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Stability of General Cubic Mapping in Fuzzy Normed Spaces

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Abstract

We establish some stability results concerning the general cubic functional equation

$$f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)=2k(k^2-1)f(y)$$

for fixed $k \in \mathbb{N}\backslash\{1\}$ in the fuzzy normed spaces. More precisely, we show under some suitable conditions that an approximately cubic function can be approximated by a cubic mapping in a fuzzy sense and we establish that the existence of a solution for any approximately cubic mapping guarantees the completeness of the fuzzy normed spaces.

1 Introduction and preliminary results

In order to construct a fuzzy structure on a linear space, in 1984, Katsaras [18] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. At the same year Wu and Fang [8] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. In [6], Biswas defined and studied fuzzy inner product spaces in a linear space. Since then some mathematicians have defined fuzzy norms on a linear space from various points of view [5, 10, 20, 27, 29]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [19].

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In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [7] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [4]).

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [28] posed the first stability problem. In the next year, Hyers [12] gave an affirmative answer to the question of Ulam. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The concept of the generalized Hyers-Ulam stability was originated from Rassias's paper [22] for the stability of functional equations. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [9, 13, 16, 23, 24, 35]. Also, the interested reader should refer to [30, 31, 32, 33] and [34] and the references therein. Following [3], we give the notion of a fuzzy norm as follows.

Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ (so-called fuzzy subset) is said to be fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N5) $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

It follows from (N2) and (N4) that if $s < t$, then

$$N(x,t) \geq \min\{N(x,s), N(0,t-s)\} = N(x,s).$$

Therefore the condition ' $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ ' in (N5) can be omitted. The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t '.

Example 1. Let $(X, \| \, . \, \|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, \\ 0, & t \leq 0 \end{cases}$$

is a fuzzy norm on X.

Example 2. Let $(X, \| \, . \, \|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 0, & t < \|x\|, \\ 1, & t \geq \|x\| \end{cases}$$

is a fuzzy norm on X.

Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. In that case, x is called the limit of the sequence x_n and we write $N - \lim x_n = x$. A sequence in a fuzzy normed space (X, N) is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$,

$$N(x_{n+p} - x_n, t) > 1 - \varepsilon.$$

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If in a fuzzy normed space, each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.
The functional equation

$$f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) = 2k(k^2 - 1)f(y) \qquad (1.1)$$

for fixed k with $k \in \mathbb{N} \setminus 1$ is called the general cubic functional equation, since the function $f(x) = x^3$ is its solution. Every solution of the general cubic functional equation is said to be cubic mapping. From (1.1), putting $x = y = 0$ yields $f(0) = 0$. Note that the left hand side of (1.1) changes sign when y is replaced by $-y$. Thus f is odd. Putting $x = 0$ and $y = x$ in (1.1). We conclude that $f(kx) = k^3 f(x)$. By induction, we infer that $f(k^n x) = k^{3n} f(x)$ for all positive integer n. The stability problem for the cubic functional equation was proved by Jun and Kim [14] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later a number of mathematicians worked on the stability of some types of the cubic equation [2, 11, 15, 17, 25, 26]. Najati in [21] established the general solution and the generalized Hyers-Ulam stability for the equation (1.1).

In the next section we proved the non-uniform version of the generalized Hyers-Ulam stability of the general cubic functional equation (1.1) in fuzzy normed spaces. The uniform version is discussed in Section 3. Finally, in section 4, we show that the existence of a conditional cubic mapping for every approximately cubic type mapping implies that our fuzzy normed space is complete.

2 Fuzzy generalized Hyers-Ulam theorem:non-uniform version

Theorem 2.1. Let $k \in \mathbb{N} \setminus \{1\}$, $\alpha \in [1, +\infty)$ and $\alpha \neq k^3$. Let X be a linear space and let (Z, N') be a fuzzy normed space. Suppose that an even function

$\varphi : X \times X \to Z$ satisfies $\varphi(k^n x, k^n y) = \alpha^n \varphi(x, y)$, for all $x, y \in X$ and for all $n \in \mathbb{N}$. Suppose that (Y, N) is a fuzzy Banach space. If a map $f : X \to Y$ satisfies

$$N\Big(f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y), t\Big) \geq N'(\varphi(x,y), t) \tag{2.1}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic map $C : X \to Y$ which satisfies (1.1) and inequality

$$N(f(x)-C(x), t) \geq \min\Big\{N'\Big(\varphi(0,x), \frac{k^3-\alpha}{3}t\Big), N'\Big(\varphi(0,x), \frac{k(k^2-1)(k^3-\alpha)}{3\alpha}t\Big)\Big\}$$

holds for all $\alpha < k^3$, $x \in X$ and $t > 0$. Also,

$$N(f(x)-C(x), t) \geq \min\Big\{N'\Big(\varphi(0,x), \frac{\alpha-k^3}{3}t\Big), N'\Big(\varphi(0,x), \frac{k(k^2-1)(\alpha-k^3)}{3\alpha}t\Big)\Big\}$$

holds for all $\alpha > k^3$, $x \in X$ and $t > 0$.

