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SPHERICAL HARMONICS EXPANSION OF THE ATMOSPHERIC GRAVITATIONAL POTENTIAL BASED ON EXPONENTIAL AND POWER MODELS OF ATMOSPHERE

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ABSTRACT. Spherical harmonic formulation of gravitational potential of the atmosphere depends on the analytical model of the atmospheric density which is used. Exponential and power models are two well-known mathematical tools which are used in atmospheric applications. This paper presents simple formulas for the harmonic coefficients of internal and external types of the atmospheric potential based on these models which can be used in most of the gravimetric aspects. It considers the atmospheric effect on the satellite gravity gradiometry data as an example for numerical investigations. The numerical studies on these data show that the maximum atmospheric effect is about 2 mE over Fennoscandia based on both models, and their differences are less than 0.1 mE. The difference between indirect atmospheric effects reaches 2 cm and 0.02 mGal on the geoid and gravity anomaly, respectively in this region.

Keywords: direct and indirect atmospheric effect, geoid, gravity anomaly, gravity gradient, atmospheric mass density model, standard atmosphere

1. INTRODUCTION

The atmospheric masses are important in gravity field investigations as they affect the terrestrial, airborne and spaceborne measurement of the gravity field. Mathematical model of the atmospheric density is of importance in approximating the gravitational potential of the atmosphere. Having assumed the atmospheric density not to change laterally, many efforts were made to formulate the atmospheric density by different geodesists. Ecker and Mittermayer (1969) used an exponential model to generate the atmospheric potential. They used the density data of the United Standard Committee (1961) in their investigations on physical geodesy and geoid. Anderson and Mather (1975) considered a linear model for the atmospheric density in their studies on the sea surface topography. In 1976 a new atmospheric density model denoted as the United States Standard Atmosphere (1976) (USSA 1976) was proposed by NOAA (National Oceanic and Atmospheric Administration), NASA (National Aeronautics and Space Administration) and USAF (United State Air Force). Novák (2000) fitted a second-order polynomial to this model and used the polynomial to consider the atmospheric effect in his geoid determination process. The polynomial proposed by Novák is valid until 10 km heights as he considered this model for formulating the atmospheric topography (atmosphere between the Earth's surface and a geocentric sphere encompassing all solid masses). Novák and Grafarend (2006) and Tenzer et al. (2006) used this simple polynomial as a density model for computing the atmospheric effect on spaceborne data and gravity anomaly, respectively. Eshagh and Sjöberg (2009) investigated further this polynomial and proposed a combination of Novák's polynomial and a power model as a density model. This power model was originally proposed by Sjöberg (1993) and used by Sjöberg (1998, 1999, 2001 and 2006), Nahavandchi (2004) and Sjöberg and Nahavandchi (1999 and 2000).

As the above review shows, it seems that only Ecker and Mittermayer (1969) have used exponential model for approximating the atmospheric density. Since this model has not been used very often due to complicated derivation process, it has not further been developed and applied. In this paper an attempt is made to formulate and develop the atmospheric potential based on this model. The model is investigated to see how the mathematical derivations to express the atmospheric potential in spherical harmonics are complicated, and if it is possible to simplify the derivations further. In the present paper, simple formulas for the harmonic coefficients of the spherical harmonic expansion of atmospheric potentials are presented. The formulas for harmonics are not as complicated as it may seem facing the problem. The power model of the atmosphere is also considered in the paper. The spectrums of the atmospheric potential based on the exponential and power models can easily be used in any gravimetric application for computing the direct and indirect effect of the atmospheric masses.

There are few analytical models for estimating the atmospheric density at different levels. The standard atmospheric model presents the atmospheric density layer by layer which is not suitable to formulate the gravitational potential of the atmospheric topography. In order to solve this problem Novák (2000) considered a second-order polynomial for those parts of the atmosphere which is below 10 km level. Those parts above 10 km should be considered layer by layer. Eshagh and Sjöberg (2009) presented another model which fits the atmospheric densities below and above 10 km and this model is fully-analytical and there is no need to consider the atmospheric masses layer by layer. A mathematical derivation of the atmospheric potential based on this model is a little bit complicated and long and this is the price that we have to pay to have more precise model. However, this paper concentrates on the exponential and power atmospheric density models. These models are two well-known and simple models in comparison with other published models and they are good enough for approximating the atmospheric potential in most geodetic applications.

In Eshagh and Sjöberg (2009) studies, the zero-degree harmonic of the atmospheric potential based on the power model, can be unbounded when the upper limit of the atmosphere increases. However, the atmospheric potential was not well-formulated (based on the power model) there and we will discuss it in Section 4. The removed atmospheric effects should be restored after computations and/or downward continuation of the geodetic data and the internal type of the formulation is obligatory to restore the effects. The indirect effect is considered on the geoid and gravity anomaly in Fennoscandia. Temporal variations of the atmospheric density are beyond the scope of this paper, but such variations will be a necessary part of data processing of GOCE as it was for GRACE as well [Flechtner et al. 2006]. Since the derivations differ with this paper it is left for future works. The static atmospheric is also beneficial and we can derive a simple mathematical formula for its density and it is good enough for geodetic applications to approximate the atmospheric effects. At least it is the first approximation of the true atmospheric density.

In the next section the atmospheric potential is presented in spherical harmonics. In Section 3 spherical harmonics coefficients of the external and internal type of the atmospheric potential are formulated in Propositions 1 and 2, respectively. Propositions 3 and 4 presented in Section 4 express the harmonics of the atmospheric potentials based on power model. In

Section 5 derived formulas are discussed and the results based on their application over a test region (in Fennoscandia) are presented. The paper is ended by some conclusions in Section 6.

2. ATMOSPHERIC GRAVITATIONAL POTENTIALS IN SPHERICAL HARMONICS

The spherical harmonic approximation of the atmospheric potential depends on the position of the computation point P. It can be formulated by external and internal series of the spherical harmonics by:

$$V_{ext}^{a}\left(P\right)=\frac{GM}{R}\sum_{n=0}^{\infty}\left(\frac{R}{r_P}\right)^{n+1}\sum_{m=-n}^{n}\left(v_{ext}^{a}\right)_{nm}Y_{nm}\left(P\right),\text{ where } r_P\geq R \tag{1}$$

and

$$V_{int}^{a}\left(P\right)=\frac{GM}{R}\sum_{n=0}^{\infty}\left(\frac{r_P}{R}\right)^{n}\sum_{m=-n}^{n}\left(v_{int}^{a}\right)_{nm}Y_{nm}\left(P\right),\text{ where } r_P<R \tag{2}$$

where GM is the geocentric gravitational constant, R stands for the mean radius of the Earth, r_P is the geocentric radius of any point P with angular spherical coordinates (spherical co-latitude and longitude), $Y_{nm}\left(P\right)$ is the fully-normalized spherical harmonics with the following property of their orthogonality [Heiskanen and Moritz, 1967]:

$$\iint_{\sigma}Y_{nm}\left(Q\right)Y_{n'm'}\left(Q\right)d\sigma=4\pi\delta_{nn'}\delta_{mm'} \tag{3}$$

$\left(v_{ext}^{a}\right)_{nm}$ and $\left(v_{int}^{a}\right)_{nm}$ are the spherical harmonics coefficients of the external and internal types of the atmospheric potential, respectively and δ is the Kronecker delta.

