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A generalized form of Ekeland's variational principle

Csaba Farkas

Abstract

In this paper we prove a generalized version of the Ekeland variational principle, which is a common generalization of Zhong variational principle and Borwein Preiss Variational principle. Therefore in a particular case, from this variational principle we get a Zhong type variational principle, and a Borwein-Preiss variational principle. As a consequence, we obtain a Caristi type fixed point theorem.

1 Introduction

In 1974 I. Ekeland formulated a variational principle in [5] having applications in many domains of Mathematics, including fixed point theory. Ekeland's variational principle (see, for instance [5] and [6]) has been widely used in nonlinear analysis, since it entails the existence of approximate solutions of minimization problems for lower semicontinuous functions on complete metric spaces (see, for instance [1]). Later,Borwein and Preiss gave a different form of this principle suitable for applications in subdifferential theory [2]. Ekeland's variational principle has many generalizations in the very recent books of Borwein, Zhu [3], Meghea [7] and the references therein.

In this paper we give a generalized form of Ekeland variational principle, which is a generalization of the variational principles given by Ekeland-Borwein-Preiss and also by Zhong. As a consequence, we obtain a Caristi type fixed point theorem in a complete metric space.

Key Words: Ekeland variational principle,Zhong variational principle, Borwein Preiss variational principle, metric spaces, Caristi type fixedpoint
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2 A generalized form of Ekeland's variational principle

In this section we give a common generalization of the variational principles of Borwein-Preiss-Ekeland [3] and Zhong [9].

First, we recall some notions used in our results, such as lower semi-continuous or proper functions. For this let (X, d) be an arbitrary metric space:

Definition 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. We say that the function f is lower semicontinuous at $x_0 \in X$ if*

$$\liminf_{x \to x_0} f(x) = f(x_0),$$

where $\liminf_{x\to x_0} f(x) = \sup_{V \in \mathcal{V}(x_0)} \inf_{x \in V} f(x)$, where $\mathcal{V}(x_0)$ is a neighborhood system of x_0.

Definition 2.2. *Let $f : X \to \overline{\mathbb{R}}$ be a function. We define the following set*

$$\mathcal{D}(f) = \{x \in X | f(x) < \infty\}.$$

We say that the function f is proper if $\mathcal{D}(f) \neq \emptyset$.

Now, we prove our main result.

Theorem 2.1. *Let $h : [0, +\infty) \to [0, +\infty)$ be continuous non-increasing function. Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous function bounded from below. Suppose that $\rho : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ is a function, satisfying:*

(i) for each $x \in X$, we have $\rho(x, x) = 0$;

(ii) for each $(y_n, z_n) \in X \times X$, such that $\rho(y_n, z_n) \to 0$ we have $d(y_n, z_n) \to 0$;

(iii) for each $z \in X$ the function $y \mapsto \rho(y, z)$ is lower semi-continuous function.

Let $\delta_n \geq 0$ $(n \in \mathbb{N}^*)$ be a nonnegative number sequence and $\delta_0 > 0$ a positive number. For every $x_0 \in X$ and $\varepsilon > 0$ with

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{2.1}$$

there exists a sequence $\{x_n\} \subset X$ which converges to some x_ε $(x_n \to x_\varepsilon)$ such that

$$h(d(x_0, x_n))\rho(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n \delta_0}, \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

If $\delta_n > 0$ for infinitely many $n \in \mathbb{N}$, then

$$f(x_\varepsilon) + h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n) \leq f(x_0), \tag{2.3}$$

and for $x \neq x_\varepsilon$ we have that

$$f(x) + h(d(x_0, x)) \sum_{n=0}^{\infty} \delta_n \rho(x, x_n) > f(x_\varepsilon) + h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n). \tag{2.4}$$

