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SOLVING THE TOPOGRAPHIC POTENTIAL BIAS AS AN INITIAL VALUE PROBLEM

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ABSTRACT. If the gravitational potential or the disturbing potential of the Earth be downward continued by harmonic continuation inside the Earth's topography, it will be biased, the bias being the difference between the downward continued fictitious, harmonic potential and the real potential inside the masses. We use initial value problem techniques to solve for the bias. First, the solution is derived for a constant topographic density, in which case the bias can be expressed by a very simple formula related with the topographic height above the computation point. Second, for an arbitrary density distribution the bias becomes an integral along the vertical from the computation point to the Earth's surface. No topographic masses, except those along the vertical through the computation point, affect the bias. (To be exact, only the direct and indirect effects of an arbitrarily small but finite volume of mass around the surface point along the radius must be considered.) This implies that the frequently computed terrain effect is not needed (except, possibly, for an arbitrarily small inner-zone around the computation point) for computing the geoid by the method of analytical continuation.

Keywords: analytical continuation, gravity, potential, topographic bias

1. INTRODUCTION

If we disregard or remove the effect of the atmosphere, the gravitational potential of the Earth is harmonic in the exterior of the Earth's surface. As this is not the case below the surface, the analytically continued exterior disturbing potential will be biased. Bjerhammar (1962) introduced a technique for quasigeoid determination with analytical continuation of the surface gravity anomaly to an internal sphere (the Bjerhammar sphere). Once the downward continued gravity anomaly on the sphere has been determined, he showed that any related classical formula that integrates the fictitious gravity anomaly on the internal sphere yields the correct function (e.g., gravity anomaly, gradient of gravity anomaly, height anomaly and deflections of the vertical) in the exterior of the Earth. (See also Heiskanen and Moritz 1967, Sect. 8-10 and Moritz 1980, Sect. 45) On the contrary, if the integral formula is applied with the computational point inside the topography, the result will be a fictitious or biased quantity. For example, if Stokes' formula is applied to determine the geoid height, the result will be biased. However, let us imagine that the disturbing potential (the gravity potential minus the normal potential) can be analytically continued to sea level, and that the topographic bias can be determined. Then, by subtracting the bias from the analytically continued potential, the true disturbing potential at sea level is obtained, and also the geoid height can be determined by applying Bruns' theorem to the potential. In this way it would be possible to determine the geoid in a very direct way by analytical continuation.

It is not proved that the downward continued disturbing potential T will exist in the strong sense within the topographic masses. Nevertheless, if we subtract the effect of a Bouguer shell (V^B) from T [and refer to the approximation theorems of Krarup-Runge (Krarup 1969; Moritz 180, Sect. 8) and Keldysh-Lavrentieff (Landkof 1972; Bjerhammar 1974)], we may expect that the reduced potential $T - V^B$ is harmonic and can be arbitrarily well approximated all the way down to the geoid. We will not be concerned with such approximations in this article. The interested reader can study, *e.g.* Moritz (1980, Sect. 45). Usually this problem is studied as that of continuing a finite series of spherical harmonics, representing the external potential of the Earth, analytically to sea level (e.g. Cook 1967, Sjöberg 1977 and Ågren 2004), but such a series will suffer not only from the topographic bias, but also from the truncation error of the series and, possibly, also from the divergence of the series when applied at sea level.

Our main concern in this study is the derivation of the topographic bias. Sjöberg (2007) performed such a study by decomposing the topographic potential into two parts (namely those generated by a Bouguer shell and the remaining topography, the terrain, respectively,) and compared the analytically continued potential with the true potential. This paper resulted in some critical remarks by Vermeer (2008), which were rather concerned with the analytical continuation error of the spherical harmonic series than with the topographic bias as pointed out by Sjöberg (2008). In two related studies by Sjöberg (2009a) and (2009b) the topographic bias was studied by employing digital terrain models as well as a Taylor series. In the present study the approach is different: we study the topographic bias as an initial value problem (IVP). An IVP is the problem of solving an ordinary differential equation related with one or several specific conditions (initial values). In the present case the differential equation is of second order, related with four conditions as specified in Eqs. (2)-(5) below. Related studies for the potential itself were performed by Holota (1994) and (1996).

