graphs are known.

Theorem 1. [6] *Let $k \geq 2$ be an integer and G a graph with n vertices. Assume that if k is odd, then n is even and G is connected. Let G satisfy*

$$n > 4k + 1 - 4\sqrt{k+2},$$

$$\delta(G) \geq \frac{(k-1)(n+2)}{2k-1} \qquad \text{and}$$

$$\delta(G) > \frac{1}{2k-2}((k-2)n + 2\alpha(G) - 2).$$

Then G has a k-factor.

Theorem 2. [13] *Let $k \geq 2$ be an even integer and G a graph of order n with $n > 4k + 1 - 4\sqrt{k+2}$. If*

$$\delta(G) \geq \frac{(k-1)(n+2)}{2k-1} \qquad \text{and}$$

$$\delta(G) > \frac{1}{2k-2}((k-2)n + 2\alpha(G) - 2),$$

then G has a fractional k-factor.

Theorem 3. [13] *Let $k \geq 3$ be an odd integer and G a graph of order n with $n \geq 4k - 5$. If*

$$\delta(G) > \frac{(k-1)(n+2)}{2k-1} \qquad \text{and}$$

$$\delta(G) > \frac{1}{2k-2}((k-2)n + 2\alpha(G) - 1),$$

then G has a fractional k-factor.

Theorem 4. [10] *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k-5$. If*

$$bind(G) > \frac{(2k-1)(n-1)}{k(n-2)},$$

then G is a fractional k-deleted graph.

In this paper, we shall proceed to research the fractional k-deleted graphs and give some new sufficient conditions for graphs to be fractional k-deleted graphs in terms of their minimum degree and independence number. Our main results are the following theorems which are some extensions of Theorem 1, Theorem 2 and Theorem 3.

Theorem 5. *Let $k \geq 2$ be an even integer and G a graph of order n with $n > 4k + 1 - 4\sqrt{k}$. If*

$$\delta(G) > \frac{(k-1)(n+2)+1}{2k-1} \qquad \text{and}$$

$$\delta(G) > \frac{(k-2)n + 2\alpha(G)}{2k-2},$$

then G is a fractional k-deleted graph.

Theorem 6. *Let $k \geq 3$ be an odd integer and G a graph of order n with $n > 4k + 1 - 4\sqrt{k-1}$. If*

$$\delta(G) > \frac{(k-1)(n+2)+2}{2k-1} \qquad \text{and}$$

$$\delta(G) > \frac{(k-2)n + 2\alpha(G) + 1}{2k-2},$$

then G is a fractional k-deleted graph.

2 The Proofs of Main Theorems

In order to prove our main theorems, we depend heavily on the following results.

Lemma 2.1. [4] *A graph G is a fractional k-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & \textit{if } T \textit{ is not independent}, \\ 1, & \textit{if } T \textit{ is independent, and } e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Lemma 2.2. [3] *Let a, b and c be integers such that $a \geq 2$, $2 \leq b \leq a-1$, $c = 0$ or 1, and let x and y be nonnegative integers. Suppose that*

$$x \leq \frac{(a-b)y + c}{2a-b}$$

and

$$x > \frac{(a-1)(y+2)+1+c}{2a-1} - h.$$

Then $y \leq 4a + 1 - 4\sqrt{a-c}$.

In the following, we shall prove our main theorems.

Proof of Theorem 5. Let G be a graph satisfying the hypothesis of Theorem 5, we prove the theorem by contradiction. Suppose that G is not a fractional k-deleted graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S,T) - 1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$.

If $T = \emptyset$, then $\varepsilon(S,T) = 0$. Combining this with (1), we have $-1 \geq \delta_G(S,T) = k|S| \geq 0$, a contradiction. Therefore, $T \neq \emptyset$. In the following, we define

$$h = \min\{d_{G-S}(x) : x \in T\}$$

and choose a vertex $x_1 \in T$ such that

$$d_{G-S}(x_1) = h.$$

Obviously, $0 \leq h \leq k$ and $\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|$. Thus, we obtain

$$|S| \geq \delta(G) - h. \tag{2}$$

We shall consider three cases by the value of h and derive contradictions.

