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TOPOLOGICAL SPACES WITH THE FREESE–NATION PROPERTY

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Abstract. We give a proposal of generalization of the Freese–Nation property for topological spaces. We introduce a few properties related to Freese–Nation property: FNS, FN, FNS*, FNI. This article presents some relationship between these concepts. We show that spaces with the FNS property satisfy ccc and any product of such spaces also satisfies ccc. We show that all metrizable spaces have the FN property.

1. Introduction

R. Freese and J.B. Nation ([2]) characterize projective lattices by four conditions. One of the conditions is called the FNS property. L. Heindorf and L.B. Shapiro ([6]) have used the FNS property to characterize Boolean algebras with a club consisting of countable relatively complete Boolean subalgebras. In other words, they showed that a family of all clopen sets of 0-dimensional compact space X has the FNS property if and only if X is openly generated. E.V. Shchepin introduced the concept of openly generated spaces in [11] and developed this theory in [12] and [13]. It is natural to generalize the Freese–Nation property to arbitrary topological spaces. The FNS property and some

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versions of it for compact spaces were studied in the papers [3], [4], [6], [9] and [10]. In [6] there are three definitions of the Freese–Nation property: FN, the interpolative version FNI and the separative version FNS.

(FN) A Boolean algebra $\mathbb{B}$ has the FN property if for each $b \in \mathbb{B}$ there are two finite sets $\mathrm{u}(b) \subseteq \{c \in \mathbb{B} : b \leq c\}$ and $\mathrm{l}(b) \subseteq \{c \in \mathbb{B} : b \geq c\}$ such that if $a \leq b$ then $\mathrm{u}(a) \cap \mathrm{l}(b) \neq \emptyset$.

(FNI) A Boolean algebra $\mathbb{B}$ has the FNI property if there is $\mathrm{I} : \mathbb{B} \to [\mathbb{B}]^{<\omega}$ such that if $a \leq b$ then $a \leq x \leq b$ for some $x \in \mathrm{I}(a) \cap \mathrm{I}(b)$.

(FNS) A Boolean algebra $\mathbb{B}$ has the FNS property if there is $\mathrm{s} : \mathbb{B} \to [\mathbb{B}]^{<\omega}$ such that if $a \wedge b = \mathbf{0}$, then $a \leq c$ and $b \leq d$ for some disjoint $c, d \in \mathrm{s}(a) \cap \mathrm{s}(b)$.

These three versions of the Freese–Nation property are equivalent for Boolean algebras. There are many possibilities to generalize the concept of the Freese–Nation property for topological spaces, depending on the subfamily of $\mathcal{P}(X)$ and the existence of finite subsets of $\mathcal{P}(X)$ with certain properties.

We give a proposal of concepts $\mathrm{FNS}, \mathrm{FN}, \mathrm{FNS}^*, \mathrm{FNI}$ for a topological space (in the next section we introduce definitions). We prove that spaces with the FNS* property satisfy ccc and any product of such spaces also satisfies ccc. All metrizable spaces have the FN property. Finally we give a topological proof of Heindorf and Shapiro's result that 0-dimensional openly generated spaces have the FNS property.

All topological spaces under consideration are assumed to be at least Tychonoff. For a topological space X let $\mathrm{Clop}(X)$ denote the set of all clopen sets in X. For the family of sets $\mathcal{B}$ we denote by $[\mathcal{B}]^{<\omega}$ the set of all finite families consisting of elements of the family $\mathcal{B}$. We assume that the readers are familiar with typical notations concerning Boolean algebras (cf., [6]) and unexplained notions can be found in [1].

2. On generalizations of the Freese–Nation property

A family $\mathcal{B}$ of open sets in a topological space X has the FNS *property* if there exists a map $\mathrm{s} : \mathcal{B} \to [\mathcal{B}]^{<\omega}$ such that if $U, V \in \mathcal{B}$ are disjoint then there are disjoint sets $W_U, W_V \in \mathrm{s}(U) \cap \mathrm{s}(V)$ such that $U \subseteq W_U, V \subseteq W_V$. A topological space has the FN *property* if there exists a base $\mathcal{B}$ such that for every $V \in \mathcal{B}$ there are two finite sets $\mathrm{u}(V) \subseteq \{U \in \mathcal{B} : V \subseteq U\}$ and $\mathrm{l}(V) \subseteq \{U \in \mathcal{B} : U \subseteq V\}$ such that if $V \subseteq W$, then $\mathrm{u}(V) \cap \mathrm{l}(W) \neq \emptyset$. We say that a topological space has the FNS *property* whenever there exists a base with the FNS property. We say that a topological space has the FNS* *property* whenever there exists a π-network with the FNS property. A family