The present paper concentrates on formulation of the spherical harmonic coefficients for the exponential and power atmospheric mass density models. Let us consider the following formulas which are frequently used throughout this paper. First the well-known Newtonian volume integral is presented for the potential at the point P.

$$V^{a}\left(P\right)=G\iint_{\sigma}\int_{r_S}^{r_Z}\frac{\rho\left(r_Q\right)r_Q^2dr_Q}{l_{PQ}}d\sigma, \tag{4}$$

where G is the Newtonian gravitational constant, l_{PQ} is the distance between any points P and Q, $r_S\approx R+H$ is the geocentric radius of the Earth surfaces (H is orthometric height of the terrain), $r_Z\approx R+Z$ is the radius of the upper bound of the atmosphere (Z is a constant). $\rho\left(r_Q\right)$ is the atmospheric density at geocentric radius of point Q and σ is the unit sphere of integration, $d\sigma$ is the integration element. Here the Earth is assumed as a ball. If a best fitted ellipsoid for the Earth is considered, the ellipsoidal layering for the atmosphere must be considered as well, otherwise a gap between the ellipsoid and spherical layered atmosphere is happened. The ellipsoidal layering of the atmosphere was well-investigated by Sjöberg (2006) and he found that the ellipsoidal correction to geoid varies between 0.3 cm and 4.0 cm from the equator to the poles. However at satellite level the ellipsoidal layering atmosphere is not important.

The inverse distance l_{PQ} can be expanded in external and internal series of Legendre polynomial as:

$$\frac{1}{l_{PQ}} = \frac{1}{r_P}\sum_{n=0}^{\infty}\left(\frac{r_Q}{r_P}\right)^n P_n\left(\cos\psi_{PQ}\right), \qquad \text{if } r_P > r_Q \tag{5}$$

$$\frac{1}{l_{PQ}} = \frac{1}{r_Q}\sum_{n=0}^{\infty}\left(\frac{r_P}{r_Q}\right)^n P_n\left(\cos\psi_{PQ}\right). \qquad \text{if } r_Q > r_P \tag{6}$$

where $P_n(\cos\psi_{PQ})$ is the Legendre polynomial of degree n and ψ_{PQ} stands for the geocentric angle between points P and Q. Another useful formula is the addition theorem of the fully-normalized spherical harmonics [Heiskanen and Moritz, 1967]:

$$P_n\left(\cos\psi_{PQ}\right) = \frac{1}{(2n+1)}\sum_{m=-n}^{n} Y_{nm}(Q)Y_{nm}(P). \tag{7}$$

3. HARMONICS BASED ON EXPONENTIAL ATMOSPHERIC DENSITY MODEL

An exponential function for the atmospheric density can be considered as [Lambeck, 1988]:

$$\rho(r) = \rho_0 e^{-\alpha(r-R)} = \rho_0 e^{\alpha R} e^{-\alpha r} \tag{8}$$

where $\rho_0 = 1.2227 \ \ kg/m^3$ is the atmospheric density at sea level, R is the mean radius of the Earth and r is the radial distance of any point inside the atmosphere, α is a constant which will be estimated in Section 5. The mathematical deviations are presented in Propositions 1 and 2.

Proposition 1. The spherical harmonic coefficients of external atmospheric potential based on the exponential atmospheric density model are

$$\left(v_{ext}^a\right)_{nm} \approx \frac{3\rho_0}{(2n+1)\rho_e}\left\{\left\{\left[1-\left(1+\frac{Z}{R}\right)e^{-\alpha Z}\right]\left(\frac{2+\alpha(R+Z)}{\alpha^2R^2}\right)+\frac{2\left(1-e^{-\alpha Z}\right)}{\alpha^3R^3}-\frac{Z}{R^2\alpha}\right\}\delta_{n0} - \right.$$

$$\left. -\left[\frac{H_{nm}}{R}+\frac{n+2-\alpha R}{2R^2}H^2{}_{nm}+\frac{(n+2)(n+1-2\alpha R)+\alpha^2R^2}{6R^3}H^3{}_{nm}\right]\right\}. \tag{9}$$

Proof. Equation (10) is obtained from Eq. (4) by substituting from Eqs. (5) and (8)

$$V_{ext}^a(P) = G\rho_0 e^{\alpha R}\sum_{n=0}^{\infty}\frac{1}{r_P^{n+1}}\iint_\sigma\int_{r_S}^{r_Z} e^{-\alpha r_Q} r_Q^{n+2} dr_Q P_n\left(\cos\psi_{PQ}\right)d\sigma. \tag{10}$$

The radial integral can be separated into

$$\int_{r_S}^{r_Z} e^{-\alpha r_Q} r_Q^{n+2} dr_Q = \int_R^{R+Z} e^{-\alpha r_Q} r_Q^{n+2} dr_Q - \int_R^{R+H} e^{-\alpha r_Q} r_Q^{n+2} dr_Q = I_n - \int_R^{R+H} e^{-\alpha r_Q} r_Q^{n+2} dr_Q. \tag{11}$$

Instead of topographic masses it is assumed there are atmospheric masses between sea and the Earth surfaces which are subtracted from the atmospheric shell potential. The solution of the first integrals is assumed I_n and we will see that we just need I_0 in our derivations and there is no need to perform partial integration to solve the integral. The exponential function in the second term of Eq. (11) is expanded by Taylor series and the integration performs:

$$\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\int_{R}^{R+H} r_Q^{n+k+2}dr_Q=\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!(n+k+3)}r_Q^{n+k+3}\Big|_{R}^{R+H}=\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^{n+k+3}}{k!(n+k+3)}\left[\left(1+\frac{H}{R}\right)^{n+k+3}-1\right]. \quad (12)$$

If we use Taylor expansion up to third order for the first term in the squared bracket of Eq. (12) we obtain

$$\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\int_{R}^{R+H_Q} r_Q^{n+k+2}dr_Q\approx\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^{n+k+3}}{k!}\left[\frac{H}{R}+\frac{n+k+2}{2}\left(\frac{H}{R}\right)^2+\frac{(n+k+2)(n+k+1)}{6}\left(\frac{H}{R}\right)^3\right]. \quad (13)$$

The error due to truncation of the Taylor series was investigated by Sun and Sjöberg (2001). They concluded when the maximum degree of the spherical harmonic expansion is 360 and the truncation of the binomial expansion to third term is less than 1% and small, if the highest elevation in the world is considered (9 km).

Inserting Eq. (13) into Eq. (11) substituting the result into Eq. (10) we derive

$$V_{ext}^a(P)\approx G\rho_0 e^{\alpha R}\sum_{n=0}^{\infty}\frac{1}{r_P^{n+1}}\iint_{\sigma}\left\{I_n-\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^{n+k+3}}{k!}\left[\frac{H}{R}+\frac{n+k+2}{2R^2}H^2+\right.\right.$$

$$\left.\left.+\frac{(n+k+2)(n+k+1)}{6R^3}H^3\right]P_n(\cos\psi_{PQ})d\sigma\right. . \quad (14)$$