Proof. Case (1): $1 \leq \alpha \leq k^3$.
Replacing $y := -y$ in (2.1) and adding the result to (2.1) yield

$$N(f(y) + f(-y), t) \geq \min\{N'(\varphi(x,y), k(k^2-1)t), N'(\varphi(x,-y), k(k^2-1)t)\}. \tag{2.2}$$

Since (2.2) and (2.1) hold for any x, let us fix $x = 0$ for convenience. Using (N4), we obtain

$$\begin{aligned} N & \quad (2f(ky) - 2k^3 f(y), t) \\ \geq & \quad \min\Big\{N'\Big(\varphi(0,y), \frac{t}{3}\Big), N\Big(f(ky)+f(-ky), \frac{t}{3}\Big), N\Big(f(y)+f(-y), \frac{t}{3k}\Big)\Big\} \\ \geq & \quad \min\Big\{N'\Big(\varphi(0,y), \frac{t}{3}\Big), N'\Big(\varphi(0,ky), \frac{k(k^2-1)}{3}t\Big), N'\Big(\varphi(0,y), \frac{k^2-1}{3}t\Big)\Big\} \\ \geq & \quad \min\Big\{N'\Big(\varphi(0,y), \frac{t}{3}\Big), N'\Big(\varphi(0,y), \frac{k(k^2-1)}{3\alpha}t\Big)\Big\}. \end{aligned} \tag{2.3}$$

It follows that

$$N\Big(\frac{2f(ky)}{k^3} - 2f(y), t\Big) \geq \min\Big\{N'\Big(\varphi(0,y), \frac{k^3}{3}t\Big), N'\Big(\varphi(0,y), \frac{k^4(k^2-1)}{3\alpha}t\Big)\Big\}. \tag{2.4}$$

Replacing y by x in (2.4), by (N3) we have

$$N\Big(\frac{f(kx)}{k^3} - f(x), t\Big) \geq \min\Big\{N'\Big(\varphi(0,x), \frac{2k^3}{3}t\Big), N'\Big(\varphi(0,x), \frac{2k^4(k^2-1)}{3\alpha}t\Big)\Big\}. \tag{2.5}$$

Replacing x by $k^n x$ in (2.5), we get

$$N\Big(\frac{f(k^{n+1}x)}{k^{3(n+1)}}-\frac{f(k^n x)}{k^{3n}},\frac{t}{k^{3n}}\Big)$$
$$\geq \min\Big\{N'\Big(\varphi(0,k^n x),\frac{2k^3}{3}t\Big),N'\Big(\varphi(0,k^n x),\frac{2k^4(k^2-1)}{3\alpha}t\Big)\Big\}$$
$$\geq \min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3\alpha^n}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha^{n+1}}t\Big)\Big\}.$$

It follows from

$$\frac{f(k^n x)}{k^{3n}}-f(x)=\sum_{i=0}^{n-1}\frac{f(k^{i+1}x)}{k^{3(i+1)}}-\frac{f(k^i x)}{k^{3i}}$$

and above inequality that

$$N\quad\Big(\frac{f(k^n x)}{k^{3n}}-f(x),\sum_{i=0}^{n-1}\frac{\alpha^i t}{k^{3i}}\Big)\geq \min\bigcup_{i=0}^{n-1}\Big\{N\Big(\frac{f(k^{i+1}x)}{k^{3(i+1)}}-\frac{f(k^i x)}{k^{3i}},\frac{\alpha^i}{k^{3i}}t\Big)\Big\}$$
$$\geq\quad \min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha}t\Big)\Big\}.$$

We replace x by $k^m x$ to prove convergence of the sequence $\{\frac{f(k^n x)}{k^{3n}}\}$. For $m,n\in\mathbb{N}$,

$$N\Big(\frac{f(k^{n+m}x)}{k^{3(n+m)}}-\frac{f(k^m x)}{k^{3m}},\sum_{i=0}^{n-1}\frac{\alpha^i}{k^{3(i+m)}}t\Big)$$
$$\geq \min\Big\{N'\Big(\varphi(0,k^m x),\frac{2k^3}{3}t\Big),N'\Big(\varphi(0,k^m x),\frac{2k^4(k^2-1)}{3\alpha}t\Big)\Big\}$$
$$\geq \min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3\alpha^m}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha^{m+1}}t\Big)\Big\}.$$

Replacing t by $\alpha^m t$ in last inequality to get

$$N\Big(\frac{f(k^{n+m}x)}{k^{3(n+m)}}-\frac{f(k^m x)}{k^{3m}},\sum_{i=m}^{n+m-1}\frac{\alpha^i}{k^{3i}}t\Big)$$
$$\geq \min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha}t\Big)\Big\}.$$

For every $n\in\mathbb{N}$ and $m\in\mathbb{N}\cup\{0\}$, we put

$$a_{mn}:=\sum_{i=m}^{n+m-1}\frac{\alpha^i}{k^{3i}}.$$

Replacing t by $\frac{t}{a_{mn}}$ in last inequality, we observe that

$$N\Big(\frac{f(k^{n+m}x)}{k^{3(n+m)}}-\frac{f(k^m x)}{k^{3m}},t\Big)$$
$$\geq \min\ \Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3a_{mn}}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha a_{mn}}t\Big)\Big\}.$$

Let $t>0$ and $\varepsilon>0$ be given. Using the fact that $\lim_{t\to\infty}N'(\varphi(0,x),t)=1$, we can find some $t_1\geq 0$ such that $N'(\varphi(0,x),t_2)>1-\varepsilon$ for every $t_2>t_1$. The convergence of the series $\sum_{i=0}^{\infty}\frac{\alpha^i}{k^{3i}}$ guarantees that there exists some m_1 such that $\min\Big\{\frac{2k^3}{3a_{mn}}t,\frac{2k^4(k^2-1)}{3\alpha a_{mn}}t\Big\}>t_1$, for every $m\geq m_1$ and $n\in\mathbb{N}$. For every $m\geq m_1$ and $n\in\mathbb{N}$, we have

$$\begin{aligned} N\quad &\Big(\frac{f(k^{n+m}x)}{k^{3(n+m)}}-\frac{f(k^{3m}x)}{k^{3m}},t\Big)\\ \geq\quad &\min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3a_{mn}}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha a_{mn}}t\Big)\Big\}\\ \geq\quad &\min\{1-\varepsilon,1-\varepsilon\}=1-\varepsilon. \end{aligned}\tag{2.6}$$

Hence $\{\frac{f(k^n x)}{k^{3n}}\}$ is a Cauchy sequence in the fuzzy Banach space (Y,N), therefore this sequence converges to some point $C(x)\in Y$.