Usually a function is defined as being harmonic, if it satisfies Laplace's equation in a domain, i.e. in a certain neighbourhood of a computation point. This condition, which may be difficult to prove/meet for the potential close to the boundary of topographic masses, is not important for our study, as our definition of topographic bias simply implies that the function does not satisfy Laplace's equation at the computation point.

In this study a spherical template will approximate the Earth's topography with arbitrary number and sizes of compartments. The IVP will be studied for a point inside the compartment (i.e. away from its boundary). The limiting case of the topographic bias (when the size of the compartment goes to zero) is studied in Sect. 2.3. The concept of the topographic bias is important in determining the geoid from the external gravity field by the method of analytical continuation.

2. FORMULATING AND SOLVING THE IVP

In this section we will first present some basic formulas in formulating the IVP, and then we proceed to solve the problem under different assumptions on the topographic density distribution.

2.1. BASIC FORMULAS

The topographic potential bias of the analytically continued disturbing potential T^* is defined by the formula

$$b = T^* - T, \tag{1}$$

where T^* and T are the analytically continued and the true disturbing potentials, respectively. A reasonable assumption on the Earth surface will be that it is star-shaped. Outside and at the Earth's surface the bias and its first radial derivative are both zero, and we can therefore introduce the following initial values to warrant continuity at the boundary:

$$b = 0 \quad , \text{ if } r = r_s \tag{2}$$

and

$$\frac{\partial b}{\partial r} = 0, \text{ if } \; r = r_s, \tag{3}$$

where r and r_s are the geocentric radii of the computation point and the Earth's surface along the radius vector through the computation point, respectively. Moreover, as the downward continuation (dwc) is based on harmonic functions, T^* is harmonic also after dwc inside the topographic masses, and it therefore obeys Laplace's equation there, i.e.

$$\Delta T^* = 0 \quad \text{for r} \geq R, \tag{4}$$

where Δ is the Laplace operator and R is the radius of sea level. On the other hand, within the masses the true disturbing potential T obeys Poisson's differential equation, so that

$$\Delta b = 4\pi\mu, \quad \text{if} \;\; R \leq r < r_s, \tag{5}$$

where μ is the product of the gravitational constant and density at the computation point.

Now, let us decompose the topographic potential V_P^t at the arbitrarily located computation point P (on or above sea level) into the Bouguer shell potential V_P^B and the terrain/ residual topographic potential V_P^{te}:

$$V_P^t = V_P^B + V_P^{te}. \tag{6}$$

Here we assume (without loss of generality) that the density distribution of the Bouguer shell is radial symmetric and determined by the radial density distribution at the computation point P. We also assume a spherical approximation of the shape of sea level with radius R, and the upper radius of the Bouguer shell as given by the radius c of topographic height at P, i.e. $c = r_S \;\; P$. Note that Eq. (6) decomposes the topographic potential into the contributions from the Bouguer shell V^B and the terrain V^{te}, and the sphere of radius c is the exterior surface of the Bouguer shell. In the computation of any derivative of V_P^t we will regard c as a constant (although this leads to an approximation in the solution of the IVP, which we will discuss in Sect. 2.3). This implies that

$$\Delta V_P^t = \Delta V_P^B = 0 \text{, if } r_P > c \text{,} \tag{7a}$$

and

$$\Delta V_P^t = \Delta V_P^B = -4\pi\rho(P) \text{, if } R < r_P < c \text{,} \tag{7b}$$

where Δ is the Laplace operator, r_P is geocentric radius of P and ρ is density. Eqs. (6), (7a) and (7b) yield that

$$\Delta V_P^{te} = \Delta V_P^t - \Delta V_P^B = 0 \text{, if } r_P > c \text{ and } R < r_P < c \text{,} \tag{8}$$

which, by our definition above, implies that the terrain does not contribute to the topographic bias, except possibly for point P located at the surface with radius c. As the second order derivatives of the potential generally do not exist at the boundary, Laplace's and Poisson's differential equations of Eqs. (7a) and (7b) are not applicable there (Kellog 1954, p. 153). However, this problem is easily circumvented by simply removing the effect dV_P^{te} of a small volume of mass of the terrain around the surface point, such that the reduced terrain potential $\tilde{V}_P^{te} = V_P^{te} - dV_P^{te}$ becomes harmonic along the radius vector, i.e.