If $\delta_k > 0$ for some $k \in \mathbb{N}^*$ and $\delta_j = 0$ for every $j > k$, then for each $x \neq x_\varepsilon$ there exists $m \in \mathbb{N}$, $m \geq k$ such that

$$f(x) + h(d(x_0, x)) \sum_{i=0}^{k-1} \delta_i \rho(x, x_i) + h(d(x_0, x)) \delta_k \rho(x, x_m) >$$

$$f(x_\varepsilon) + h(d(x_0, x_\varepsilon)) \sum_{i=0}^{k-1} \delta_i \rho(x_\varepsilon, x_i) + h(d(x_0, x_\varepsilon)) \delta_k \rho(x_\varepsilon, x_m). \tag{2.5}$$

Proof. In the first case, for infinitely many $n \in \mathbb{N}$, without loss of generality, we can assume that $\delta_n > 0$, for every $n \in \mathbb{N}$. We can define the following set:

$$\mathcal{W}(x_0) = \{x \in X | f(x) + h(d(x_0, x)) \delta_0 \rho(x, x_0) \leq f(x_0)\}. \tag{2.6}$$

By the assumption (i), we have that $\rho(x_0, x_0) = 0$, so $x_0 \in \mathcal{W}(x_0)$. Therefore the set $\mathcal{W}(x_0) \neq \emptyset$. From the lower semi-continuity of the functions f and $\rho(y, \cdot)$ and continuity of function h, we have that $\mathcal{W}(x_0)$ is closed subset of X. We can choose $x_1 \in \mathcal{W}(x_0)$, such that

$$f(x_1) + h(d(x_0, x_1)) \delta_0 \rho(x_1, x_0) \leq \inf_{x \in \mathcal{W}(x_0)} \{f(x) + h(d(x_0, x)) \delta_0 \rho(x, x_0)\} + \frac{\varepsilon \cdot \delta_1}{2\delta_0}$$

and set again:

$$\mathcal{W}(x_1) = \left\{ x \in \mathcal{W}(x_0) | f(x) + h(d(x_0, x)) \sum_{i=0}^{1} \delta_i \rho(x, x_i) \leq f(x_1) + h(d(x_0, x_1)) \delta_0 \rho(x_1, x_0) \right\}$$

Similarly as above, we have that $\mathcal{W}(x_1) \neq \emptyset$ (since $x_1 \in \mathcal{W}(x_1)$), and $\mathcal{W}(x_1)$ is non-empty closed subset of $\mathcal{W}(x_0)$, which means that $\mathcal{W}(x_1)$ is a non-empty closed subset of X as well.

Using the mathematical induction we can define a sequence $x_{n-1} \in \mathcal{W}(x_{n-2})$ and $\mathcal{W}(x_{n-1})$ such that:

$$\mathcal{W}(x_{n-1}) = \{x \in \mathcal{W}(x_{n-2}) | f(x) + h(d(x_0, x)) \sum_{i=0}^{n-1} \delta_i \rho(x, x_i) \leq$$

$$f(x_{n-1})+h(d(x_0,x_{n-1}))\sum_{i=0}^{n-2}\delta_i\rho(x_{n-1},x_i)\}.$$

It is easy to see that $\mathcal{W}(x_{n-1})\neq\emptyset$, and $\mathcal{W}(x_{n-1})$ is closed subset of X. We can choose $x_n\in\mathcal{W}(x_{n-1})$ such that

$$f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)\leq$$

$$\leq\inf_{x\in\mathcal{W}(x_{n-1})}\left\{f(x)+h(d(x_0,x))\sum_{i=0}^{n-1}\delta_i\rho(x,x_i)\right\}+\frac{\delta_n\cdot\varepsilon}{2^n\delta_0},$$

and we can define the set

$$\mathcal{W}(x_n)=\left\{x\in\mathcal{W}(x_{n-1})|f(x)+h(d(x_0,x))\sum_{i=0}^{n}\delta_i\rho(x,x_i)\leq f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)\right\}$$

which is closed subset of X.