Fix $x\in X$ and put $m=0$ in (2.6) to obtain

$$N\Big(\frac{f(k^n x)}{k^{3n}}-f(x),t\Big)\geq\min\Big\{N'\Big(\varphi(0,x),\frac{2k^3}{3a_{0n}}t\Big),N'\Big(\varphi(0,x),\frac{2k^4(k^2-1)}{3\alpha a_{0n}}t\Big)\Big\}.$$

For every $n\in\mathbb{N}$,

$$N(C(x)-f(x),t)\geq\min\Big\{N\Big(C(x)-\frac{f(k^n x)}{k^{3n}},\frac{t}{2}\Big),N\Big(\frac{f(k^n x)}{k^{3n}}-f(x),\frac{t}{2}\Big)\Big\}.$$

The first two terms on the right hand side of the above inequality tend to 1 as $n\to\infty$. Therefore

$$\begin{aligned} N(C(x)-f(x),t)&\geq\min\Big\{N\Big(C(x)-\frac{f(k^n x)}{k^{3n}},\frac{t}{2}\Big),N\Big(\frac{f(k^n x)}{k^{3n}}-f(x),\frac{t}{2}\Big)\Big\}\\ &\geq\min\Big\{N'\Big(\varphi(0,x),\frac{k^3}{3a_{0n}}t\Big),N'\Big(\varphi(0,x),\frac{k^4(k^2-1)}{3\alpha a_{0n}}t\Big)\Big\} \end{aligned}$$

for n large enough. By last inequality, we have

$$\begin{aligned} N(C(x)-f(x),t)\geq\min\Big\{&N'\Big(\varphi(0,x),\frac{k^3-\alpha}{3}t\Big),\\ &N'\Big(\varphi(0,x),\frac{k(k^2-1)(k^3-\alpha)}{3\alpha}t\Big)\Big\} \end{aligned}\tag{2.7}$$

Replacing x, y by $k^n x$ and $k^n y$, respectively in (2.1) to get

$$N\Big(\frac{f(k^n(x+ky))}{k^{3n}} - \frac{kf(k^n(x+y))}{k^{3n}} + \frac{kf(k^n(x-y))}{k^{3n}} - \frac{f(k^n(x-ky))}{k^{3n}} - \frac{2k(k^2-1)f(k^n y)}{k^{3n}}, t\Big)$$
$$\geq N'(\varphi(k^n x, k^n y), k^{3n} t) \geq N'\Big(\varphi(x,y), \frac{k^{3n}}{\alpha^n} t\Big)$$

for all $x, y \in X$ and all $t > 0$. Since $1 \leq \alpha < k^3$, by (N5)

$$\lim_{n\to\infty} N'\Big(\varphi(x,y), \frac{k^{3n}}{\alpha^n} t\Big) = 1.$$

We conclude that C fulfills (1.1). To prove the uniqueness of the cubic function C, assume that there exists cubic function $C' : X \to Y$ which satisfies (2.7). Fix $x \in X$. Obviously

$$C(k^n x) = k^{3n} C(x), \quad C'(k^n x) = k^{3n} C'(x)$$

for all $n \in \mathbb{N}$. It follows from (2.7) that

$$N(C(x) - C'(x), t) = N\Big(\frac{C(k^n x)}{k^{3n}} - \frac{C'(k^n x)}{k^{3n}}, t\Big)$$
$$\geq \min\Big\{N\Big(\frac{C(k^n x)}{k^{3n}} - \frac{f(k^n x)}{k^{3n}}, \frac{t}{2}\Big), N\Big(\frac{f(k^n x)}{k^{3n}} - \frac{C'(k^n x)}{k^{3n}}, \frac{t}{2}\Big)\Big\}$$
$$\geq \min\Big\{N'\Big(\varphi(0, k^n x), \frac{(k^3-\alpha)k^{3n}}{6} t\Big), N'\Big(\varphi(0, k^n x), \frac{k(k^2-1)(k^3-\alpha)k^{3n}}{6\alpha} t\Big)\Big\}$$
$$\geq \min\Big\{N'\Big(\varphi(0, x), \frac{(k^3-\alpha)k^{3n}}{6\alpha^n} t\Big), N'\Big(\varphi(0, x), \frac{k(k^2-1)(k^3-\alpha)k^{3n}}{6\alpha^{n+1}} t\Big)\Big\}.$$

Since $1 \leq \alpha < k^3$, we obtain

$$\lim_{n\to\infty} N'\Big(\varphi(0,x), \frac{(k^3-\alpha)k^{3n}}{6\alpha^n} t\Big) = \lim_{n\to\infty} N'\Big(\varphi(0,x), \frac{k(k^2-1)(k^3-\alpha)k^{3n}}{6\alpha^{n+1}} t\Big) = 1.$$

Therefore $N'(C(x) - C'(x), t) = 1$ for all $t > 0$, whence $C(x) = C'(x)$.
Case (2): $\alpha > k^3$. We can state the proof in same pattern as we did in first case. Replacing x, t by $\frac{x}{k}$ and $2t$, respectively in (2.3) to get

$$N\Big(f(x) - k^3 f(\frac{x}{k}), t\Big) \geq \min\Big\{N'\Big(\varphi(0, \frac{x}{k}), \frac{2t}{3}\Big), N'\Big(\varphi(0, \frac{x}{k}), \frac{2k(k^2-1)}{3\alpha} t\Big)\Big\}.$$