$$\Delta \tilde{V}_P^{te} = 0 \quad \text{for } r > R \text{.} \tag{9}$$

This implies also that the disturbing potential T_P (to be downward continued) is reduced to $\tilde{T}_P = T_P - dT_P^{te}$, and the indirect effect dV_P^{te} must be added to the computed/downward continued potential in order not to introduce an additional bias. It is obvious that the smaller we make the (arbitrary) mass volume generating dV_P^{te}, the smaller will be the sum of the direct and indirect effects of this volume (but, admittedly, the more critical/difficult becomes the dwc of $\tilde{T}_P$). This is even more obvious, if we consider the approximation the exterior surface of this volume by a sloping plane centred at the Earth's surface. In case of a constant topographic density, the terrain effect of this innermost mass volume will be exactly zero, because the potential generated by the masses below the surface of the Bouguer shell (to be subtracted for) exactly compensates the potential of the positive mass contribution above the Bouguer shell. Hence, in practice there is no contribution to the topographic bias of T_P from the terrain.

All these derivations lead to the result that the topographic bias of Eq. (1) is a function of the mass of the Bouguer shell alone. Below we will determine the topographic bias of this potential in the cases of a constant topographic density (Sect. 2.2) and for a general density distribution (Sect. 2.3).

In the solutions that follow we will start from the following tentative general model for the topographic bias of the potential $\tilde{T}$ within the topographic masses:

$$b = \sum_{n=-\infty}^{\infty} b_n r^n \text{,} \quad R \le r \le r_S = c \text{,} \tag{10}$$

where the coefficients b_n should be consistent with Laplace's and Poisson's equations (4) and (5) and the initial values provided by Eqs. (2) and (3). (Note that the model is chosen under the assumption that the coefficients are constants. Cf. Sect. 2.3.) The Laplace operator working on Eq. (10) yields

$$\Delta b = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) \sum_{n=-\infty}^{\infty} b_n r^n = \sum_{n=-\infty}^{\infty} n(n+1) b_n r^{n-2} . \tag{11}$$

Hence, it remains to determine the coefficients b_n under the initial values of Eqs. (2)-(5) and by considering Eq. (11). We will consider both a constant and radially variable density distribution in the topography (Sects. 2.2-2.3 and 2.4, respectively).

2.2. SOLUTION FOR A CONSTANT TOPOGRAPHIC DENSITY

Let μ be constant within the topography, implying that the left hand side of Eq. (5) must be a constant for any *r* in the interval $R \le r < r_s$, i.e. the left hand-side must be independent of *r*. Hence, from Eq. (11) follows that only terms with b_{-1}, b_0 and b_2 of the series for *b* in Eq. (10) are different from zero. Thus one obtains

$$b = \frac{b_{-1}}{r} + b_0 + b_2 r^2 , \tag{12}$$

and from Eq. (5) follows also that $b_2 = 2\pi\mu/3$. Furthermore, by inserting Eq. (12) into Eqs. (2) and (3) we can also solve for b_{-1} and b_0, resulting in the following solution for the bias:

$$b = \frac{2\pi\mu}{3} \left(\frac{2r_s^3}{r} - 3r_s^2 + r^2 \right) , \ R \le r \le r_s \tag{13}$$

or, (Sjöberg 2007)

$$b = 2\pi\mu \left[\Delta H^2 + \frac{2}{3} \frac{\Delta H^3}{r} \right] \tag{14a}$$

where

$$\Delta H = \begin{cases} r_s - r & ,if \quad R \le r \le r_s \\ 0 & ,if \quad r > r_s \end{cases} . \tag{14b}$$

The topographic bias solved above is that for an Earth model, whose topography is a spherical template, and the solution is found for any point inside a compartment of the template. Although this would be the case for any numerical application, the question arises, what happens to the topographic bias in the limit, when the size of the compartment vanishes? The solution to this problem is discussed in the next subsection.