Let z be an arbitrary element of $\mathcal{W}(x_n)$. Then from the definition of $\mathcal{W}(x_n)$ we have the following inequality

$$f(z)+h(d(x_0,z))\sum_{i=0}^{n}\delta_i\rho(z,x_i)\leq f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)\Leftrightarrow$$

$$\Leftrightarrow f(z)+h(d(x_0,z))\delta_n\rho(z,x_n)+h(d(x_0,z))\sum_{i=0}^{n-1}\delta_i\rho(z,x_i)\leq f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i).$$

Then, we obtain

$$\begin{aligned}
h(d(x_0,z))\delta_n\rho(z,x_n) &\leq \left[f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)\right]-\\
&\quad -\left[f(z)+h(d(x_0,z))\sum_{i=0}^{n-1}\delta_i\rho(z,x_i)\right]\leq\\
&\leq \left[f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)\right]\\
&\quad -\inf_{x\in\mathcal{W}(x_{n-1})}\left[f(x)+h(d(x_0,x))\sum_{i=0}^{n-1}\delta_i\rho(x,x_i)\right]\\
&\leq \frac{\delta_n\varepsilon}{2^n\delta_0},
\end{aligned}$$

therefore
$$h(d(x_0,z))\rho(z,x_n) \leq \frac{\varepsilon}{2^n\delta_0}. \tag{2.7}$$

So, if $n \to \infty$, then $\rho(z,x_n) \to 0$. Then from (ii) it follows that $d(z,x_n) \to 0$. Therefore $diam(\mathcal{W}(x_n)) \to 0$, whenever $n \to \infty$ and we obtain a descending sequence $\{\mathcal{W}(x_n)\}_{n\geq 0}$ of nonempty closed subsets of X,

$$\mathcal{W}(x_0) \supset \mathcal{W}(x_1) \supset .... \supset \mathcal{W}(x_n) \supset ...$$

such that $diam(\mathcal{W}(x_n)) \to 0$, as $n \to \infty$. Applying the Cantor intersection theorem for the set sequence $\{\mathcal{W}(x_n)\}_{n\in\mathbb{N}}$, we have that there exists an $x_\varepsilon \in X$ such that
$$\bigcap_{n=0}^{\infty} \mathcal{W}(x_n) = \{x_\varepsilon\}.$$

We can observe that $z = x_\varepsilon$ satisfies the inequality (2.7), therefore $x_n \to x_\varepsilon$. If $x \neq x_\varepsilon$, then there exists $m \in \mathbb{N}$ such that

$$f(x)+h(d(x_0,x))\sum_{i=0}^{m}\delta_i\rho(x,x_i) > f(x_m)+h(d(x_0,x_m))\sum_{i=0}^{m-1}\delta_i\rho(x_m,x_i). \tag{2.8}$$

It is clear that if $q \geq m$ then

$$\begin{aligned} f(x_m)+h(d(x_0,x_m))\sum_{i=0}^{m-1}\delta_i\rho(x,x_i) \quad &\geq \quad f(x_q)+h(d(x_0,x_q))\sum_{i=0}^{q-1}\delta_i\rho(x_q,x_i) \geq \\ &\geq \quad f(x_\varepsilon)+h(d(x_0,x_\varepsilon))\sum_{i=0}^{q-1}\delta_i\rho(x_\varepsilon,x_i). \end{aligned}$$

using the inequality (2.8) we get the following estimate

$$f(x)+h(d(x_0,x))\sum_{i=0}^{m}\delta_i\rho(x,x_i) \geq f(x_\varepsilon)+h(d(x_0,x_\varepsilon))\sum_{i=0}^{q}\delta_i\rho(x_\varepsilon,x_i),$$

from where if $q,m \to \infty$, we have the claimed (2.4) relation.

Now, we assume the existence of a $k \in \mathbb{N}$ such that $\delta_k > 0$ and $\delta_j = 0$ for each $j > k \geq 0$. Without loss of generality we can assume that $\delta_i > 0$ for every $i \leq k$. If $n \leq k$ then we can take x_n and $\mathcal{W}(x_n)$ similarly as above. If $n > k$, we can choose $x_n \in \mathcal{W}(x_{n-1})$ so that

$$f(x_n)+h(d(x_0,x_n))\sum_{i=0}^{k-1}\delta_i\rho(x_n,x_i) \leq \inf_{x\in\mathcal{W}(x_{n-1})}\left\{f(x)+h(d(x_0,x))\sum_{i=0}^{k-1}\delta_i\rho(x,x_i)\right\}+\frac{\delta_k\varepsilon}{2^n\delta_0},$$

