Annales Mathematicae Silesianae **33** (2019), 55–65
DOI: 10.2478/amsil-2019-0005

SOME ADDITION FORMULAS FOR FIBONACCI, PELL AND JACOBSTHAL NUMBERS

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Abstract. In this paper, we obtain a closed form for $F_{\sum_{i=1}^{k}}$, $P_{\sum_{i=1}^{k}}$ and $J_{\sum_{i=1}^{k}}$ for some positive integers k where F_r, P_r and J_r are the rth Fibonacci, Pell and Jacobsthal numbers, respectively. We also give three open problems for the general cases $F_{\sum_{i=1}^{n}}$, $P_{\sum_{i=1}^{n}}$ and $J_{\sum_{i=1}^{n}}$ for any arbitrary positive integer n.

1. Introduction

Fibonacci, Pell and Jacobsthal numbers create well-known integer sequences and satisfy the following recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1,$$
$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1,$$

and

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad J_1 = 1,$$

respectively.

Received: 21.11.2018. Accepted: 01.05.2019. Published online: 13.06.2019.
(2010) Mathematics Subject Classification: 11B39, 11K31, 11Y55.
Key words and phrases: Fibonacci numbers, Pell numbers, Jacobsthal numbers.

In [5], Mana gave the addition formula

$$F_{a+b} = F_{a+1}F_{b+1} - F_{a-1}F_{b-1}. \tag{1.1}$$

Similarly, in [3, p.89], the following addition formula was given

$$F_{a+b+c} = F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1}. \tag{1.2}$$

These two identities inspired us to investigate the general case. Eqs. (1.1) and (1.2) can be obtained by using Fibonacci **Q**-matrix

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

A detailed history of **Q**-matrix can be found in [1]. **Q**-matrix satisfies some interesting relations. For example, for $n \geq 1$, Koshy [3, p.363] gave

$$\mathbf{Q}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \quad \mathbf{Q}^{n+1} = \mathbf{Q}^n\mathbf{Q}, \quad \text{and} \quad \mathbf{Q}^{m+n} = \mathbf{Q}^m\mathbf{Q}^n.$$

We use **Q**-matrix to calculate desired identities.

2. Addition Formulas for Fibonacci Numbers

Firstly, we can write the known results as follows:

$$F_{\sum_{i=1}^{1} k_i} = \prod_{i=1}^{1} F_{k_i+1} - \prod_{i=1}^{1} F_{k_i-1} = F_{k_1} = F_{k_1+1} - F_{k_1-1},$$

$$F_{\sum_{i=1}^{2} k_i} = \prod_{i=1}^{2} F_{k_i+1} - \prod_{i=1}^{2} F_{k_i-1} = F_{k_1+k_2} = F_{k_1+1}F_{k_2+1} - F_{k_1-1}F_{k_2-1},$$

$$\begin{aligned} F_{\sum_{i=1}^{3} k_i} &= \prod_{i=1}^{3} F_{k_i+1} - \prod_{i=1}^{3} F_{k_i-1} + \prod_{i=1}^{3} F_{k_i} = F_{k_1+k_2+k_3} \\ &= F_{k_1+1}F_{k_2+1}F_{k_3+1} - F_{k_1-1}F_{k_2-1}F_{k_3-1} + F_{k_1}F_{k_2}F_{k_3}. \end{aligned}$$

Similar forms for $F_{k_1+k_2+\cdots+k_n}$ where $n = 4, 5$ and 6 are in the following theorem.