We replacing y and t by $\frac{x}{k^n}$ and $\frac{t}{k^{3n}}$ in last inequality, respectively, we find that

$$\begin{aligned}N\Big(k^{3n}f(\frac{x}{k^n})-k^{3(n+1)}f(\frac{x}{k^{n+1}}),t\Big)\\ &\geq \min\Big\{N'\Big(\varphi(0,\frac{x}{k^{n+1}}),\frac{2t}{3k^{3n}}\Big),N'\Big(\varphi(0,\frac{x}{k^{n+1}}),\frac{2k(k^2-1)}{3k^{3n}\alpha}t\Big)\Big\}\\ &\geq \min\Big\{N'\Big(\varphi(0,x),\frac{2\alpha^{n+1}}{3k^{3n}}t\Big),N'\Big(\varphi(0,x),\frac{2k(k^2-1)\alpha^n}{3k^{3n}}t\Big)\Big\}.\end{aligned}$$

For each $n\in\mathbb{N}$, one can deduce

$$N\Big(k^{3n}f(\frac{x}{k^n})-f(x),t\Big)\geq \min\Big\{N'\Big(\varphi(0,x),\frac{2\alpha}{3b_{0n}}t\Big),N'\Big(\varphi(0,x),\frac{2(k^2-1)}{3b_{0n}}t\Big)\Big\}$$

where $b_{0n}=\sum_{i=0}^{n-1}\frac{k^{3i}}{\alpha^i}$. It is easy to see that $\{k^{3n}f(\frac{x}{k^n})\}$ is a Cauchy sequence in (Y,N). Therefore this sequence converges to some point $C(x)\in Y$ in the Banach space Y. Moreover, C satisfies (1.1) and

$$N(f(x)-C(x),t)\geq \min\Big\{N'\Big(\varphi(0,x),\frac{\alpha-k^3}{3}t\Big),N'\Big(\varphi(0,x),\frac{k(k^2-1)(\alpha-k^3)}{3\alpha}t\Big)\Big\}.$$

The proof for uniqueness of C for this case, proceeds similarly to that in the previous case, hence it is omitted. □

Corollary 2.2. Let X be a Banach space and $\varepsilon>0$ be a real number. Suppose that a function $f:X\to X$ satisfies

$$\begin{aligned}&\|f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y)\|\\ &\leq \varepsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)\end{aligned}$$

for all $x,y\in X$ where $0<p<\frac{1}{2}$ and $k\in\mathbb{N}\setminus\{1\}$. Then there exists a unique cubic function $C:X\to X$ which satisfying (1.1) and the inequality

$$\|C(x)-f(x)\|<\frac{3\varepsilon\|x\|^p}{k^3-k^{2p}}$$

for all $x\in X$.

Proof. Define $N:X\times\mathbb{R}\to[0,1]$ by

$$N(x,t)=\begin{cases}\frac{t}{t+\|x\|}, & t>0,\\ 0, & t\leq 0\end{cases}$$

It is easy to see that (X,N) is a fuzzy Banach space. Denote by $\varphi:X\times X\to\mathbb{R}$ the map sending each (x,y) to $\varepsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)$. By assumption

$$N(f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y),t)\geq N'(\varphi(x,y),t),$$

note that $N' : \mathbb{R} \times \mathbb{R} \to [0,1]$ given by

$$N'(x,t) = \begin{cases} \frac{t}{t+|x|}, & t > 0, \\ 0, & t \leq 0 \end{cases}$$

is a fuzzy norm on $\mathbb{R}$. By Theorem 2.1, there exists a unique cubic function $C : X \to X$ satisfies (1.1) and inequality

$$\begin{aligned} \tfrac{t}{t+\|f(x)-C(x)\|} &= N(f(x)-C(x),t) \\ &\geq \min\Big\{N'\Big(\varphi(0,x), \tfrac{k^3-k^{2p}}{3}t\Big), N'\Big(\varphi(0,x), \tfrac{k(k^2-1)(k^3-k^{2p})}{3k^{2p}}t\Big)\Big\} \\ &= \min\Big\{\tfrac{(k^3-k^{2p})t}{(k^3-k^{2p})t+3\varepsilon\|x\|^p}, \tfrac{k(k^2-1)(k^3-k^{2p})t}{k(k^2-1)(k^3-k^{2p})t+3\varepsilon\|x\|^p}\Big\} \\ &= \tfrac{(k^3-k^{2p})t}{(k^3-k^{2p})t+3\varepsilon\|x\|^p}. \end{aligned}$$

for all $x \in X$ and $t > 0$. Consequently $\|f(x) - C(x)\| \leq \frac{3\varepsilon}{k^3-k^{2p}}\|x\|^p$.

□

Corollary 2.3. Let X be a Banach space and $\varepsilon > 0$ be a real number. Suppose that a function $f : X \to X$ satisfies

$$\|f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) - 2k(k^2-1)f(y)\|$$
$$\leq \varepsilon(\|x\| + \|y\| - \|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}})$$

for all $x, y \in X$ where $k \in \mathbb{N} \setminus \{1\}$. Then there exists a unique cubic function $C : X \to X$ which satisfying (1.1) and the inequality

$$\|C(x) - f(x)\| < \frac{3\|x\|\varepsilon}{k^3 - k}$$

for all $x \in X$.