and we define the following set

$$\mathcal{W}(x_n) = \{x \in \mathcal{W}(x_{n-1}) | f(x)+h(d(x_0,x))\sum_{i=0}^{k-1}\delta_i\rho(x,x_i)+h(d(x_0,x))\delta_k\rho(x,x_n) \leq$$

$$f(x_n) + h(d(x_0,x))\sum_{i=0}^{k-1}\delta_i\rho(x_n,x_i)\}.$$

In the same way as above, we can see that the statement of Theorem 2.1 holds.

But, if we have $x \neq x_\varepsilon$, then there exists $m > k$ such that

$$f(x)+h(d(x_0,x))\sum_{i=0}^{k-1}\delta_i\rho(x,x_i)+h(d(x_0,x))\delta_k(x,x_m) > f(x_m)+h(d(x_0,x))\sum_{i=0}^{k-1}\delta_i\rho(x_m,x_i)$$

$$\geq f(x_\varepsilon) + h(d(x_0,x_\varepsilon))\sum_{i=0}^{k-1}\delta_i\rho(x_\varepsilon,x_i) + h(d(x_0,x_\varepsilon))\delta_k\rho(x_\varepsilon,x_m),$$

which concludes the proof. □

3 Relation with the Zhong variational principle and the Ekeland-Borwein-Preiss variational principle

We show that in a special case of the Theorem 2.1 we get Zhong's variational principle (see for instance [9] and [10]), and in another special case we get the generalized form of Ekeland-Borwein-Preiss variational principle given by Li Yongxin and Shi Shuzhong(see [2], [8] and [3]).

3.1 Relation with Ekeland-Borwein-Preiss variational principle

From the theorem 2.1 we have that

$$f(x) + h(d(x_0,x))\sum_{n=0}^{\infty}\delta_n\widetilde{\rho}(x,x_n) > f(x_\varepsilon) + h(d(x_0,x_\varepsilon))\sum_{n=0}^{\infty}\delta_n\widetilde{\rho}(x_\varepsilon,x_n).$$

We choose $h \equiv \varepsilon > 0$ and $\widetilde{\rho} = \frac{1}{\varepsilon}\rho$. This means that theorem 2.1 gets the following form:

Corollary 3.1. *(Yongxin-Shuzong [8]) Let (X,d) be a complete metric space and $f : X \to \mathbb{R}\cup\{\infty\}$ be a lower semi-continuous function bounded from below, such that $\mathcal{D}(f) \neq \emptyset$. Suppose that $\rho : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ is a function, satisfying:*

(i) for each $x \in X$, *we have* $\rho(x,x) = 0$;

(ii) for each $(y_n, z_n) \in X \times X$, *such that* $\rho(y_n, z_n) \to 0$ *we have* $d(y_n, z_n) \to 0$;

(iii) for each $z \in X$ *the function* $y \mapsto \rho(y,z)$ *is lower semi-continuous function.*

And let $\delta_n \geq 0 (n \in \mathbb{N}^*)$ *be a nonnegative number sequence,* $\delta_0 > 0$. *Then for every* $x_0 \in X$ *and* $\varepsilon > 0$ *with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{3.9}$$

there exists a sequence $\{x_n\} \subset X$ *which converges to some* x_ε $(x_n \to x_\varepsilon)$ *such that*

$$\rho(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n \delta_0} \quad n \in \mathbb{N}. \tag{3.10}$$

If $\delta_n > 0$ *for infinitely many* n, *then*

$$f(x_\varepsilon) + \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n) \leq f(x_0), \tag{3.11}$$

and for $x \neq x_\varepsilon$ *we have*

$$f(x) + \sum_{n=0}^{\infty} \delta_n \rho(x, x_n) > f(x_\varepsilon) + \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n). \tag{3.12}$$