Using Eq. (7) which is the addition theorem of the fully-normalized spherical harmonics, Eq. (14) can further be simplified

$$V_{ext}^a(P)=4\pi G\rho_0 e^{\alpha R}R^2\sum_{n=0}^{\infty}\left(\frac{R}{r_P}\right)^{n+1}\frac{1}{(2n+1)}\sum_{m=-n}^{n}\left\{\frac{I_n\delta_{n0}}{R^{n+3}}-\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}\left[\frac{H_{nm}}{R}+\frac{n+k+2}{2R^2}H^2{}_{nm}+\right.\right.$$

$$\left.\left.+\frac{(n+k+2)(n+k+1)}{6R^3}H^3{}_{nm}\right]\right\}Y_{nm}(P), \quad (15)$$

where H_{nm}, $H^2{}_{nm}$ and $H^3{}_{nm}$ are spherical harmonic coefficients H, H^2 and H^2, respectively, which can be derived by using the simple global spherical harmonic analysis. Considering $GM=4\pi GR^3\rho_e/3$ (ρ_e=5500 kg/m^3 is the mean density of the Earth) and after further simplifications we obtain

$$\left(v_{ext}^a\right)_{nm}\approx\frac{3\rho_0}{(2n+1)\rho_e}\left\{\frac{I_n e^{\alpha R}}{R^{n+3}}\delta_{n0}-e^{\alpha R}\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}\left[\frac{H_{nm}}{R}+\frac{n+k+2}{2R^2}H^2{}_{nm}+\right.\right.$$

$$\left.\left.+\frac{(n+k+2)(n+k+1)}{6R^3}H^3{}_{nm}\right]\right\}=\frac{3\rho_0}{(2n+1)\rho_e}\{A\delta_{n0}-B\}. \quad (16)$$

The first term in the bracket consists of the solution of the first radial integral in Eq. (11) and we have (a proof is given in Appendix A)

$$A=\frac{I_0 e^{\alpha R}}{R^3}=\left\{\left[1-\left(1+\frac{Z}{R}\right)e^{-\alpha Z}\right]\left[\frac{2+\alpha(R+Z)}{\alpha^2R^2}\right]+\frac{2\left(1-e^{-\alpha Z}\right)}{\alpha^3R^3}-\frac{Z}{R^2\alpha}\right\}. \quad (17)$$

The second term can be simplified to (see appendix A for a proof)

$$B=\left\{\frac{H_{nm}}{R}+\frac{n+2-\alpha R}{2R^2}H^2{}_{nm}+\frac{(n+2)(n+1-2\alpha R)+\alpha^2R^2}{6R^3}H^3{}_{nm}\right\}. \tag{18}$$

Inserting Eqs. (18) and (17) into Eq. (16) the proposition is proved.

Proposition 2. The harmonics of internal atmospheric potential based on the exponential atmospheric density model are

$$\left(v_{int}^{a}\right)_{nm}\approx\frac{3\rho_0}{(2n+1)\rho_e}\left\{\left\{\frac{1}{\alpha R}\left[1-\left(1+\frac{Z}{R}\right)e^{-\alpha Z}\right]+\frac{\left(1-e^{-\alpha Z}\right)}{\alpha^2R^2}\right\}\delta_{n0}-\left[\frac{H_{nm}}{R}+\frac{-n+1-\alpha R}{2R^2}H^2{}_{nm}+\right.\right.$$

$$\left.\left.+\frac{(n-1)(n+2\alpha R)+\alpha^2R^2}{6R^3}H^3{}_{nm}\right]\right\}. \tag{19}$$

Proof. Considering Eqs. (4) and (6) as the internal type of the Legendre expansion for $1/l_{PQ}$ we can write

$$V_{int}^{a}(P)=G\rho_0e^{\alpha R}\iint_{\sigma}\sum_{n=0}^{\infty}r_P^n\int_{r_S}^{r_Z}e^{-\alpha r_Q}r_Q^{-n+1}dr_QP_n\left(\cos\psi_{PQ}\right)d\sigma\ . \tag{20}$$

Similar to Proposition 1 we can divide the radial integral into two parts; see Eq. (11). Having considered the solution of the first integral by $J_n=\int_R^{R+Z}e^{-\alpha r_Q}r_Q^{-n+1}dr_Q$, the exponential function in the second radial integral is expanded into the Taylor series and integration performs as:

$$\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\int_R^{R+H}r_Q^{k-n+1}dr_Q=\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!(k-n+2)}r_Q^{k-n+2}\Bigg|_R^{R+H}=\sum_{k=0}^{\infty}\frac{(-\alpha)^kR^{k-n+2}}{k!(k-n+2)}\left[\left(1+\frac{H}{R}\right)^{n+k+3}-1\right]. \tag{21}$$

After expanding the first term in the square bracket in a binomial series up to third order we derive

$$\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\int_R^{R+H}r_Q^{k-n+1}dr_Q\approx\sum_{k=0}^{\infty}\frac{(-\alpha)^kR^{k-n+2}}{k!}\left[\frac{H}{R}+\frac{k-n+1}{2}\left(\frac{H}{R}\right)^2+\frac{(k-n+1)(k-n)}{6}\left(\frac{H}{R}\right)^3\right]. \tag{22}$$

Inserting Eq. (22) into Eq. (20) and using Eq. (6) the internal potential is

$$V_{int}^{a}(P)\approx4\pi GR^2\rho_0e^{\alpha R}\sum_{n=0}^{\infty}\left(\frac{r_P}{R}\right)^n\frac{1}{(2n+1)}\sum_{m=-n}^{n}\left\{\frac{J_n\delta_{n0}}{R^{-n+2}}-\sum_{k=0}^{\infty}\frac{(-\alpha)^kR^k}{k!}\left[\frac{H_{nm}}{R}+\frac{k-n+1}{2R^2}H^2{}_{nm}+\right.\right.$$

$$\left.\left.+\frac{(k-n+1)(k-n)}{6R^3}H^3{}_{nm}\right]\right\}Y_{nm}(P), \tag{23}$$

considering $GM=4\pi GR^3\rho_e/3$ and further simplifications we obtain

$$\left(V_{\text{int}}^{a}\right)_{nm} \approx \frac{3\rho_0}{(2n+1)\rho_e}\left\{\frac{J_n e^{\alpha R}}{R^{-n+2}}\delta_{n0} - e^{\alpha R}\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}\left[\frac{H_{nm}}{R} + \frac{k-n+1}{2R^2}H^2{}_{nm} + \right.\right.$$

$$\left.\left. + \frac{(k-n+1)(k-n)}{6R^3}H^3{}_{nm}\right]\right\} = \frac{3\rho_0}{(2n+1)\rho_e}\left\{C\delta_{n0} - D\right\} . \qquad (24)$$

The first term in the bracket is

$$C = \frac{J_0 e^{\alpha R}}{R^2} = \frac{1}{\alpha R}\left[1 - \left(1 + \frac{Z}{R}\right)e^{-\alpha Z}\right] + \frac{\left(1 - e^{-\alpha Z}\right)}{\alpha^2 R^2} \qquad (25)$$

and similar to the proof of Proposition 1 and Appendix A the second term is simplified as

$$D = \left\{\frac{H_{nm}}{R} + \frac{-n+1-\alpha R}{2R^2}H^2{}_{nm} + \frac{(n-1)(n+2\alpha R) + \alpha^2 R^2}{2R^2}H^3{}_{nm}\right\}. \qquad (26)$$

Substituting Eq. (26) and Eq. (25) into Eq. (24) and after further simplifications the proposition is proved.

4. HARMONICS BASED ON POWER ATMOSPHERIC DENSITY MODEL

The power model has the following mathematical expression:

$$\rho(r) = \rho_0\left(\frac{R}{r}\right)^{\nu} . \qquad \text{where} \qquad R \le r \qquad (27)$$

All the parameters were introduced in previous section.