Proof. Consider the fuzzy norms defined by Corollary 2.2. We define

$$\varphi(x,y) = \varepsilon(\|x\| + \|y\| - \|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}})$$

for all $x, y \in X$. By assumption

$$N(f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y),t) \geq N'(\varphi(x,y),t),$$

for all $x, y \in X$ and $t > 0$. Therefore there exists a unique cubic function $C : X \to X$ satisfies (1.1) and inequality

$$\begin{aligned}
\tfrac{t}{t+\|f(x)-C(x)\|} &= N(f(x)-C(x),t) \\
&\geq \min\left\{N'\Big(\varphi(0,x),\tfrac{k^3-k}{3}t\Big), N'\Big(\varphi(0,x),\tfrac{k(k^2-1)(k^3-k)}{3k}t\Big)\right\} \\
&= \min\left\{\tfrac{(k^3-k)t}{(k^3-k)t+3\|x\|\varepsilon}, \tfrac{k(k^2-1)(k^3-k)t}{k(k^2-1)(k^3-k)t+3k\|x\|\varepsilon}\right\} \\
&= \tfrac{(k^3-k)t}{(k^3-k)t+3\|x\|\varepsilon}.
\end{aligned}$$

for all $x \in X$ and $t > 0$. Consequently $\|C(x) - f(x)\| < \frac{3\|x\|\varepsilon}{k^3-k}$. □

Let X be a Banach space. Denote by N and N' the fuzzy norms obtained as Corollary 2.2 on X and $\mathbb{R}$, respectively. Let $\epsilon > 0$ and let $\varphi : X \times X \to \mathbb{R}$ be defined by $\varphi(x, y) = \varepsilon$ for all $x, y \in X$. Let $f : X \to X$ be a φ-approximately cubic mapping in the sense that

$$\|f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) - 2k(k^2-1)f(y)\| < \varepsilon,$$

then there exists a unique cubic function $C : X \to X$ which satisfies

$$\|f(x) - C(x)\| \leq \frac{3\varepsilon}{k^3-1}$$

for all $x \in X$.

Let f be a mapping from X to Y. For each $k \in \mathbb{N}$, let $Df_k : X \times X \to Y$ be a mapping defined by

$$Df_k(x,y) = f(x+ky) - f(x-ky) - kf(x+y) + kf(x-y) - 2k(k^2-1)f(y)$$

Proposition 2.4. Let X be a linear space and let Y be a normed space. Let ε be a nonnegative real number and let f be a mapping from X to Y. Suppose that

$$\|Df_2(x,y)\| \leq \varepsilon \tag{2.8}$$

for all $x, y \in X$. Then there exists a sequence of nonnegative real numbers $\{\varepsilon_k\}_{k=0}^{\infty}$ such that $\varepsilon_0 = \varepsilon, \varepsilon_1 = 4\varepsilon, \varepsilon_2 = 10\varepsilon, ..., \varepsilon_k = 2\varepsilon_{k-1} + (k+1)\varepsilon + \varepsilon_{k-2}$ $(k \geq 3)$ and

$$\|Df_k(x,y)\| \leq \varepsilon_{k-2}. \tag{2.9}$$

Proof. Replacing x by $x+y$ and $x-y$ in (2.8), respectively, we get from (2.8) that

$$\|Df_3(x,y)\| \leq 4\varepsilon = \varepsilon_1 \tag{2.10}$$

for all $x, y \in X$. Replacing x by $y+x$ and $x-y$ in (2.10), respectively, we get from (2.8) that

$$\|Df_4(x,y)\| \leq 10\varepsilon = \varepsilon_2 \tag{2.11}$$

for all $x, y \in X$. Replacing x by $y+x$ and $x-y$ in (2.11), respectively, we get from (2.8) and (2.10)

$$\|Df_5(x,y)\| \leq 28\varepsilon = 2\varepsilon_2 + 4\varepsilon + \varepsilon_1 = \varepsilon_3,$$

for all $x, y \in X$. Therefore by using this method, by induction we infer (2.9). □

From about argument we have the following Corollary.

Corollary 2.5. Let X be a linear space and let Y be a Banach space. Let f be a mapping from X to Y and let ε be a nonnegative real number. Suppose that $\|Df_2(x,y)\| \leq \varepsilon$ holds for all $x, y \in X$. Then for each positive integer $k > 1$, there exists a unique cubic mapping $C_k : X \to Y$ such that

$$\|C_k(x) - f(x)\| \leq \frac{3\varepsilon_{k-2}}{k^3 - 1}$$

for all $x \in X$.

3 Fuzzy Generalized Hyers-Ulam theorem: uniform version

In this section, we deal with a fuzzy version of the generalized Hyers-Ulam stability in which we have uniformly approximate cubic mapping.

Theorem 3.1. Let X be a linear space and (Y, N) be a fuzzy Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$\phi(x,y) = \sum_{n=0}^{\infty} k^{-3n} \varphi(k^n x, k^n y) < \infty \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a uniformly approximately cubic function respect to φ in the sense that

$$\lim_{t\to\infty} N(f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) - 2k(k^2-1)f(y), t\varphi(x,y)) = 1 \tag{3.2}$$

uniformly on $X \times X$. Then $T(x) := N - \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$ for each $x \in X$ exists and defines a cubic mapping $T : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$,

$$N(f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y), \delta\varphi(x,y)) > \alpha \tag{3.3}$$

for all $x, y \in X$, then

$$N\Big(T(x) - f(x), \frac{\delta}{k^3}\phi(0,x)\Big) > \alpha$$

for all $x \in X$

Proof. Let $\varepsilon > 0$, by (3.2), we can find $t_0 > 0$ such that

$$N(f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y), t\varphi(x,y)) \geq 1-\varepsilon \tag{3.4}$$

for all $x, y \in X$ and all $t \geq t_0$. By induction on n, we shall show that

$$N\Big(f(k^n x) - k^{3n}f(x), t\sum_{m=0}^{n-1} k^{3(n-m-1)}\varphi(0, k^m x)\Big) \geq 1-\varepsilon \tag{3.5}$$

for all $t \geq t_0$, all $x \in X$ and all positive integers n. Putting $x = 0$ and $y = x$ in (3.4), we get (3.5) for $n = 1$. Let (3.5) holds for some positive integers n. Then

$$\begin{aligned}
&N\Big(f(k^{n+1}x) - k^{3(n+1)}f(x), t\sum_{m=0}^{n} k^{3(n-m)}\varphi(0, k^m x)\Big)\\
&\qquad\geq \min\Big\{N(f(k^{n+1}x) - k^3 f(k^n x), t\varphi(0, k^n x)),\\
&\qquad N(k^3 f(k^n x) - k^{3(n+1)}f(x), t\sum_{m=0}^{n-1} k^{3(n-m)}\varphi(0, k^m x))\Big\}\\
&\qquad\geq \min\{1-\varepsilon, 1-\varepsilon\} = 1-\varepsilon
\end{aligned}$$