2.3. THE TOPOGRAPHIC BIAS IN THE LIMITING CASE

Let us now use the notations $\bar{b}$ and $\bar{r}_s$ for the topographic bias and surface radius of a compartment within the Earth template model. Then it holds that

$$\overline{b} = k\left(\frac{2\overline{r}_S^3}{r_P} - 3\overline{r}_S^2 + r_P^2\right), \tag{15}$$

where $k = 2\pi\mu/3$ and r_P is the radius of the computation point P with $R \le r_P \le c = r_s$ P . Denoting the real Earth radius by r_S, it follows directly from Eq. (15) that

$$\lim \overline{b} = k\left(\frac{2r_S^3}{r_P} - 3r_S^2 + r_P\right), \quad \text{as} \quad \overline{r}_S \to r_S . \tag{16}$$

Hence, $\overline{b}$ approaches smoothly the limiting case. Although, in this case the condition of Eq. (5) is violated, there is no other option for the topographic bias of the real Earth than that of Eq. (16).

2.4. SOLUTION FOR AN ARBITRARY TOPOGRAPHIC DENSITY DISTRIBUTION

We now generalize the topographic density distribution along the radial direction at the computation point to the following equation, consistent with Eq. (11):

$$\mu = \mu(r) = \sum_{n=-\infty}^{\infty} a_n r^{n-2} , \ R \le r < r_s , \tag{17}$$

where we assume that the series is absolutely convergent and the coefficients a_n govern the density distribution. Also in this case we assume that b can be represented by Eq. (10), and by inserting Eqs. (11) and (17) into Eq. (5) we arrive at

$$\sum_{n=-\infty}^{\infty} n(n+1) b_n r^{n-2} = 4\pi \sum_{n=-\infty}^{\infty} a_n r^{n-2} , \ R \le r < r_s , \tag{18}$$

where we can identify most of the coefficients b_n. The result is

$$b_n = \frac{4\pi a_n}{n(n+1)} , \text{ if } n \ne 0 \text{ and } n \ne -1 . \tag{19}$$

Hence, the solution can now be written

$$b = \frac{b_{-1}}{r} + b_0 + db , \tag{20a}$$

where the known part becomes

$$db = 4\pi \sum_{\substack{n=-\infty \\ n\ne -1,0}}^{\infty} \frac{a_n}{n(n+1)} r^n . \tag{20b}$$

In order to determine the remaining coefficients we reconsider Eqs. (2) and (3). From Eqs. (3), (20a) and (20b) we obtain

$$b_{-1} = 4\pi \sum_{\substack{n=-\infty \\ n\ne -1,0}}^{\infty} \frac{a_n}{n+1} r_s^{n+1} , \tag{21}$$

and Eq. (2) yields

$$b_0 = -4\pi \sum_{\substack{n=-\infty \\ n\neq -1,0}}^{\infty} \frac{a_n}{n} r_s^n \,. \tag{22}$$

The solution is thus provided by Eqs. (20a, b)-(22). It requires that the coefficients a_n be known. As we assumed that the series of Eq. (17) is absolutely convergent, the convergences of Eqs. (20b), (21) and (22) follow.

A more general form of the solution is provided by the following proposition.