Theorem 2.1. *For all positive integers $k_1, k_2, \ldots, k_6$, we have*

$$F_{\sum_{i=1}^{4} k_i} = \prod_{i=1}^{4} F_{k_i+1} - \prod_{i=1}^{4} F_{k_i-1} + 2\prod_{i=1}^{4} F_{k_i} + \sum_{j=1}^{4} \prod_{\substack{i=1\\ i\neq j}}^{4} F_{k_j-1}F_{k_i}, \tag{2.1}$$

$$F_{\sum_{i=1}^{5} k_i} = \prod_{i=1}^{5} F_{k_i+1} - \prod_{i=1}^{5} F_{k_i-1} + 4\prod_{i=1}^{5} F_{k_i} + 2\sum_{j=1}^{5} \prod_{\substack{i=1\\ i\neq j}}^{5} F_{k_j-1}F_{k_i}$$
$$+ \sum_{l=1}^{4} \sum_{j=l+1}^{5} \prod_{\substack{i=1\\ i\neq j\\ i\neq l}}^{5} F_{k_l-1}F_{k_j-1}F_{k_i},$$

and

$$F_{\sum_{i=1}^{6} k_i} = \prod_{i=1}^{6} F_{k_i+1} - \prod_{i=1}^{6} F_{k_i-1} + 7\prod_{i=1}^{6} F_{k_i} + 4\sum_{j=1}^{6} \prod_{\substack{i=1\\ i\neq j}}^{6} F_{k_j-1}F_{k_i}$$
$$+ 2\sum_{l=1}^{5} \sum_{j=l+1}^{6} \prod_{\substack{i=1\\ i\neq j\\ i\neq l}}^{6} F_{k_l-1}F_{k_j-1}F_{k_i}$$
$$+ \sum_{m=1}^{4} \sum_{l=m+1}^{5} \sum_{j=l+1}^{6} \prod_{\substack{i=1\\ i\neq j\\ i\neq l\\ i\neq m}}^{6} F_{k_m-1}F_{k_l-1}F_{k_j-1}F_{k_i}.$$

Proof. From the matrix theory, we have

$$\mathbf{Q}^{k_1+k_2+k_3+k_4} = \mathbf{Q}^{k_1+k_2+k_3}\mathbf{Q}^{k_4}.$$

The entry $\mathbf{Q}^{k_1+k_2+k_3+k_4}$ ($[1,2]$) gives

$$\begin{aligned}
&F_{k_1+k_2+k_3+k_4}\\
&\quad= \Big[(F_{k_1+1}F_{k_2+1} + F_{k_1}F_{k_2})F_{k_3+1} + (F_{k_1+1}F_{k_2} + F_{k_1}F_{k_2-1})F_{k_3}\Big]F_{k_4}\\
&\qquad+ \Big[F_{k_1+1}F_{k_2+1}F_{k_3+1} + F_{k_1}F_{k_2}F_{k_3} - F_{k_1-1}F_{k_2-1}F_{k_3-1}\Big]F_{k_4-1}\\
&= F_{k_1+1}F_{k_2+1}F_{k_3+1}F_{k_4} + F_{k_1}F_{k_2}F_{k_3+1}F_{k_4} + F_{k_1+1}F_{k_2}F_{k_3}F_{k_4} + F_{k_1}F_{k_2-1}F_{k_3}F_{k_4}\\
&\quad+ F_{k_1+1}F_{k_2+1}F_{k_3+1}F_{k_4-1} + F_{k_1}F_{k_2}F_{k_3}F_{k_4-1} - F_{k_1-1}F_{k_2-1}F_{k_3-1}F_{k_4-1}
\end{aligned}$$

$$= F_{k_1+1}F_{k_2+1}F_{k_3+1}F_{k_4+1} - F_{k_1-1}F_{k_2-1}F_{k_3-1}F_{k_4-1} + F_{k_1+1}F_{k_2}F_{k_3}F_{k_4}$$
$$+ F_{k_1}F_{k_2-1}F_{k_3}F_{k_4} + F_{k_1}F_{k_2}F_{k_3+1}F_{k_4} + F_{k_1}F_{k_2}F_{k_3}F_{k_4-1}$$
$$= F_{k_1+1}F_{k_2+1}F_{k_3+1}F_{k_4+1} - F_{k_1-1}F_{k_2-1}F_{k_3-1}F_{k_4-1} + 2F_{k_1}F_{k_2}F_{k_3}F_{k_4}$$
$$+ F_{k_1-1}F_{k_2}F_{k_3}F_{k_4} + F_{k_1}F_{k_2-1}F_{k_3}F_{k_4} + F_{k_1}F_{k_2}F_{k_3-1}F_{k_4} + F_{k_1}F_{k_2}F_{k_3}F_{k_4-1}.$$

The final equality gives Eq. (2.1). The remaining two identities in the theorem can be proved similarly. □

Above identities can help us to give the following conjecture for the general case.