Case 1. $h = 0$.

Set $X = \{x \in T : d_{G-S}(x) = 0\}$, $Y = \{x \in T : d_{G-S}(x) = 1\}$, $Y_1 = \{x \in Y : N_{G-S}(x) \subseteq T\}$ and $Y_2 = Y - Y_1$. Then the graph induced by Y_1 in $G - S$ has maximum degree at most 1. Let Z be a maximum independent set of the graph. Obviously, $|Z| \geq \frac{1}{2}|Y_1|$. According to the definitions, $X \cup Z \cup Y_2$ is an independent set of G. Therefore, we have

$$\alpha(G) \geq |X| + |Z| + |Y_2| \geq |X| + \frac{1}{2}|Y_1| + \frac{1}{2}|Y_2| = |X| + \frac{1}{2}|Y|. \tag{3}$$

Using (1), (3) and $|S| + |T| \leq n$, we obtain

$$\begin{aligned}
1 &\geq \varepsilon(S,T) - 1 \geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\
&= k|S| + d_{G-S}(T \setminus (X \cup Y)) - k|T| + |Y| \\
&\geq k|S| + 2|T - (X \cup Y)| - k|T| + |Y| \\
&= k|S| + 2|T| - k|T| - 2|X| - |Y| \\
&= k|S| - (k-2)|T| - 2(|X| + \frac{1}{2}|Y|) \\
&\geq k|S| - (k-2)(n - |S|) - 2(|X| + \frac{1}{2}|Y|) \\
&= (2k-2)|S| - (k-2)n - 2(|X| + \frac{1}{2}|Y|) \\
&\geq (2k-2)|S| - (k-2)n - 2\alpha(G),
\end{aligned}$$

that is,

$$(2k-2)|S| - (k-2)n - 2\alpha(G) \leq 1. \tag{4}$$

Note that k is even. Therefore, the left-hand side of (4) is even. Thus, we obtain

$$(2k-2)|S| - (k-2)n - 2\alpha(G) \leq 0,$$

which implies

$$|S| \leq \frac{(k-2)n + 2\alpha(G)}{2k-2}. \tag{5}$$

On the other hand, from (2), $h = 0$ and $\delta(G) > \frac{(k-2)n+2\alpha(G)}{2k-2}$, we get

$$|S| \geq \delta(G) - h > \frac{(k-2)n + 2\alpha(G)}{2k-2},$$

which contradicts (5).

Case 2. $1 \leq h \leq k-1$.

Claim 1.[12] $|S| \leq \frac{(k-h)n}{2k-h}$.

On the other hand, by (2) and $\delta(G) > \frac{(k-1)(n+2)+1}{2k-1}$, we get

$$|S| \geq \delta(G) - h > \frac{(k-1)(n+2)+1}{2k-1} - h. \tag{6}$$

If $h = 1$, then (6) contradicts Claim 1. In the following, we assume that $2 \leq h \leq k-1$. Applying Lemma 2.2 with $a = k$, $b = h$, $c = 0$, $x = |S|$ and $y = n$, we get

$$n \leq 4k + 1 - 4\sqrt{k},$$

which contradicts the hypothesis that $n > 4k + 1 - 4\sqrt{k}$.

Case 3. $h = k$.

It is easy to see that $4k + 1 - 4\sqrt{k} \geq 2k - 1$. Hence, we have $n > 2k - 1$. Thus, we obtain

$$\delta(G) > \frac{(k-1)(n+2)+1}{2k-1} = \frac{(k-1)n}{2k-1} + 1 > k.$$

In terms of the integrity of $\delta(G)$, we obtain

$$\delta(G) \geq k + 1. \tag{7}$$

Claim 2. $S \neq \emptyset$.