Proposition 1: For $R \le r \le r_s$ the topographic potential bias can be written

$$b = 4\pi \int_r^{r_s} \mu(x) \left(\frac{x^2}{r} - x \right) dx \,. \tag{23}$$

Proof: Let us first introduce the abbreviation

$$k(x) = 4\pi\mu(x) \,. \tag{24}$$

We want to verify that the possible solution of the bias

$$b(r) = \int_r^{r_s} k(x) \left(\frac{x^2}{r} - x \right) dx \tag{25}$$

satisfies Eqs. (2), (3) and (5). Eq. (2) follows directly from Eq. (25). Moreover, from

$$b'(r) = \frac{db(r)}{dr} = -\int_r^{r_s} k(x) \frac{x^2}{r} dx \tag{26}$$

it follows also that Eq. (3) is satisfied. Finally, as

$$b''(r) = 2\int_r^{r_s} k(x) \frac{x^2}{r^3} dx + k(r) \,, \tag{27}$$

it follows that

$$\Delta b(r) = b''(r) + \frac{2}{r} b'(r) = k(r) \,, \tag{28}$$

and we have thus verified that the candidate solution b satisfies the proposition.

However, it remains to show that b is the one and only solution. For that reason we assume that there is also another solution

$$g(r) = b(r) + f(r) \tag{29}$$

for some function $f(r)$. As function g must satisfy Eqs. (2), (3) and (29), it follows that $f(r_s) = f'(r_s) = 0$, and f must also be harmonic in the interval $R \le r < r_s$. However, as a harmonic function attains it extreme values at the boundary, it follows that f vanishes in the whole interval.

The solution for f can also be obtained as follows: Let us assume that

$$f(r) = \sum_{n=-\infty}^{\infty} f_n r^n \,. \tag{30}$$

Then it holds that

$$\Delta f(r) = f''(r) + \frac{2}{r} f'(r) = \sum_{n=-\infty}^{\infty} n(n+1) f_n r^{n-2} \tag{31}$$

As f must satisfy Laplace's equation for any *r* in the interval, it follows that $f_n = 0$ for all *n* except for *n* = 0 and *n* = 2. Hence,

$$f(r) = f_0 + f_2 r^2 \tag{32}$$

and

$$f'(r) = 2 f_2 r \, . \tag{33}$$

Equating $f \;\; r_S$ and $f \;' \;\; r_S$ to zero yields that

$$f_0 = f_2 = f(r) = 0 \, , \tag{34}$$

which is exactly what we postulated.

Note. As with the solution discussed in Subsect. 2.2, the solution derived in this subsection holds strictly for the template model, while for the real Earth topography, we refer to the limiting case of Subsect.2.3.

3. CONCLUDING REMARKS

In the case of a constant topographic density we have shown that the topographic potential bias can be expressed by the simple formula of Eq. (13). For an arbitrary density distribution the formula is also simple, being a one-dimensional integral along the vertical through the computation point. Ågren (2004, Sect. 8.4.3) compared Eq. (13) with Rapp's (1997) approach to correct a series representation in external type spherical harmonics of the geoid height, and he concluded that the two techniques are practically identical.

By adding the topographic potential bias to the downward continued disturbing potential at sea level one can easily use Bruns' formula to determine the geoid height. The downward continuation of the disturbing potential can be performed by gravimetric data alone (e.g., Sjöberg 2003a). Hence, from a theoretical point of view, by applying the technique of analytical continuation the frequently applied terrain effect is not needed to determine the geoid height. (Admittedly, we have removed and restored the effect of an arbitrarily small mass of the terrain at the surface point.) On the other hand, the terrain effect can be numerically efficient to smooth the observed gravity anomaly or disturbing potential to allow better interpolation and Stokes' integration as well and in particular to allow a practical solution to the analytical downward continuation problem. However, the removal and restoration (r-r) of the terrain effect must therefore not necessarily be carried out with theoretical rigour, but only in such a way that the r-r technique be consistent.)

Finally, it should be stated that this article dealt with the topographic bias of the downward continued disturbing potential, and we did not dwell upon the common (Molodensky) problem to compute the disturbing potential at the surface from gravity anomalies. A scheme to carry out all these steps is given by the KTH (acronym for Royal Institute of Technology) method for geoid determination, presented, e.g., in Sjöberg (2003b), and the origin of the present article was actually to determine the potential bias of this method with clarity.

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