CONJECTURE 2.2. *For all positive integers k_i, we have*

$$F_{\sum_{i=1}^{n} k_i} = \prod_{i=1}^{n} F_{k_i+1} - \prod_{i=1}^{n} F_{k_i-1} + (F_n - 1)\prod_{i=1}^{n} F_{k_i}$$
$$+ (F_{n-1} - 1)\sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{n} F_{k_j-1}F_{k_i}$$
$$+ (F_{n-2} - 1)\sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \prod_{\substack{i=1 \\ i\neq j_1 \\ i\neq j_2}}^{n} F_{k_{j_2}-1}F_{k_{j_1}-1}F_{k_i}$$
$$+ \dots$$
$$+ (F_3 - 1)\sum_{j_1=1}^{4} \sum_{j_2=j_1+1}^{5} \sum_{j_3=j_2+1}^{6} \cdots \sum_{j_{n-3}=j_{n-4}+1}^{n} \prod_{\substack{i=1 \\ i\neq j_m}}^{n} F_{k_{j_1}-1}F_{k_{j_2}-1}\cdots F_{k_{j_{n-3}}-1}F_{k_i}.$$

3. Addition Formulas for Pell Numbers

In this section, we extend results of the previous section to the Pell numbers. We need the following matrix similar to **Q**-matrix

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Properties of **P**-matrix can be found in [4]. **P**-matrix satisfies

$$\mathbf{P}^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}, \quad \mathbf{P}^{n+1} = \mathbf{P}^n\mathbf{P}, \quad \text{and} \quad \mathbf{P}^{m+n} = \mathbf{P}^m\mathbf{P}^n.$$

The following addition formula ([4, p. 30, Exercise 48]) can be obtained by using **P**-matrix:

$$P_{m+n} = P_m P_{n+1} + P_{m-1}P_n.$$

Now we transform the last equation to another form:

$$\begin{aligned} P_{m+n} &= P_m P_{n+1} + P_{m-1}P_n \\ &= \frac{1}{2}(P_{n+1} - P_{n-1}P_{m+1}) + \frac{1}{2}(P_{m+1} - P_{m-1}P_{n-1}) \\ &= \frac{1}{2}(P_{n+1}P_{m+1} - P_{n-1}P_{m+1} + P_{m+1}P_{n-1} - P_{m-1}P_{n-1}), \end{aligned}$$

and we have

$$P_{m+n} = \frac{1}{2}(P_{m+1}P_{n+1} - P_{m-1}P_{n-1}). \tag{3.1}$$

This form of addition formula is very useful for us. The main results of this section are in the following theorem.

Theorem 3.1. *For all positive integers $k_1, k_2, \ldots, k_5$, we have*

$$\begin{aligned} P_{\sum_{i=1}^{1} k_i} &= \frac{1}{2}\prod_{i=1}^{1} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{1} P_{k_i-1} = \frac{1}{2}P_{k_1+1} - \frac{1}{2}P_{k_1-1}, \\ P_{\sum_{i=1}^{2} k_i} &= \frac{1}{2}\prod_{i=1}^{2} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{2} P_{k_i-1} = \frac{1}{2}P_{k_1+1}P_{k_2+1} - \frac{1}{2}P_{k_1-1}P_{k_2-1}, \\ P_{\sum_{i=1}^{3} k_i} &= \frac{1}{2}\prod_{i=1}^{3} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{3} P_{k_i-1} + \prod_{i=1}^{3} P_{k_i}, \\ P_{\sum_{i=1}^{4} k_i} &= \frac{1}{2}\prod_{i=1}^{4} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{4} P_{k_i-1} + 4\prod_{i=1}^{4} P_{k_i} + \sum_{j=1}^{4}\prod_{\substack{i=1 \\ i\neq j}}^{4} P_{k_j-1}P_{k_i}, \end{aligned}$$