Proof. If $S = \emptyset$, then by (1) and (7) we have

$$\begin{aligned} \varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq \delta(G)|T| - k|T| \geq |T| \geq \varepsilon(S,T), \end{aligned}$$

it is a contradiction. The proof of Claim 2 is complete.

According to Claim 2, $h = k$ and $k \geq 2$, we obtain

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h|T| - k|T| = k|S| \geq k \geq 2 \geq \varepsilon(S,T), \end{aligned}$$

which contradicts (1).

From the contradictions above, we deduce that G is a fractional k-deleted graph. This completes the proof of Theorem 5.

The proof of Theorem 6 is quite similar to that of Theorem 5 and is omitted.

3 Remarks

Remark 1. We now show that the conditions $\delta(G) > \frac{(k-1)(n+2)+1}{2k-1}$ and $\delta(G) > \frac{(k-2)n+2\alpha(G)}{2k-2}$ in Theorem 5 are best possible. Let $k \geq 2$ be an integer and $G = K_{2k-2} \bigvee kK_2$. We denote by n the order of the graph G. Then $n = 4k-2 > 4k+1-4\sqrt{k}$ and $\alpha(G) = k$. Thus, we have $\delta(G) = 2k-1 = \frac{(k-1)(n+2)+1}{2k-1}$ and $\delta(G) = 2k-1 = \frac{(2k-1)(2k-2)}{2k-2} = \frac{4k^2-6k+2}{2k-2} > \frac{4k^2-8k+4}{2k-2} = \frac{4k^2-10k+4+2k}{2k-2} = \frac{(k-2)(4k-2)+2k}{2k-2} = \frac{(k-2)n+2\alpha(G)}{2k-2}$. Let $S = V(K_{2k-2})$, $T = V(kK_2)$. Then $|S| = 2k-2$, $|T| = 2k$, and $d_{G-S}(T) = 2k$. Since $T = V(kK_2)$ is not independent, $\varepsilon(S,T) = 2$. Thus, we get

$$\begin{aligned}
\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\
&= k(2k-2) + 2k - k \cdot 2k \\
&= 0 < 2 = \varepsilon(S,T).
\end{aligned}$$

Then by Lemma 2.1, G is not a fractional k-deleted graph. In the above sense, the condition $\delta(G) > \frac{(k-1)(n+2)+1}{2k-1}$ in Theorem 5 is best possible.

Let $k \geq 2$ is even. Obviously, $\frac{k}{2}$ is a positive integer. Put $G = K_{3k-1} \bigvee (2kK_1 \bigcup \frac{k}{2}K_2)$. We use n to denote the order of the graph G. Then $n = 6k-1 > 4k+1-4\sqrt{k}$ and $\alpha(G) = 2k + \frac{k}{2} = \frac{5k}{2}$. Thus, $\delta(G) = 3k-1 = \frac{(3k-1)(2k-2)}{2k-2} = \frac{6k^2-8k+2}{2k-2} = \frac{(k-2)(6k-1)+5k}{2k-2} = \frac{(k-2)n+2\alpha(G)}{2k-2}$ and $\delta(G) = 3k-1 = \frac{(3k-1)(2k-1)}{2k-1} = \frac{(k-1)(6k+1)+2}{2k-1} = \frac{(k-1)(n+2)+2}{2k-1} > \frac{(k-1)(n+2)+1}{2k-1}$. Let $S = V(K_{3k-1})$, $T = V(2kK_1 \bigcup \frac{k}{2}K_2)$. Clearly, $|S| = 3k-1$, $|T| = 3k$, and $d_{G-S}(T) = k$. Since $T = V(2kK_1 \bigcup \frac{k}{2}K_2)$ is not independent, $\varepsilon(S,T) = 2$. Thus, we have

$$\begin{aligned}
\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\
&= k(3k-1) + k - k \cdot 3k \\
&= 0 < 2 = \varepsilon(S,T).
\end{aligned}$$

Then by Lemma 2.1, G is not a fractional k-deleted graph. In the above sense, the condition $\delta(G) > \frac{(k-2)n+2\alpha(G)}{2k-2}$ in Theorem 5 is best possible.