This completes the induction argument. Let $t = t_0$ and put $n = p$. Then by replacing x with $k^n x$ in (3.5), we obtain

$$N\Big(\frac{f(k^{n+p}x)}{k^{3(n+p)}} - \frac{f(k^n x)}{k^{3n}}, t_0\sum_{m=0}^{p-1} k^{-3(n+m+1)}\varphi(0, k^{n+m}x)\Big) \geq 1-\varepsilon \tag{3.6}$$

for all integers $n \geq 0$, $p > 0$. The convergence of (3.1) and the equation

$$\sum_{m=0}^{p-1} k^{-3(n+m+1)}\varphi(0, k^{n+m}x) = \frac{1}{k^3}\sum_{m=n}^{n+p-1} k^{-3m}\varphi(0, k^m x),$$

guarantees that for given $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{k^3} \sum_{m=n}^{n+p-1} k^{-3m} \varphi(0, k^m x) < \delta,$$

for all $n \geq n_0$ and $p > 0$. It follows from (3.6) that

$$\begin{aligned} & N\Big(\tfrac{f(k^{n+p}x)}{k^{3(n+p)}} - \tfrac{f(k^n x)}{k^{3n}}, \delta\Big) \\ \geq N\Big(\tfrac{f(k^{n+p}x)}{k^{3(n+p)}} - \tfrac{f(k^n x)}{k^{3n}}, & t_0 \textstyle\sum_{m=0}^{p-1} k^{-3(n+m+1)} \varphi(0, k^{n+m}x)\Big) \\ & \geq \qquad 1 - \varepsilon \end{aligned} \tag{3.7}$$

for each $n \geq n_0$ and all $p > 0$. Hence $\{\frac{f(k^n x)}{k^{3n}}\}$ is a Cauchy sequence in Y. Since Y is a fuzzy Banach space, this sequence converges to some $T(x) \in Y$. Hence we can define a mapping $T : X \to Y$ by $T(x) := N - \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$, namely. For each $t > 0$ and $x \in X$, $\lim_{n\to\infty} N\Big(T(x) - \frac{f(k^n x)}{k^{3n}}, t\Big) = 1$. Now, let $x, y \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} k^{-3n}\varphi(k^n x, k^n y) = 0$, there is some $n_1 > n_0$ such that $t_0\varphi(k^n x, k^n y) < \frac{k^{3n}t}{6}$ for all $n \geq n_1$. Hence for each $n \geq n_1$, we infer that

$$\begin{aligned} & N(T(x+ky) - kT(x+y) + kT(x-y) - T(x-ky) - 2k(k^2-1)T(y), t) \\ & \quad \geq \min\Big\{N\Big(T(x+ky) - \frac{f(k^n(x+ky))}{k^{3n}}, \frac{t}{6}\Big), N\Big(T(x+y) - \frac{f(k^n(x+y))}{k^{3n}}, \frac{t}{6k}\Big), \\ & \quad N\Big(T(x-y) - \frac{f(k^n(x-y))}{k^{3n}}, \frac{t}{6k}\Big), N\Big(T(x-ky) - \frac{f(k^n(x-ky))}{k^{3n}}, \frac{t}{6}\Big), \\ & \quad N\Big(T(y) - \frac{f(k^n y)}{k^{3n}}, \frac{t}{12k(k^2-1)}\Big), N\Big(f(k^n(x+ky)) - kf(k^n(x+y)) \\ & \quad + kf(k^n(x-y)) - f(k^n(x-ky)) - 2k(k^2-1)f(k^n y), \frac{k^{3n}t}{6}\Big)\Big\}. \end{aligned}$$

As $n \to \infty$, we see that the first five terms on the right-hand side of the above inequality tend to 1 and the sixth term is greater than $N(D_k f(k^n x, k^n y), t_0\varphi(k^n x, k^n y))$, that is, by (3.4), greater than or equal to $1 - \varepsilon$. Thus

$$N(T(x+ky) - kT(x+y) + kT(x-y) - T(x-ky) - 2k(k^2-1)T(y), t) \geq 1 - \varepsilon,$$

for all $t > 0$ and $0 < \varepsilon < 1$. It follows that

$$N(T(x+ky) - kT(x+y) + kT(x-y) - T(x-ky) - 2k(k^2-1)T(y), t) = 1,$$

for all $t > 0$ and by (N2), we have

$$T(x+ky) - kT(x+y) + kT(x-y) - T(x-ky) = 2k(k^2-1)T(y).$$

To end the proof, let for some positive δ and α, (3.3) holds. Let

$$\varphi_n(x,y) := \sum_{m=0}^{n-1} k^{3(-m-1)}\varphi(k^m x, k^m y) \quad (x, y \in X).$$

Let $x \in X$. By a similar discussion as in the beginning of the proof we can obtain from (3.3)

$$N\Big(f(k^n x) - k^{3n} f(x), \delta \sum_{m=0}^{n-1} k^{3(n-m-1)}\varphi(0, k^m x)\Big) \geq \alpha, \tag{3.8}$$

for all positive integers n. Let $s > 0$. We have

$$N\Big(f(x) - T(x), \delta\varphi_n(0,x) + s\Big) \tag{3.9}$$

$$\geq \min\Big\{N\Big(f(x) - \frac{f(k^n x)}{k^{3n}}, \delta\varphi_n(0,x)\Big), N\Big(\frac{f(k^n x)}{k^{3n}} - T(x), s\Big)\Big\}. \tag{3.10}$$