$$P_{\sum_{i=1}^{5} k_i} = \frac{1}{2}\prod_{i=1}^{5} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{5} P_{k_i-1} + 13\prod_{i=1}^{5} P_{k_i} + 4\sum_{j=1}^{5}\prod_{\substack{i=1\\ i\neq j}}^{5} P_{k_j-1}P_{k_i},$$
$$+\sum_{l=1}^{4}\sum_{j=l+1}^{5}\prod_{\substack{i=1\\ i\neq j\\ i\neq l}}^{5} P_{k_l-1}P_{k_j-1}P_{k_i}.$$

Proof. We proved the second identity (see (3.1)) at the beginning of this section. Now, we start with the proof of the third identity. **P**-matrix satisfies

$$\mathbf{P}^{k_1+k_2+k_3} = \mathbf{P}^{k_1+k_2}\mathbf{P}^{k_3}.$$

This equation gives

$$\begin{bmatrix} P_{k_1+k_2+k_3+1} & P_{k_1+k_2+k_3} \\ P_{k_1+k_2+k_3} & P_{k_1+k_2+k_3-1} \end{bmatrix}$$
$$= \begin{bmatrix} P_{k_1+1}P_{k_2+1} + P_{k_1}P_{k_2} & P_{k_1+1}P_{k_2} + P_{k_1}P_{k_2-1} \\ P_{k_1}P_{k_2+1} + P_{k_1-1}P_{k_2} & P_{k_1}P_{k_2} + P_{k_1-1}P_{k_2-1} \end{bmatrix} \times \begin{bmatrix} P_{k_3+1} & P_{k_3} \\ P_{k_3} & P_{k_3-1} \end{bmatrix}$$

and

$$\begin{aligned} P_{k_1+k_2+k_3} &= P_{k_1+1}P_{k_2+1}P_{k_3} + P_{k_1}P_{k_2}P_{k_3} + P_{k_1+1}P_{k_2}P_{k_3-1} + P_{k_1}P_{k_2-1}P_{k_3-1} \\ &= P_{k_1+1}P_{k_2+1}\left(\frac{1}{2}P_{k_3+1} - \frac{1}{2}P_{k_3-1}\right) + P_{k_1}P_{k_2}P_{k_3} \\ &\quad + P_{k_1+1}\left(\frac{1}{2}P_{k_2+1} - \frac{1}{2}P_{k_2-1}\right)P_{k_3-1} \\ &\quad + \left(\frac{1}{2}P_{k_1+1} - \frac{1}{2}P_{k_1-1}\right)P_{k_2-1}P_{k_3-1} \\ &= \frac{1}{2}P_{k_1+1}P_{k_2+1}P_{k_3+1} - \frac{1}{2}P_{k_1-1}P_{k_2-1}P_{k_3-1} + P_{k_1}P_{k_2}P_{k_3}. \end{aligned}$$

The last equality completes the proof of the third identity. The remaining two addition formulas can be proved in a similar way. □

We guess the general case in next conjecture.

Conjecture 3.2. *For all positive integers k_i, we have*

$$
\begin{aligned}
P_{\sum_{i=1}^n k_i} &= \frac{1}{2}\prod_{i=1}^{n} P_{k_i+1} - \frac{1}{2}\prod_{i=1}^{n} P_{k_i-1} + A_{n-2}\prod_{i=1}^{n} P_{k_i} \\
&+ A_{n-3}\sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n} P_{k_j-1}P_{k_i} + A_{n-4}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\prod_{\substack{i=1\\ i\neq j_1\\ i\neq j_2}}^{n} P_{k_{j_2}-1}P_{k_{j_1}-1}P_{k_i} \\
&+ \dots \\
&+ A_1\sum_{j_1=1}^{4}\sum_{j_2=j_1+1}^{5}\sum_{j_3=j_2+1}^{6}\cdots\sum_{j_{n-3}=j_{n-4}+1}^{n}\prod_{\substack{i=1\\ i\neq j_m}}^{n} P_{k_{j_1}-1}P_{k_{j_2}-1}\cdots P_{k_{j_{n-3}}-1}P_{k_i}
\end{aligned}
$$