Remark 2. We show that the conditions $\delta(G) > \frac{(k-1)(n+2)+2}{2k-1}$ and $\delta(G) > \frac{(k-2)n+2\alpha(G)+1}{2k-2}$ in Theorem 6 are best possible. Let $k \geq 3$ be an odd integer and $G = K_{3k-2} \bigvee \frac{3k+1}{2}K_2$. Clearly, $\frac{3k+1}{2}$ is a positive integer. We denote by n the order of the graph G. Then $n = 6k-1 > 4k+1-4\sqrt{k-1}$ and $\alpha(G) = \frac{3k+1}{2}$. Thus, we have $\delta(G) = 3k-1 = \frac{(3k-1)(2k-1)}{2k-1} = \frac{6k^2-5k+1}{2k-1} = \frac{(k-1)(6k+1)+2}{2k-1} = \frac{(k-1)(n+2)+2}{2k-1}$ and $\delta(G) = 3k-1 = \frac{(3k-1)(2k-2)}{2k-2} = \frac{6k^2-8k+2}{2k-2} > \frac{6k^2-10k+4}{2k-2} = \frac{(k-2)(6k-1)+3k+2}{2k-2} = \frac{(k-2)n+2\alpha(G)+1}{2k-2}$. Let $S = V(K_{3k-2})$, $T = V(\frac{3k+1}{2}K_2)$. Then $|S| = 3k-2$, $|T| = 3k+1$, and $d_{G-S}(T) = 3k+1$. Since $T = V(\frac{3k+1}{2}K_2)$

is not independent, $\varepsilon(S,T) = 2$. Thus, we obtain

$$\begin{aligned}\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= k(3k-2) + 3k + 1 - k(3k+1) \\ &= 1 < 2 = \varepsilon(S,T).\end{aligned}$$

Then by Lemma 2.1, G is not a fractional k-deleted graph. In the above sense, the condition $\delta(G) > \frac{(k-1)(n+2)+2}{2k-1}$ in Theorem 6 is best possible.

Let $k \geq 3$ is odd. Obviously, $\frac{5k+1}{2}$ is a positive integer. Put $G = K_{3k} \bigvee (2kK_1 \bigcup \frac{k+1}{2} K_2)$. We use n to denote the order of the graph G. Then $n = 6k + 1 > 4k + 1 - 4\sqrt{k-1}$ and $\alpha(G) = 2k + \frac{k+1}{2} = \frac{5k+1}{2}$. Thus, $\delta(G) = 3k = \frac{3k(2k-2)}{2k-2} = \frac{6k^2-6k}{2k-2} = \frac{(k-2)(6k+1)+(5k+1)+1}{2k-2} = \frac{(k-2)n+2\alpha(G)+1}{2k-2}$ and $\delta(G) = 3k = \frac{3k(2k-1)}{2k-1} = \frac{(k-1)(6k+3)+3}{2k-1} = \frac{(k-1)(n+2)+3}{2k-1} > \frac{(k-1)(n+2)+2}{2k-1}$. Let $S = V(K_{3k})$, $T = V(2kK_1 \bigcup \frac{k+1}{2} K_2)$. Clearly, $|S| = 3k$, $|T| = 3k + 1$, and $d_{G-S}(T) = k + 1$. Since $T = V(2kK_1 \bigcup \frac{k+1}{2} K_2)$ is not independent, $\varepsilon(S,T) = 2$. Thus, we have

$$\begin{aligned}\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= k \cdot 3k + k + 1 - k(3k+1) \\ &= 1 < 2 = \varepsilon(S,T).\end{aligned}$$

Then by Lemma 2.1, G is not a fractional k-deleted graph. In the above sense, the condition $\delta(G) > \frac{(k-2)n+2\alpha(G)+1}{2k-2}$ in Theorem 6 is best possible.

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