The constant ν in the power model can be estimated by a simple fitting to the logarithm of the atmospheric density. Sjöberg's fitting was based on the USSA1961 [Reference Atmosphere Committee, 1961], this model was updated based on the US standard atmospheric model (1976) by Eshagh and Sjöberg (2009); in the former and latter cases the exponent ν was derived 850 and 890, respectively. They used different strategy that is proposed in the current paper to generate the atmospheric potential as they considered the radial integral limits in Eq. (4) directly from surface of the terrain to the upper bound of the atmosphere. This assumption will lead to a very large and unrealistic value for the zero-degree harmonic coefficient. They formulated this harmonic separately to overcome this problem. However, if instead of topographic masses, it is assumed the atmospheric masses between sea and the Earth surfaces are subtracted from the atmospheric shell (the shell between sea surface and upper bound of the atmosphere) potential. The derivation of the potential includes all degrees without problem. In the following we derive the spherical harmonics coefficients of the external and internal types of the atmospheric potential by Propositions 3 and 4, respectively.

Proposition 3. The spherical harmonic coefficients of the external atmospheric potential based on power density model are:

$$\left(V_{\text{ext}}^{a}\right)_{nm} \approx \frac{3\rho_0}{(2n+1)\rho_e}\left\{\left[\left(1+\frac{Z}{R}\right)^{3-\nu} - 1\right]\frac{\delta_{n0}}{3-\nu} - \frac{H_{nm}}{R} - \frac{n+2-\nu}{2R^2}H^2{}_{nm} - \frac{(n+2-\nu)(n+1-\nu)}{6R^3}H^3{}_{nm}\right\}. \qquad (28)$$

Proof. Considering Eqs. (4) and (5) we can write the external potential as

$$V_{ext}^{a}(P)=G\rho_0\sum_{n=0}^{\infty}\iint_{\sigma}\frac{R^{\nu}}{r_P^{n+1}}\int_{r_S}^{r_Z} r_Q^{n+2-\nu}dr_Q P_n\left(\cos\psi_{PQ}\right)d\sigma . \tag{29}$$

The radial integral can be separated into two parts

$$\int_{r_S}^{r_Z} r_Q^{n+2-\nu}dr_Q=\int_{R}^{R+Z} r_Q^{n+2-\nu}dr_Q-\int_{R}^{R+H} r_Q^{n+2-\nu}dr_Q . \tag{30}$$

Solution of the above integral is

$$\int_{r_S}^{r_Z} r_Q^{n+2-\nu}dr_Q\approx\frac{R^{n+3-\nu}}{n+3-\nu}\left[\left(1+\frac{Z}{R}\right)^{n+3-\nu}-1\right]-R^{n+3-\nu}\left[\frac{H}{R}+\frac{n+2-\nu}{2R^2}H^2+\frac{(n-\nu+2)(n-\nu+1)}{6R^3}H^3\right] \tag{31}$$

It should be emphasized that Eq. (31) is an approximate expression obtained after a truncation of the Taylor series. Equation (31) is substituted into Eq. (29) and we have

$$V_{ext}^{a}(P)\approx G\rho_0R^2\sum_{n=0}^{\infty}\left(\frac{R}{r_P}\right)^{n+1}\iint_{\sigma}\left\{\left[\left(1+\frac{Z}{R}\right)^{n+3-\nu}-1\right]\frac{1}{n+3-\nu}-\left[\frac{H}{R}+\frac{n+2-\nu}{2R^2}H^2+\right.\right.$$
$$\left.\left.+\frac{(n-\nu+2)(n-\nu+1)}{6R^3}H^3\right]\right\}P_n\left(\cos\psi_{PQ}\right)d\sigma . \tag{32}$$

By using Eq. (7) and further simplifications the proposition is proven.

Proposition 4. The harmonics of the internal atmospheric potential based on the power density model are:

$$\left(V_{int}^{a}\right)_{nm}\approx\frac{3\rho_0}{(2n+1)\rho_e}\left\{\left[1-\left(1+\frac{Z}{R}\right)^{2-\nu}\right]\frac{\delta_{n0}}{\nu-2}-\frac{H_{nm}}{R}+\frac{n+\nu-1}{2R^2}H^2{}_{nm}-\frac{(n+\nu-1)(n+\nu)}{6R^3}H^3{}_{nm}\right\} \tag{33}$$

Proof. According to Eqs. (4) and (6) and also Eq. (27) the internal type of the atmospheric potential is obtained:

$$V_{int}^{a}(P)=G\rho_0\sum_{n=0}^{\infty}R^{\nu}r_P^{n}\iint_{\sigma}\int_{R+H}^{R+Z} r_Q^{1-\nu-n}dr_Q P_n\left(\cos\psi_{PQ}\right)d\sigma\ . \tag{34}$$

Similar to Eq. (30) we can divide the radial integral into two parts. Solution of the radial integral is

$$\int_{R+H}^{R+Z} r_Q^{1-\nu-n}dr_Q\approx\frac{R^{2-\nu-n}}{2-\nu-n}\left[\left(1+\frac{Z}{R}\right)^{2-\nu-n}-1\right]-R^{2-\nu-n}\left[\frac{H}{R}+\frac{1-\nu-n}{2R^2}H^2-\frac{(1-\nu-n)(-\nu-n)}{6R^3}H^3\right]. \tag{35}$$

It should be mentioned again that this is an approximate expressions obtained after truncation of the Taylor expansion.

Inserting Eqs. (35) into Eq. (34) we derive

$$V_{int}^{a}(P) \approx G\rho_0 R^2 \sum_{n=0}^{\infty}\left(\frac{r_P}{R}\right)^n \iint_{\sigma}\left\{\left[\left(1+\frac{Z}{R}\right)^{2-\nu-n}-1\right]\frac{1}{2-\nu-n}-\frac{H}{R}-\frac{1-\nu-n}{2R^2}H^2+\right.$$

$$\left.-\frac{(1-\nu-n)(-\nu-n)}{6R^3}H^3\right\}P_n\left(\cos\psi_{PQ}\right)d\sigma\,. \tag{36}$$

By considering Eq. (7) and simplification the proposition is proved.

5. DISCUSSIONS AND NUMERICAL INVESTIGATIONS

Comparing Propositions 1 and 3, the main difference between these harmonic coefficients is related to the first terms in the brackets, which are the potential of the atmospheric shells from sea surface to an arbitrary point P outside the atmosphere. In other words, an imaginary atmosphere which is obtained by replacing the terrain, down to the sea level, with air is considered. The values of these terms are 1.132×10^{-3} and 1.127×10^{-3} derived according to the exponential and power models, respectively. In the following it is explained how to estimate α in the exponential model. The second terms in Propositions 1 and 3 are related to the atmospheric topography.

For more interpretations the formulas are numerically compared. First α is estimated in Eq. (8). In this case the United State Standard atmospheric model (United States Atmosphere, 1976) is considered as the true model. The atmospheric density is generated based on this model up to 86 km (maximum elevation for the atmospheric density in this model) and by fitting a line into the logarithm of the atmospheric density versus elevation 1.3886×10^{-4} is estimated for α. Also by similar fitting 890 is estimated for ν in Eq. (27). The exponential and power models are equations of two nearly coinciding lines in logarithmic scale of the atmospheric density; see Figure 1.