where $A_n = \frac{3^n-1}{2}$ (series A003462 in [6]*).*

4. Addition Formulas for Jacobsthal Numbers

In previous sections, we gave addition formulas for the Fibonacci and Pell numbers. We investigate these addition formulas for Jacobsthal numbers in this section. As usual, we need the **J**-matrix:

$$\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

Similar to **Q**-matrix and **P**-matrix, **J**-matrix satisfies

$$\mathbf{J}^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}, \quad \mathbf{J}^{m+1} = \mathbf{J}^n\mathbf{J}, \quad \text{and} \quad \mathbf{J}^{m+n} = \mathbf{J}^m\mathbf{J}^n.$$

The **J**-matrix was defined and studied in [2]. We have the next identity from [2]:

$$J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n.$$

Let us change this equation a little bit. From the recurrence relation of Jacobsthal numbers, we have

$$
\begin{aligned}
J_{m+n} &= J_{n+1}(J_{m+1} - 2J_{m-1}) + 2J_{m-1}J_n \\
&= J_{m+1}J_{n+1} - 2J_{m-1}(J_{n+1} - J_n)
\end{aligned}
$$

and

$$J_{m+n} = J_{m+1}J_{n+1} - 4J_{m-1}J_{n-1}. \tag{4.1}$$

This equality is our reference point, i.e., we write the other addition formulas similar to the last equation. We calculate one more identity for Jacobsthal numbers in order to estimate the general case.

THEOREM 4.1. *For all positive integers $k_1, k_2, \dots, k_7$, we have*

$$J_{\sum_{i=1}^{1} k_i} = \prod_{i=1}^{1} J_{k_i+1} - 2\prod_{i=1}^{1} J_{k_i-1},$$

$$J_{\sum_{i=1}^{2} k_i} = \prod_{i=1}^{2} J_{k_i+1} - 2^2\prod_{i=1}^{2} J_{k_i-1},$$

$$J_{\sum_{i=1}^{3} k_i} = \prod_{i=1}^{3} J_{k_i+1} - 2^3\prod_{i=1}^{3} J_{k_i-1} + 2\prod_{i=1}^{3} J_{k_i},$$

$$J_{\sum_{i=1}^{4} k_i} = \prod_{i=1}^{4} J_{k_i+1} - 2^4\prod_{i=1}^{4} J_{k_i-1} + 4\prod_{i=1}^{4} J_{k_i} + 4\sum_{t=1}^{4}\prod_{\substack{i=1\\ i\neq t}}^{4} J_{k_t-1}J_{k_i},$$

$$\begin{aligned} J_{\sum_{i=1}^{5} k_i} &= \prod_{i=1}^{5} J_{k_i+1} - 2^5\prod_{i=1}^{5} J_{k_i-1} + 10\prod_{i=1}^{5} J_{k_i} + 8\sum_{t=1}^{5}\prod_{\substack{i=1\\ i\neq t}}^{5} J_{k_t-1}J_{k_i} \\ &\quad + 8\sum_{l=1}^{4}\sum_{t=l+1}^{5}\prod_{\substack{i=1\\ i\neq t\\ i\neq l}}^{5} J_{k_l-1}J_{k_t-1}J_{k_i}, \end{aligned}$$