Combining (3.7), (3.8) and the fact that $\lim_{n\to\infty} N\Big(\frac{f(k^n x)}{k^{3n}} - T(x), s\Big) = 1$, we obtain that

$$N\Big(f(x) - T(x), \delta\varphi_n(0,x) + s\Big) \geq \alpha$$

for large enough n. By the (upper semi) continuity of the real function $N(f(x) - T(x), .)$, we obtain that $N\Big(f(x) - T(x), \frac{\delta}{k^3}\phi(0,x) + s\Big) \geq \alpha$. Taking the limit as $s \to 0$, we conclude that

$$N\Big(f(x) - T(x), \frac{\delta}{k^3}\phi(0,x)\Big) \geq \alpha.$$

□

Corollary 3.2. Let X be a linear space and (Y, N) be a fuzzy Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a function satisfying (3.1). Let $f : X \to Y$ be a uniformly approximately cubic function with respect to φ. Then there is a unique cubic mapping $T : X \to Y$ such that

$$\lim_{t\to\infty} N(f(x) - T(x), t\phi(0,x)) = 1 \tag{3.11}$$

uniformly on X.

Proof. The existence of uniform limit (3.9) immediately follows Theorem 3.1. It remains to prove the uniqueness assertion. Let S be another cubic mapping satisfying (3.9). Fix $c > 0$. Given $\varepsilon > 0$, by (3.9) for T and S, we obtain some $t_0 > 0$ such that

$$N\Big(f(x) - T(x), \frac{t}{2}\phi(0, x)\Big) \geq 1 - \varepsilon, \quad N\Big(f(x) - S(x), \frac{t}{2}\phi(0, x)\Big) \geq 1 - \varepsilon,$$

for all $x \in X$ and all $t \geq t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 \sum_{m=n}^{\infty} k^{-3m}\varphi(0, k^m x) < \frac{c}{2}, \qquad \forall n \geq n_0.$$

Since

$$\begin{aligned}
\sum_{m=n}^{\infty} k^{-3n}\varphi(0, k^m x) &= \frac{1}{k^{3n}} \sum_{m=n}^{\infty} k^{-3(m-n)}\varphi(0, k^{m-n}(k^n x)) \\
&= \frac{1}{k^{3n}} \sum_{i=0}^{\infty} k^{-3i}\varphi(0, k^i(k^n x)) \\
&= \frac{1}{k^{3n}}\phi(0, k^n x),
\end{aligned}$$

we have

$$\begin{aligned}
N(S(x) - T(x), c) &\geq \min\Big\{N\Big(\frac{f(k^n x)}{k^{3n}} - T(x), \frac{c}{2}\Big), N\Big(S(x) - \frac{f(k^n x)}{k^{3n}}, \frac{c}{2}\Big)\Big\} \\
&= \min\Big\{N\Big(f(k^n x) - T(k^n x), \frac{k^{3n}}{2}c\Big), N\Big(S(k^n x) - f(k^n x), \frac{k^{3n}}{2}c\Big)\Big\} \\
&\geq \min\Big\{N\Big(f(k^n x) - T(k^n x), k^{3n}t_0 \sum_{m=n}^{\infty} k^{-3n}\varphi(0, k^m x)\Big), \\
&\qquad N\Big(S(k^n x) - f(k^n x), k^{3n}t_0 \sum_{m=n}^{\infty} k^{-3n}\varphi(0, k^m x)\Big)\Big\} \\
&= \min\{N(f(k^n x) - T(k^n x), t_0\phi(0, k^n x)), N(S(k^n x) - f(k^n x), t_0\phi(k^n x))\} \\
&\geq 1 - \varepsilon
\end{aligned}$$

It follows that $N(S(x) - T(x), c) = 1$ for all $c > 0$. Thus $T(x) = S(x)$ for all $x \in X$. □

4 Fuzzy Completeness

We proved that under suitable conditions including the completeness of a space, for every approximately cubic function there exists a unique cubic mapping which is close to it. It is natural to ask whether the converse of this result holds. More precisely, under what conditions involving approximately cubic functions, our fuzzy normed space is complete. The following result gives a partial answer to this question.

Definition 4.1. Let (X, N) be a fuzzy normed space. A mapping $f : \mathbb{N} \cup \{0\} \to X$ is said to be approximately cubic if for each $\alpha \in (0, 1)$ there is some $n_\alpha \in \mathbb{N}$ such that

$$N(f(i+kj) - kf(i+j) + kf(i-j) - f(i-kj) - 2k(k^2-1)f(j), t) \geq \alpha,$$

for all $i \geq 2j \geq n_\alpha$.

Definition 4.2. Let (X, N) be a fuzzy normed space. A mapping $f : \mathbb{N} \cup \{0\} \to X$ is said to be a conditional cubic if

$$f(i+kj) - kf(i+j) + kf(i-j) - f(i-kj) = 2k(k^2-1)f(j),$$

for all $i \geq 2j$.