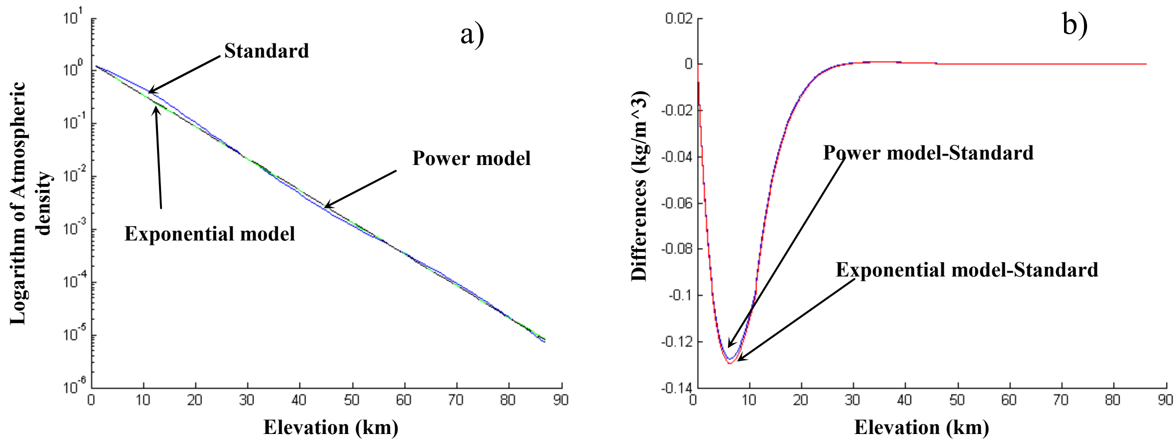

Fig 1. (a) fitting to the atmospheric density of the standard model, (b) difference between power and exponential models and standard model

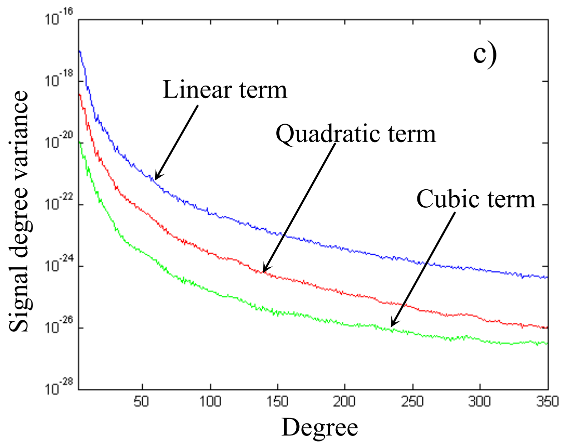

Fig. 2. linear, quadratic and cubic terms of spherical harmonics coefficients of Proposition 1

The atmospheric density which is estimated by the standard model is not linear versus elevation, and the power and exponential models both underestimate the most massive part of the atmospheric density below 10 km height see Fig.1(a). It is also possible to fit these models this massive part (10 km) of the atmosphere but in this case they overestimate the densities above this level. However the presented formula does not change except the values of ν and α in the models. In such a case, the models will overestimate the densities above this level which are not so massive.

Fig. 1(b) shows the differences between the density estimated by the power and exponential models with the standard model. It shows the exponential model is slightly better than the power model as their root mean square error of the fitting is 0.0374 while it is 0.0380 for the power model. Figure 2 illustrates the signal degree variance of the atmospheric potential considering linear, quadratic and cubit terms of Proposition 1, respectively to show the convergence of the binomial expansion which was used in approximating the atmospheric potential. Now the topographic terms of propositions 1 and 3 are compared. Comparison of the topographic terms is useful to clarify the difference of the atmospheric potential and how the exponential and power models are related to each other. The first terms are exactly the same and the second topographic terms are very similar. The value of αR is equal to 886 (R=6378137 m) and very close to 890 for ν thus the second terms are approximately the same. The coefficients of the third topographic term is $\alpha^2 R^2 + (n+2)(n+1-2\alpha R)$ which corresponds to

$$\nu^2 + (n+2)\left[n+1-\nu\left(\frac{n+1}{n+2}+1\right)\right] \tag{37}$$

in Proposition 3 (after simplification). In the proposition if n=0 then Eq. (37) changes to $\nu^2 + (n+2)(n+1-1.5\nu)$ and when n increases it goes to 2. Therefore in high degrees the harmonic coefficients of external potential of the atmosphere bases on both models are more or less the same as $\alpha R \approx \nu$. Similar comparison can be made for the harmonics of the internal type.

Some conclusions can be made from the presented formulas which are briefly summarized as follow:

-There is no exponential function (e) in topographic terms of the formulated spherical harmonic coefficients based on the exponential model but the first term (which contributes the zero-degree harmonic) is involved with $e^{-\alpha Z}$.

-No partial integration is needed for generating the potential of the atmosphere based in the exponential model. The constant ν in power model is approximately equal to αR of exponential model. This implies the attenuation factor (R/r) in power model corresponds to $\exp(1 - r/R)$ in the exponential model.

-The difference in the third topographic terms is related to the coefficients $2\alpha R$ and $[(n+1)/(n+2)+1]\,\nu$ but when n is equal to 0 the latter coefficient is equal to 1.5 and when it increases the coefficient goes to 2. However, this difference is negligible as both coefficients are divided by $6R^3$.

In order to compare the models practically, the atmospheric effect on the satellite gradiometry data are considered. A satellite orbit at 250 km altitude (altitude of GOCE) is generated by 4-th order Runge-Kutta algorithm for two month revolutions at sampling rate of 30 seconds. The harmonics coefficients of the height function H and its powers H_{nm}, H^2_{nm} and H^3_{nm} are generated through the global spherical harmonic synthesis and analysis. The SRTM global topographic height model [Wieczorek 2007] with resolution $0.5^\circ \times 0.5^\circ$ corresponding to degree of 360 is considered in this process. The non-singular expression for the gravitational gradients in orbital frame [Petrovskaya and Vershkov, 2006] are used for generating the atmospheric effect on the satellite gradiometric data on the integrated orbit (the orbital frame is defined by u, v and w axes so that w axis coincides with z and upward, v points towards the instantaneous angular momentum vector and u complements the right-handed triad). It is just needed to insert the harmonic coefficients of the atmospheric potential into these relations to compute the atmospheric effect on satellite gradiometric data. Figure 3 shows the maps of the atmospheric effect based on the exponential model on satellite gradiometric data over Fennoscandia.