$$\begin{aligned} J_{\sum_{i=1}^{6} k_i} &= \prod_{i=1}^{6} J_{k_i+1} - 2^6\prod_{i=1}^{6} J_{k_i-1} + 20\prod_{i=1}^{6} J_{k_i} + 20\sum_{t=1}^{6}\prod_{\substack{i=1\\ i\neq t}}^{6} J_{k_t-1}J_{k_i} \\ &\quad + 16\sum_{l=1}^{5}\sum_{t=l+1}^{6}\prod_{\substack{i=1\\ i\neq t\\ i\neq l}}^{6} J_{k_l-1}J_{k_t-1}J_{k_i} \\ &\quad + 16\sum_{m=1}^{4}\sum_{l=m+1}^{5}\sum_{n=l+1}^{6}\prod_{\substack{i=1\\ i\neq n\\ i\neq l\\ i\neq m}}^{6} J_{k_m-1}J_{k_l-1}J_{k_n-1}J_{k_i} \end{aligned}$$

and

$$J_{\sum_{i=1}^{7} k_i} = \prod_{i=1}^{7} J_{k_i+1} - 2^7 \prod_{i=1}^{7} J_{k_i-1} + 42 \prod_{i=1}^{7} J_{k_i} + 40 \sum_{t=1}^{7} \prod_{\substack{i=1 \\ i\neq t}}^{7} J_{k_t-1} J_{k_i}$$

$$+40 \sum_{l=1}^{6} \sum_{t=l+1}^{7} \prod_{\substack{i=1 \\ i\neq t \\ i\neq l}}^{7} J_{k_l-1} J_{k_t-1} J_{k_i} + 32 \sum_{m=1}^{5} \sum_{l=m+1}^{6} \sum_{n=l+1}^{7} \prod_{\substack{i=1 \\ i\neq n \\ i\neq l \\ i\neq m}}^{7} J_{k_m-1} J_{k_l-1} J_{k_n-1} J_{k_i}$$

$$+ 32 \sum_{j_1=1}^{4} \sum_{j_2=j_1+1}^{5} \sum_{j_3=j_2+1}^{6} \sum_{j_4=j_3+1}^{7} \prod_{\substack{i=1 \\ i\neq j_m}}^{7} J_{k_{j_1}-1} J_{k_{j_2}-1} J_{k_{j_3}-1} J_{k_{j_4}-1} J_{k_i}.$$

Proof. We gave the proof of the second identity (see (4.1)) at the beginning of this section. Now, we prove the third equality of this theorem and the others can be proved by following a similar course. Let us remind the following property of **J** matrix:

$$\mathbf{J}^{k_1+k_2+k_3} = \mathbf{J}^{k_1+k_2} \mathbf{J}^{k_3}.$$

From this identity, we write

$$\begin{bmatrix} J_{k_1+k_2+k_3+1} & 2J_{k_1+k_2+k_3} \\ J_{k_1+k_2+k_3} & 2J_{k_1+k_2+k_3-1} \end{bmatrix}$$

$$= \begin{bmatrix} J_{k_1+1}J_{k_2+1} + 2J_{k_1}J_{k_2} & 2J_{k_1+1}J_{k_2} + 4J_{k_1}J_{k_2-1} \\ J_{k_1}J_{k_2+1} + 2J_{k_1-1}J_{k_2} & 2J_{k_1}J_{k_2} + 4J_{k_1-1}J_{k_2-1} \end{bmatrix} \times \begin{bmatrix} J_{k_3+1} & 2J_{k_3} \\ J_{k_3} & 2J_{k_3-1} \end{bmatrix}.$$

As in the case of Fibonacci and Pell numbers, we obtain

$$\begin{aligned} J_{k_1+k_2+k_3} &= J_{k_1+1}J_{k_2+1}J_{k_3+1} - 4J_{k_1-1}J_{k_2-1}J_{k_3+1} \\ &\quad + 2J_{k_1}J_{k_2}J_{k_3} + 4J_{k_1-1}J_{k_2-1}J_{k_3} \\ &= J_{k_1+1}J_{k_2+1}J_{k_3+1} - 4J_{k_1-1}J_{k_2-1}(J_{k_3+1} - J_{k_3}) + 2J_{k_1}J_{k_2}J_{k_3} \\ &= J_{k_1+1}J_{k_2+1}J_{k_3+1} - 8J_{k_1-1}J_{k_2-1}J_{k_3-1} + 2J_{k_1}J_{k_2}J_{k_3}. \end{aligned}$$

The last equality completes the proof. □

Our guess for the general case of the Jacobsthal numbers is more complicated than those of Fibonacci and Pell numbers.