3.2 Relation with Zhong variational principle

To obtain the Zhong's variational principle as a special case of Theorem 2.1 we choose the functions h, ρ, and the sequence δ_n as follows. Let $\delta_0 = 1$ and $\delta_n = 0$, for every $n > 0$. Let $\varepsilon, \lambda > 0$ and $h(t) = \dfrac{\varepsilon}{\lambda(1+g(t))}$, where $g : [0,\infty) \to [0,\infty)$ is a continuous non-decreasing function. Then, in this case

$$\sum_{n=0}^{\infty} \delta_n \rho(x, x_n) = \delta_0 \rho(x, x_0) = \rho(x, x_0).$$

If $\rho = d$ then the Theorem 2.1 has the following form:

$$f(x) \geq f(x_\varepsilon) + \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x_\varepsilon, x_0) - \frac{\varepsilon}{\lambda(1+g(d(x_0,x)))} d(x, x_0). \tag{3.13}$$

In the sequel, we examine the conditions when the following inequality holds:

$$\frac{d(x_0,x)}{1+g(d(x_0,x))} - \frac{d(x_0,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))} \leq \frac{d(x,x_\varepsilon)}{1+g(d(x_\varepsilon,x_0))} \tag{3.14}$$

We use the notations

$$\begin{cases} d(x_0, x) = a, \\ d(x_0, x_\varepsilon) = c, \\ d(x, x_\varepsilon) = b. \end{cases}$$

It is easy to see that a, b, c are exactly the sides of a triangle. The inequality (3.14) is equivalent with the following

$$\frac{a}{1+g(a)} \leq \frac{b+c}{1+g(c)} \Leftrightarrow$$

$$a + ag(c) \leq (b+c) + (b+c)g(a). \tag{3.15}$$

Now, we distinguish two cases, whether $a \geq c$ or $a < c$.

If $a \geq c$, then by the choice of g, we have $g(a) \geq g(c)$, so $ag(c) \leq ag(a) \leq (b+c)g(a)$. So, if $x \notin B(x_\varepsilon, d(x_0, x_\varepsilon))$, then

$$f(x) \geq f(x_\varepsilon) + \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x_\varepsilon, x_0) - \frac{\varepsilon}{\lambda(1+g(d(x_0,x)))} d(x, x_0) \geq$$

$$\geq f(x_\varepsilon) - \frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))} d(x, x_\varepsilon) \tag{3.16}$$

Now, we examine the case when $a < c$. We can observe that, if $x \mapsto \dfrac{g(x)}{x}$ is a non-increasing function, then $\dfrac{g(c)}{c} \leq \dfrac{g(a)}{a}$ and we obtain

$$a + ag(c) \leq a + cg(a) \leq (b+c) + cg(a) \leq (b+c) + (b+c)g(a).$$

So, in this case, the (3.14) inequality holds assuming that $x \mapsto \frac{g(x)}{x}$ is non-increasing.

Now, we can announce the following corollary of the Theorem 2.1.

Corollary 3.2. *Let $g : [0, +\infty) \to [0, +\infty)$ continuous non-decreasing function. Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function bounded from below, such that $\mathcal{D}(f) \neq \emptyset$. Then for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{3.17}$$

there exists a sequence $\{x_n\} \subset X$ which converges to some x_ε ($x_n \to x_\varepsilon$) such that

$$h(d(x_0, x_n))d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n} \; n \in \mathbb{N}. \tag{3.18}$$

then if $x \notin B(x_0, d(x_0, x_\varepsilon))$,

$$f(x) \geq f(x_\varepsilon) - \frac{\varepsilon}{\lambda(1 + g(d(x_0, x_\varepsilon)))} d(x, x_\varepsilon). \tag{3.19}$$

If $\frac{g(x)}{x}$ *is decreasing on* $(0, d(x_0, x_\varepsilon)]$ *then for all* $x \neq x_\varepsilon$

$$f(x) \geq f(x_\varepsilon) - \frac{\varepsilon}{\lambda(1 + g(d(x_0, x_\varepsilon)))} d(x, x_\varepsilon).$$

Remark 3.1. *If* g *is differentiable then we have* $\left(\frac{g(x)}{x}\right)' \leq 0$, *which means that* $g(x) \leq x$.