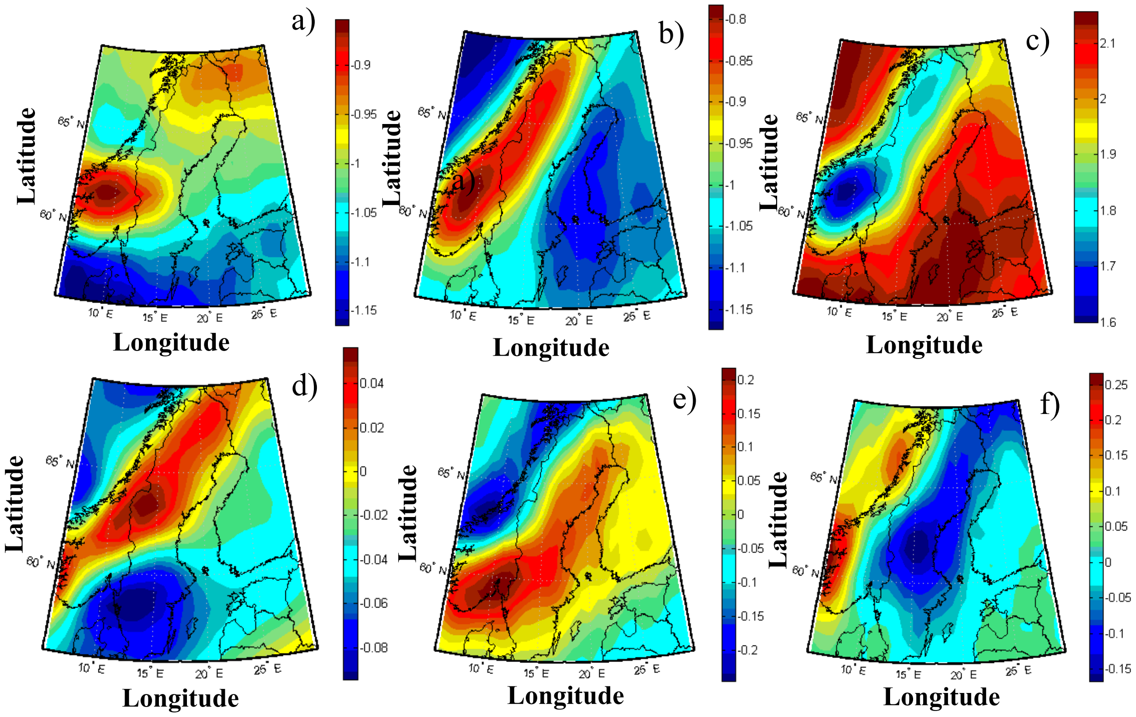

Fig 3. Atmospheric effect on satellite gradiometric data at 250km elevation, based on exponential model in Fennoscandia, (a), (b), (c), (d), (e) and (f) stands for V^a_{uu}, V^a_{vv}, V^a_{ww}, V^a_{uv}, V^a_{uw} and V^a_{vw}, respectively. Unit: 1 mE

V^a_{uu}, V^a_{vv}, V^a_{ww}, V^a_{uv}, V^a_{uw} and V^a_{vw} are the atmospheric effects on the gravitational gradients in orbital frame; for the readers benefit the mathematical formulas are presented in Appendixes B and C. V^a_{vw} is the largest since it includes the zero-degree harmonic, and V^a_{uv} is the smallest as it excludes zero- and first-degree harmonics. The statistics of the atmospheric effects on satellite gradiometric data over Fennoscandia are presented in Table 1.

Table 1. Statistics of atmospheric effect on the satellite gradiometric data at 250 km level over Fennoscandia based on exponential and power density models. Unit: 1 mE

	Exponential density model				Power density model			
	max	mean	min	std	max	mean	min	std
V^a_{uu}	-0.8279	-1.0146	-1.1719	$\pm$0.0692	-0.8237	-1.0104	-1.1675	$\pm$0.0691
V^a_{vv}	-0.7482	-1.0145	-1.1800	$\pm$0.0964	-0.7441	-1.0102	-1.1756	$\pm$0.0964
V^a_{ww}	2.1984	2.0291	1.5816	$\pm$0.1390	2.1897	2.0206	1.5733	$\pm$0.1389
V^a_{uv}	0.0679	-0.0229	-0.1002	$\pm$0.0370	0.0678	-0.0228	-0.1002	$\pm$0.0370
V^a_{uw}	0.2627	0.0062	-0.2701	$\pm$0.1036	0.2624	0.0062	-0.2699	$\pm$0.1035
V^a_{vw}	0.3207	0.0140	-0.1817	$\pm$0.0808	0.3205	0.0140	-0.1817	$\pm$0.0807

Table 1 shows small differences between the atmospheric effects on satellite gravity gradiometric data computed based on the exponential and power models. As it is already mentioned, these models are very similar and there is not doubt that the estimated atmospheric effects are similar as well. Having considered the statistics of Table 1, it is seen that the differences are below 0.1 mE which mean that either exponential or power model can be used in practical consideration in satellite gradiometry respect.

The gravity anomaly and disturbing potential can be derived from the satellite gradiometric data in local gravity field determination (using inversion of the second-order derivatives of the extended Stokes and Abel-Poisson integrals). In this case the removed atmospheric effects should be restored as the indirect atmospheric effect. The indirect effect depends on the type of the atmospheric density model as well. The maps of the differences between indirect effects are presented in Figs. 4(a) and 4(b) in Fennoscandia.

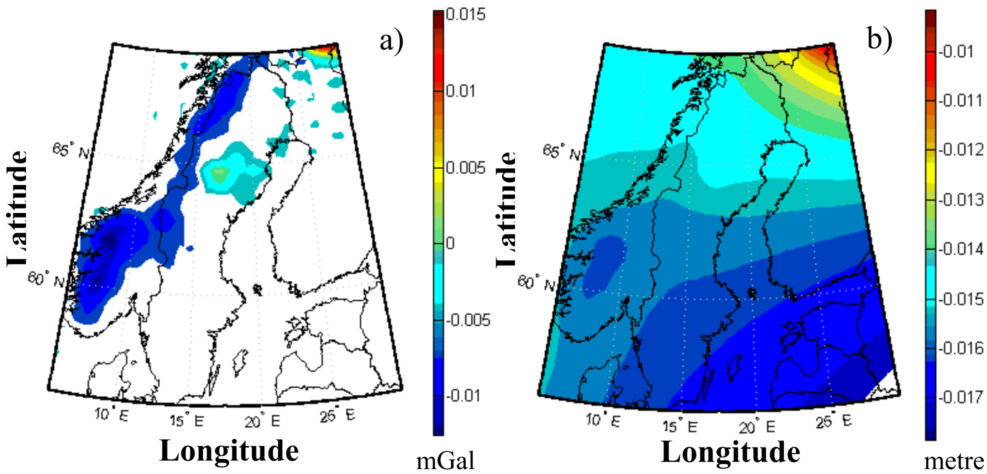

Fig. 4. (a) and (b) difference between indirect atmospheric effects due to exponential and power models, on gravity anomaly and geoid, respectively.

The formulas of the indirect effects on gravity anomaly and geoid are given in Appendix B. It is about -4.6 m and 1.4 mGal on the geoid and gravity anomaly, respectively (using both exponential and power models). The maximum difference between the indirect effects is less than 2 cm on the geoid and 0.02 mGal on gravity anomaly in mountainous regions. It should be emphasized that the presented values of the differences are not the total error of the atmospheric effect. They are the differences between two independent indirect effects which were obtained based on exponential and power models. Definitely the approximation error of the atmospheric densities by these models is larger, but it is not very significant comparing to the total indirect effect of the atmosphere on the geoid which is about -4.6 m. If a precise atmospheric effect is desired, the numerical models should be used which are not simple. The idea of splitting the atmosphere into several layers is feasible for those parts of the atmospheric masses which are above 10 km height. For considering the atmospheric topography we have to consider an analytical model, otherwise modeling of the atmospheric density in spherical harmonics will not be easy.

6. CONCLUSIONS

The spherical harmonic coefficients of the external and internal types of the atmospheric potential were formulated based on the exponential and power density models. The constant parameters of both models were estimated based on the standard atmospheric model. Mathematical derivation of the spherical harmonic based on exponential model is slightly longer and complicated than the power model. We presented that the exponential model is slightly better than the power model according to the fitting error. The formulas presented for the spherical harmonic coefficients of the atmospheric potential can be used in every geodetic aspect. However we selected satellite gravity gradiometry as an example in this paper. Numerical studies on the atmospheric effects on the satellite gradiometric data over Fennoscandia show small differences for the direct atmospheric effects due to these density models (less than 0.1 mE). The maximum atmospheric effect is about 2 mE when the satellite passes over this region. The indirect atmospheric effect on the geoid and gravity anomaly depends on the type of atmospheric density model too. This study found the maximum difference 2 cm and 0.02 mGal on the geoid and gravity anomaly in roughest part of Fennoscandia.