CONJECTURE 4.2. *For all positive integers k_i, if n is odd, we have*

$$\begin{aligned}
J_{\sum_{i=1}^{n} k_i} &= \prod_{i=1}^{n} J_{k_i+1} - 2^n \prod_{i=1}^{n} J_{k_i-1} + (J_n - J_1) \prod_{i=1}^{n} J_{k_i} \\
&+ (J_n - J_3)\left(\sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{n} J_{k_j-1} J_{k_i} + \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \prod_{\substack{i=1 \\ i\neq j_1 \\ i\neq j_2}}^{n} J_{k_{j_2}-1} J_{k_{j_1}-1} J_{k_i} \right) + \cdots \\
&+ (J_n - J_{n-2})\left(\sum_{j_1=1}^{5} \sum_{j_2=j_1+1}^{6} \sum_{j_3=j_2+1}^{7} \cdots \sum_{j_{n-4}=j_{n-5}+1}^{n} \prod_{\substack{i=1 \\ i\neq j_m}}^{n} J_{k_{j_1}-1} J_{k_{j_2}-1} \cdots J_{k_{j_{n-3}}-1} P_{k_i} \right. \\
&\left. + \sum_{j_1=1}^{4} \sum_{j_2=j_1+1}^{5} \sum_{j_3=j_2+1}^{6} \cdots \sum_{j_{n-3}=j_{n-4}+1}^{n} \prod_{\substack{i=1 \\ i\neq j_m}}^{n} J_{k_{j_1}-1} J_{k_{j_2}-1} \cdots J_{k_{j_{n-3}}-1} P_{k_i} \right),
\end{aligned}$$

if n is even, we have

$$\begin{aligned}
J_{\sum_{i=1}^{n} k_i} &= \prod_{i=1}^{n} J_{k_i+1} - 2^n \prod_{i=1}^{n} J_{k_i-1} + (J_n - J_2)\left(\prod_{i=1}^{n} J_{k_i} + \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{n} J_{k_j-1} J_{k_i} \right) \\
&+ (J_n - J_4)\left(\sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} \prod_{\substack{i=1 \\ i\neq j_1 \\ i\neq j_2}}^{n} J_{k_{j_2}-1} J_{k_{j_1}-1} J_{k_i} \right. \\
&\left. + \sum_{j_1=1}^{n-2} \sum_{j_2=j_1+1}^{n-1} \sum_{j_3=j_2+1}^{n} \prod_{\substack{i=1 \\ i\neq j_1 \\ i\neq j_2 \\ i\neq j_2}}^{n} J_{k_{j_3}-1} J_{k_{j_2}-1} J_{k_{j_1}-1} J_{k_i} \right) + \cdots \\
&+ (J_n - J_{n-2})\left(\sum_{j_1=1}^{5} \sum_{j_2=j_1+1}^{6} \sum_{j_3=j_2+1}^{7} \cdots \sum_{j_{n-4}=j_{n-5}+1}^{n} \prod_{\substack{i=1 \\ i\neq j_m}}^{n} J_{k_{j_1}-1} J_{k_{j_2}-1} \cdots J_{k_{j_{n-3}}-1} P_{k_i} \right. \\
&\left. + \sum_{j_1=1}^{4} \sum_{j_2=j_1+1}^{5} \sum_{j_3=j_2+1}^{6} \cdots \sum_{j_{n-3}=j_{n-4}+1}^{n} \prod_{\substack{i=1 \\ i\neq j_m}}^{n} J_{k_{j_1}-1} J_{k_{j_2}-1} \cdots J_{k_{j_{n-3}}-1} P_{k_i} \right),
\end{aligned}$$

Acknowledgements. All authors contributed equally to the writing of this paper.

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