4 An extension of Caristi fixed point theorem

In this section we give an extension of Caristi fixed point theorem. In the sequel let $\xi = \sum_{n=0}^{\infty} \delta_n < \infty$, then we have the following:

Theorem 4.2. *Let* (X, d) *be a complete metric space, such that the function* ρ *is continuous. Let* $\varphi : X \to X$ *be an operator for which there exists a lower semi-continuous mapping* $f : X \to \mathbb{R}_+ \cup \{\infty\}$, *such that*

(i) $h(d(x_0, \varphi(x)))\rho(\varphi(x), y) - h(d(x_0, x))\rho(x, y) \leq \rho(x, \varphi(x))$,

(ii) $\xi\rho(u, \varphi(u)) \leq f(u) - f(\varphi(u))$.

Then φ *has at least one fixed point.*

Proof. We argue by contradiction. We assume that

$$\varphi(x) \neq x, \text{for all } x \in X. \tag{4.20}$$

Using Corollary 3.1 we have that for each $\varepsilon > 0$ there exists a δ_j sequence of positive real numbers and a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \to x_\varepsilon$ as $n \to \infty$, $x_\varepsilon \in X$ such that for every $x \in X$, $x \neq x_\varepsilon$ we have

$$f(x) + h(d(x_0, x)) \sum_{n=0}^{\infty} \delta_n \rho(x, x_n) > f(x_\varepsilon) + h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n). \tag{4.21}$$

In (4.21) we can put $x := \varphi(x_\varepsilon)$, because $\varphi(x_\varepsilon) \neq x_\varepsilon$. So, we get the following inequality:

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < h(d(x_0, \varphi(x_\varepsilon))) \sum_{n=0}^{\infty} \delta_n \rho(\varphi(x_\varepsilon), x_n) - h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n) \Leftrightarrow$$

$$f(x_\varepsilon)-f(\varphi(x_\varepsilon)) < \sum_{n=0}^{\infty}\delta_n\left[h(d(x_0,\varphi(x_\varepsilon)))\rho(\varphi(x_\varepsilon),x_n)-h(d(x_0,x_\varepsilon))\rho(x_\varepsilon,x_n)\right]. \tag{4.22}$$

Using (i), we get the following

$$f(x_\varepsilon)-f(\varphi(x_\varepsilon)) < \sum_{n=0}^{\infty}\delta_n\left[h(d(x_0,\varphi(x_\varepsilon)))\rho(\varphi(x_\varepsilon),x_n)-h(d(x_0,x_\varepsilon))\rho(x_\varepsilon,x_n)\right]$$

$$\leq \sum_{n=0}^{\infty}\delta_n\left[\rho(x_\varepsilon,\varphi(x_\varepsilon))\right] = \rho(x_\varepsilon,\varphi(x_\varepsilon))\sum_{n=0}^{\infty}\delta_n = \xi\rho(x_\varepsilon,\varphi(x_\varepsilon)). \tag{4.23}$$

If in (ii) we choose $u = x_\varepsilon$ we get the following inequality

$$\xi\rho(x_\varepsilon,\varphi(x_\varepsilon)) \leq f(x_\varepsilon) - f(\varphi(x_\varepsilon)). \tag{4.24}$$

From the (4.23) we have

$$f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \xi\rho(x_\varepsilon,\varphi(x_\varepsilon)). \tag{4.25}$$

If we compare the inequalities (4.25) and (4.24), we have that

$$\xi\rho(x_\varepsilon,\varphi(x_\varepsilon)) \leq f(x_\varepsilon) - f(\varphi(x_\varepsilon)) < \xi\rho(x_\varepsilon,\varphi(x_\varepsilon)),$$

which is a contradiction.

Thus, there exists $\tilde{x} \in X$ such that $\tilde{x} \in \varphi(\tilde{x})$. □

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Csaba Farkas,
Department of Mathematics,
Babeş-Bolyai University
1 Kogălniceanu Str., 400084, Cluj-Napoca, Romania. Email: farkas.csaba2008@gmail.com