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APPENDIX A

We have shown that because of appearing δ_{n0} in the first term of Eq. (16), it is enough to consider the zero-degree harmonics and there is not need to perform partial integration. We have

$$\frac{I_0 e^{\alpha R}}{R^3} = \frac{e^{\alpha R}}{R^3}\int_R^{R+Z} r_Q e^{-\alpha r_Q} dr_Q = \frac{e^{\alpha R}}{R^3}\left(-\frac{r_Q^2}{\alpha} e^{-\alpha r_Q} - \frac{2r_Q}{\alpha^2} e^{-\alpha r_Q} - \frac{2}{\alpha^3} e^{-\alpha r_Q} \right)_R^{R+Z}, \quad \text{(A.1)}$$

After substituting the limits of the integral into Eq. (A.1) we obtain

$$\frac{I_0 e^{\alpha R}}{R^3} = \frac{e^{\alpha R}}{R^3}\left\{ \frac{1-\left(1+Z/R\right)^2 e^{-\alpha Z}}{\alpha R} + \frac{2}{R^2\alpha^2}\left[\left(1+\frac{Z}{R}\right) e^{-\alpha Z} - 1\right] + \frac{2\left(1-e^{-\alpha Z}\right)}{R^3\alpha^3} \right\}, \quad \text{(A.2)}$$

And after further manipulations Eq. (17) is obtained. Equation (25) can also be proved in a very similar way.

The topographic term in the harmonic coefficients see Eq. (16) is simplified considering

$$e^{-\alpha R} = \sum_{k=0}^{\infty} \frac{(-\alpha)^k R^k}{k!} \quad \text{(A.3)}$$

as Taylor expansion of $e^{-\alpha R}$, as:

$$B = e^{\alpha R}\frac{H_{nm}}{R}\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!} + e^{\alpha R}\left\{ \frac{H_{nm}^2}{2R^2}\left[(n+2)\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!} + (-\alpha R)\sum_{k=1}^{\infty}\frac{(-\alpha)^{k-1} R^{k-1}}{(k-1)!} \right] + \right.$$

$$\left. + \frac{H_{nm}^3}{6R^3}\left[\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}k^2 + (2n+3)\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}k + (n+2)(n+1)\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!} \right] \right\}. \quad \text{(A.4)}$$

The first term of the second square bracket can be simplified more as

$$\sum_{k=0}^{\infty}\frac{(-\alpha)^k R^k}{k!}k^2 = -\alpha R + \sum_{k=2}^{\infty}\frac{(-\alpha)^k R^k}{(k-1)!}(k-1+1) = -\alpha R + \sum_{k=2}^{\infty}\frac{(-\alpha)^k R^k}{(k-1)!}(k-1) + \sum_{k=2}^{\infty}\frac{(-\alpha)^k R^k}{(k-1)!}$$

$$= (-\alpha)^2 R^2 \sum_{k=2}^{\infty}\frac{(-\alpha)^{k-2} R^{k-2}}{(k-2)!} - \alpha R\sum_{k=1}^{\infty}\frac{(-\alpha)^{k-1} R^{k-1}}{(k-1)!} = \alpha R(\alpha R - 1)e^{-\alpha R}. \quad \text{(A.5)}$$

Considering Eq. (A.3) we have

$$B = e^{\alpha R}\frac{H_{nm}}{R}e^{-\alpha R} + e^{\alpha R}\left\{ \frac{H_{nm}^2}{2R^2}\left[(n+2)e^{-\alpha R} + (-\alpha R)e^{-\alpha R} \right] + \right.$$

$$\left. + \frac{H_{nm}^3}{6R^3}\left[\left(-\alpha R + \alpha^2 R^2\right)e^{-\alpha R} + (2n+3)(-\alpha R)e^{-\alpha R} + (n+2)(n+1)e^{-\alpha R} \right] \right\}. \quad \text{(A.6)}$$

After further simplification we derive Eq. (18). Equation (24) can be proved in a similar way.

APPENDIX B

Non-singular expression of the gravitational gradient in orbital frame [Petrovskaya and Vershkov 2006]

$$V_{uu}\left(P\right)=\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}\left\{Q_{m}\left(\lambda_{P}\right)\left[\cos 2\alpha f_{n,m,1}-\cos^{2}\alpha\left(n+1\right)\left(n+2\right)\overline{P}_{n,|m|}\right]+\right.$$
$$\left.+Q_{-m}\left(\lambda_{P}\right)\sin 2\alpha f_{nm,2}\right\} \tag{B.1}$$

$$V_{vv}\left(P\right)=-\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}\left\{Q_{m}\left(\lambda_{P}\right)\left[\cos 2\alpha f_{nm,1}+\sin^{2}\alpha\left(n+1\right)\left(n+2\right)\overline{P}_{n,|m|}\right]+\right.$$
$$\left.+Q_{-m}\left(\lambda_{P}\right)\sin 2\alpha f_{nm,2}\right\} \tag{B.2}$$

$$V_{uv}\left(P\right)=-\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}\left\{Q_{m}\left(\lambda_{P}\right)\left[\sin 2\alpha f_{nm,1}-\cos\alpha\sin\alpha\left(n+1\right)\left(n+2\right)\overline{P}_{n,|m|}\right]+\right.$$
$$\left.-Q_{-m}\left(\lambda_{P}\right)\cos 2\alpha f_{nm,2}\right\} \tag{B.3}$$

$$V_{uw}\left(P\right)=\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}\left[Q_{m}\left(\lambda_{P}\right)\cos\alpha f_{nm,3}+Q_{-m}\left(\lambda_{P}\right)\sin\alpha f_{nm,4}\right] \tag{B.4}$$

$$V_{vw}\left(P\right)=-\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}\left[Q_{m}\left(\lambda_{P}\right)\sin\alpha f_{nm,3}-Q_{-m}\left(\lambda_{P}\right)\cos\alpha f_{nm,4}\right] \tag{B.5}$$

$$V_{ww}\left(P\right)=\frac{GM}{R^{3}}\sum_{n=2}^{N}\sum_{m=-n}^{n}\left(n+1\right)\left(n+2\right)\left(\frac{R}{r_{P}}\right)^{n+3}\left(v_{ext}^{a}\right)_{nm}Q_{m}\left(\lambda_{P}\right)\overline{P}_{n,|m|} \tag{B.6}$$

where, $\overline{P}_{n,|m|}=\overline{P}_{n,|m|}\left(\cos\theta_{P}\right)$ and $f_{nm,1}=f_{nm,1}\left(\theta_{P}\right)$

$$f_{nm,1}\left(\theta_{P}\right)=a_{nm}\overline{P}_{n,|m|-2}+b_{nm}\overline{P}_{n,|m|}+c_{nm}\overline{P}_{n,|m|+2}, \tag{B.7}$$

$$f_{nm,2}\left(\theta_{P}\right)=d_{nm}\overline{P}_{n-1,|m|-2}+g_{nm}\overline{P}_{n-1,|m|}+h_{nm}\overline{P}_{n-1,|m|+2}, \tag{B.8}$$

$$f_{nm,3}\left(\theta_{P}\right)=\beta_{nm}\overline{P}_{n,|m|-1}+\gamma_{nm}\overline{P}_{n,|m|+1}, \tag{B.9}$$