Theorem 4.3. Let (X, N) be a fuzzy normed space such that for each approximately cubic type mapping $f : \mathbb{N} \cup \{0\} \to X$, there is a conditional cubic mapping $C : \mathbb{N} \cup \{0\} \to X$ such that $\lim_{n\to\infty} N(C(n) - f(n), 1) = 1$. Then (X, N) is a fuzzy Banach space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, N). There is an increasing sequence $\{n_l\}$ of natural numbers

$$N\Big(x_n - x_m, \frac{1}{(10kl)^3}\Big) \geq 1 - \frac{1}{l}$$

for each $n, m \geq n_l$. Put $y_l = x_{n_l}$ and define $f : \mathbb{N} \cup \{0\} \to X$ by $f(l) = l^3 y_l$. Let $\alpha \in (0, 1)$ and find some $n_0 \in \mathbb{N}$ such that $1 - \frac{1}{n_0} > \alpha$. Since

$$\begin{aligned}
&f(i+kj) - kf(i+j) + kf(i-j) - f(i-kj) - 2k(k^2-1)f(j)\\
&\quad = kj^3[y_{i+kj} - y_{i+j}] + kj^3[y_{i+kj} - y_{i-j}] + (k^3-2k)j^3[y_{i+kj} - y_j]\\
&\quad + k^3j^3[y_{i-kj} - y_j] + 3k^2ij^2[y_{i+kj} - y_{i-kj}] + 3kij^2[y_{i+j} - y_{i-j}]\\
&\quad + 3ki^2j[y_{i+kj} - y_{i+j}] + 3ki^2j[y_{i-kj} - y_{i-j}] + i^3[y_{i+kj} - y_{i-kj}]\\
&\quad + ki^3[y_{i-j} - y_{i+j}],
\end{aligned}$$

for each $i \geq 2kj$, hence

$$N(f(i+kj) - kf(i+j) + kf(i-j) - f(i-kj) - 2k(k^2-1)f(j), 1) \geq$$

$$\min\Big\{N\Big(y_{i+kj} - y_{i+j}, \tfrac{1}{10kj^3}\Big), N\Big(y_{i+kj} - y_{i-j}, \tfrac{1}{10kj^3}\Big), N\Big(y_{i+kj} - y_j, \tfrac{1}{10(k^3-2k)j^3}\Big),$$

$$N\Big(y_{i-kj} - y_j, \tfrac{1}{10k^3j^3}\Big), N\Big(y_{i+kj} - y_{i-kj}, \tfrac{1}{30k^2ij^2}\Big), N\Big(y_{i+j} - y_{i-j}, \tfrac{1}{30kij^2}\Big),$$

$$N\Big(y_{i+kj} - y_{i+j}, \tfrac{1}{30ki^2j}\Big), N\Big(y_{i-kj} - y_{i-j}, \tfrac{1}{30ki^2j}\Big), N\Big(y_{i+kj} - y_{i-kj}, \tfrac{1}{10i^3}\Big),$$

$$N\Big(y_{i-j} - y_{i+j}, \tfrac{1}{10ki^3}\Big)\Big\}.$$

Let $i > n_0$. Then

$$N\Big(y_{i+kj} - y_{i+j}, \frac{1}{10kj^3}\Big) \geq N\Big(y_{i+kj} - y_{i+j}, \frac{1}{10^3k^3(i+j)^3}\Big) \geq \alpha,$$

$$N\Big(y_{i+kj} - y_{i-j}, \frac{1}{10kj^3}\Big) \geq N\Big(y_{i+kj} - y_{i-j}, \frac{1}{10^3k^3(i-j)^3}\Big) \geq \alpha,$$

$$N\Big(y_{i+kj} - y_j, \frac{1}{10(k^3-2k)j^3}\Big) \quad \geq \quad N\Big(y_{i+kj} - y_j, \frac{1}{10^3k^3j^3}\Big) \geq \alpha,$$

$$N\Big(y_{i-kj} - y_j, \frac{1}{10k^3j^3}\Big) \quad \geq \quad N\Big(y_{i-kj} - y_j, \frac{1}{10^3k^3j^3}\Big) \geq \alpha,$$

$$N\Big(y_{i+kj} - y_{i+j}, \frac{1}{30ki^2j}\Big) \geq N\Big(y_{i+kj} - y_{i+j}, \frac{1}{10^3k^3(i+j)^3}\Big) \geq \alpha.$$

Since $i - kj \geq kj$ and $i - \frac{i}{2} \geq kj$, we infer that

$$N\Big(y_{i+kj} - y_{i-kj}, \frac{1}{30k^2ij^2}\Big) \geq N\Big(y_{i+kj} - y_{i-kj}, \frac{1}{10^3k^3(i-kj)^3}\Big) \geq \alpha,$$

$$N\Big(y_{i+j} - y_{i-j}, \frac{1}{30kij^2}\Big) \geq N\Big(y_{i+j} - y_{i-j}, \frac{1}{10^3k^3(i-j)^3}\Big) \geq \alpha,$$

$$N\Big(y_{i-kj}-y_{i-j},\frac{1}{30ki^2j}\Big)\geq N\Big(y_{i-kj}-y_{i-j},\frac{1}{10^3k^3(i-j)^3}\Big)\geq\alpha,$$

$$N\Big(y_{i+kj}-y_{i-kj},\frac{1}{10i^3}\Big)\geq N\Big(y_{i+kj}-y_{i-kj},\frac{1}{10^3k^3(i-kj)^3}\Big)\geq\alpha,$$

$$N\Big(y_{i-j}-y_{i+j},\frac{1}{10ki^3}\Big)\geq N\Big(y_{i-j}-y_{i+j},\frac{1}{10^3k^3(i-j)^3}\Big)\geq\alpha.$$

Therefore

$$N(f(i+kj)-kf(i+j)+kf(i-j)-f(i-kj)-2k(k^2-1)f(j),t)\geq\alpha.$$

This means that f is an approximately cubic type mapping. By our assumption, there is a conditional cubic mapping $C:\mathbb{N}\cup\{0\}\to X$ such that $\lim_{n\to\infty}N(C(n)-f(n),1)=1$. In particular, $\lim_{n\to\infty}N(C(k^n)-f(k^n),1)=1$. This means that $\lim_{n\to\infty}N\Big(C(1)-y_{k^n},\frac{1}{k^{3n}}\Big)=1$. Hence the subsequence $\{y_{k^n}\}$ of the cauchy sequence $\{x_n\}$ converges to $y=C(1)$. Therefore $\{x_n\}$ also converges to y. $\square$

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