$$f_{nm,4}\left(\theta_{P}\right)=\mu_{nm}\overline{P}_{n-1,|m|-1}+\nu_{nm}\overline{P}_{n-1,|m|+1}, \tag{B.10}$$

$$Q_{m}\left(\lambda_{P}\right)=\begin{cases}\cos m\lambda_{P} & m\geq 0\\ \sin|m|\lambda_{P} & m<0\end{cases} \tag{B.11}$$

where, $\left(v_{ext}^{a}\right)_{nm}$ is the spherical harmonic coefficients of the external atmospheric potential, θ_P and λ_P and r_P are the co-latitude, longitude and geocentric radius of the point P or the satellite position. N is the maximum degree of harmonic expansion, and $\overline{P}_{n,|m|}$ is the fully-normalized associated Legendre function of degree n and order m. α is the satellite track azimuth. a_{nm}, b_{nm}, c_{nm}, d_{nm}, g_{nm}, h_{nm}, β_{nm}, γ_{nm}, μ_{nm} and ν_{nm} are the constant coefficients:

$$a_{nm} = \begin{cases} 0 & |m| = 0,1 \\ \dfrac{\sqrt{1+\delta_{|m|,2}}}{4}\sqrt{n^2-\left(|m|-1\right)^2}\sqrt{n+|m|}\sqrt{n-|m|+2} & 2 \le |m| \le n \end{cases} \tag{B.12}$$

$$b_{nm} = \begin{cases} \dfrac{\left(n+|m|+1\right)\left(n+|m|+2\right)}{2\left(|m|+1\right)} & |m| = 0,1 \\ \dfrac{n^2+m^2+3n+2}{2} & 2 \le |m| \le n \end{cases} \tag{B.13}$$

$$c_{nm} = \begin{cases} \dfrac{\sqrt{1+\delta_{|m|,0}}}{4}\sqrt{n^2-\left(|m|+1\right)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, & |m| = 0,1 \\ \dfrac{1}{4}\sqrt{n^2-\left(|m|+1\right)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, & 2 \le |m| \le n \end{cases} \tag{B.14}$$

$$d_{nm} = \begin{cases} 0 & |m| = 1 \\ -\dfrac{m}{4|m|}\sqrt{\dfrac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,2}}\sqrt{n^2-\left(|m|-1\right)^2}\sqrt{n+|m|}\sqrt{n+|m|-2}, & 2 \le |m| \le n \end{cases} \tag{B.15}$$

$$g_{nm} = \begin{cases} \dfrac{m}{4|m|}\sqrt{\dfrac{2n+1}{2n-1}}\sqrt{n+1}\sqrt{n-1}\left(n+2\right), & |m| = 1 \\ \dfrac{m}{2}\sqrt{\dfrac{2n+1}{2n-1}}\sqrt{n+|m|}\sqrt{n-|m|}, & 2 \le |m| \le n \end{cases} \tag{B.16}$$

$$h_{nm} = \begin{cases} \dfrac{m}{4|m|}\sqrt{\dfrac{2n+1}{2n-1}}\sqrt{n-3}\sqrt{n-2}\sqrt{n-1}\sqrt{n+2}, & |m| = 1 \\ \dfrac{m}{4|m|}\sqrt{\dfrac{2n+1}{2n-1}}\sqrt{n^2-\left(|m|+1\right)^2}\sqrt{n-|m|}\sqrt{n-|m|-2}, & 2 \le |m| \le n \end{cases} \tag{B.17}$$

$$\beta_{nm} = \begin{cases} 0 & |m| = 0 \\ \dfrac{n+2}{2}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n-|m|+1}, & 1 \le |m| \le n \end{cases} \tag{B.18}$$

$$\gamma_{nm} = \begin{cases} -\left(n+2\right)\sqrt{\dfrac{n\left(n+1\right)}{2}}, & |m| = 0 \\ -\dfrac{\left(n+2\right)}{2}\sqrt{n-|m|}\sqrt{n+|m|+1}, & 1 \le |m| \le n \end{cases} \tag{B.19}$$

$$\mu_{nm} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n+|m|-1} \tag{B.20}$$

$$v_{nm} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{n-|m|}\sqrt{n-|m|-1} \quad \text{(B. 21)}$$

where δ is Kronecker's delta. Contribution of the zero- and first-degrees is presented in Appendix C. The indirect atmospheric effect on gravity anomaly and geoid is

$$\delta\Delta g^{a}_{ind}(P) = -\frac{GM}{R^2}\sum_{n=0}^{N}(n+2)\sum_{m=-n}^{n}\left(v^{a}_{int}\right)_{nm} Y_{nm}(P), \text{ and} \quad \text{(B.22)}$$

$$\delta N^{a}_{ind}(P) = -\frac{GM}{R\gamma}\sum_{n=0}^{N}\sum_{m=-n}^{n}\left(v^{a}_{int}\right)_{nm} Y_{nm}(P). \quad \text{(B.23)}$$

APPENDIX C

The contribution of the zero- and the first-degree harmonics to Eqs (B.1)-(B.6) can be derived based on the original formulas of the gravitational gradients in the orbital frame as (see also Petrovskaya and Vershkov 2006, Eq. 45):

$$V^{0,1}_{uu}(P) = -\frac{GMR}{r_P^4}\left\{\frac{r_P}{R}C_{00} + 3\sqrt{3}\left[C_{10}\cos\theta_P + \left(C_{11}\cos\lambda_P + S_{11}\sin\lambda_P\right)\sin\theta_P\right]\right\}, \quad \text{(C.1)}$$

$$V^{0,1}_{vv}(P) = -\frac{GMR}{r_P^4}\left\{\frac{r_P}{R}C_{00} + 3\sqrt{3}\left[C_{10}\cos\theta_P + \left(C_{11}\cos\lambda_P + S_{11}\sin\lambda_P\right)\sin\theta_P\right]\right\}, \quad \text{(C.2)}$$

$$V^{0,1}_{ww}(P) = \frac{2GMR}{r_P^4}\left\{\frac{r_P}{R}C_{00} + 3\sqrt{3}\left[C_{10}\cos\theta_P + \left(C_{11}\cos\lambda_P + S_{11}\sin\lambda_P\right)\sin\theta_P\right]\right\}, \quad \text{(C.3)}$$

$$V^{0,1}_{uv}(P) = 0, \quad \text{(C.4)}$$

$$V^{0,1}_{uw}(P) = \frac{3\sqrt{3}GMR}{r_P^4}\left\{\cos\alpha\left[C_{10}\sin\theta_P + \left(C_{11}\cos\lambda_P + S_{11}\sin\lambda_P\right)\cos\theta_P\right] + \right.$$

$$\left. + \sin\alpha\left(-C_{11}\sin\lambda_P + S_{11}\cos\lambda_P\right)\right., \quad \text{(C.5)}$$

$$V^{0,1}_{vw}(P) = \frac{3\sqrt{3}GMR}{r_P^4}\left\{-\sin\alpha\left[-C_{10}\sin\theta_P + \left(C_{11}\cos\lambda_P + S_{11}\sin\lambda_P\right)\cos\theta\right] + \right.$$

$$\left. + \cos\alpha\left(-C_{11}\sin\lambda_P + S_{11}\cos\lambda_P\right)\right., \quad \text{(C.6)}$$

where superscripts of 0 and1 stand for the zero- and the first-degree harmonics, respectively. C_{00}, C_{10}, C_{11}, S_{10} and S_{11} are the fully-normalized zero- and the first-degree geopotential coefficients.

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