$\mathcal{N}$ of subsets of a topological space X is a π-*network* for X if for any open non-empty set U there is an $M \in \mathcal{N}$ such that $M \subseteq U$. We say that a topological space has the FNI *property* whenever there exists a base $\mathcal{B}$ of open sets in a topological space X and there is $\mathrm{I}\colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ such that if $V \subseteq U$ then $V \subseteq W \subseteq U$ for some $W \in \mathrm{I}(V) \cap \mathrm{I}(U)$. It is obvious that the FNI property is equivalent to the FN property for any topological space. Since every base is a π-network we have

Lemma 2.1. *Every space with the* FNS *property has the* FNS* *property.*

It turns out that if the FNS and FN properties are defined on a special basis, they are equivalent.

Proposition 2.2. *If a base $\mathcal{B}$ has the* FNS *property, then the base* $\{\operatorname{int}\operatorname{cl} V : V \in \mathcal{B}\}$ *has also the* FNS *property.*

Proof. Let $\mathrm{s}\colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ be witness on the FNS property. For each $W \in \{\operatorname{int}\operatorname{cl} V : V \in \mathcal{B}\}$ we choose $V_W \in \mathcal{B}$ such that $\operatorname{int}\operatorname{cl}(V_W) = W$ and put $\mathrm{s}'(W) = \{\operatorname{int}\operatorname{cl} U : U \in \mathrm{s}(V_W)\}$. The function s' is as we desired. □

Proposition 2.3. *If a base $\mathcal{B}$, for a topological space X, has the* FNS *property and $X \setminus \operatorname{cl} V \in \mathcal{B}$, whenever $V \in \mathcal{B}$, then X has the* FN *property.*

Proof. Let $\mathrm{s}\colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ be a witness on the property FNS. By Proposition 2.2, without loss of generality, we can assume that $\mathcal{B}$ consists of regular open sets. Put

$$\mathrm{u}(U) = \mathrm{s}(U) \quad \text{and} \quad \mathrm{l}(U) = \mathrm{s}(X \setminus \operatorname{cl} U),$$

for each $U \in \mathcal{B}$. Suppose that $U, V \in \mathcal{B}$ and $U \subseteq V$, i.e. $U \cap (X \setminus \operatorname{cl} V) = \emptyset$. By the FNS property there are disjoint open sets

$$U', V' \in \mathrm{s}(U) \cap \mathrm{s}(X \setminus \operatorname{cl} V) = \mathrm{u}(U) \cap \mathrm{l}(V)$$

such that $U \subseteq U'$ and $X \setminus \operatorname{cl} V \subseteq V'$. Finally we get $U \subseteq U' \subseteq X \setminus \operatorname{cl} V' \subseteq \operatorname{int}\operatorname{cl} V = V$. □

Proposition 2.4. *If a base $\mathcal{B}$, for a topological space X, has the* FN *property and $X \setminus \operatorname{cl} V \in \mathcal{B}$, whenever $V \in \mathcal{B}$, then X has the* FNS *property.*

Proof. Let $\mathrm{u}, \mathrm{l}\colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ be witnesses on the FN property. For each $U \in \mathcal{B}$ let $\mathrm{s}(U)$ be a family

$$\mathrm{u}(U) \cup \mathrm{l}(X \setminus \operatorname{cl} U) \cup \{X \setminus \operatorname{cl} V : V \in \mathrm{u}(U)\} \cup \{X \setminus \operatorname{cl} V : V \in \mathrm{l}(X \setminus \operatorname{cl} U)\}.$$

Suppose $U, V \in \mathcal{B}$ are disjoint, i.e. $U \subseteq X \setminus \operatorname{cl} V$. By the FN property for $\mathcal{B}$, choose

$$W \in \mathrm{u}(U) \cap \mathrm{l}(X \setminus \operatorname{cl} V) \subseteq \mathrm{s}(U) \cap \mathrm{s}(V)$$

such that $U \subseteq W \subseteq X \setminus \operatorname{cl} V$. Then check that $X \setminus \operatorname{cl} W$ belongs to

$$\{X \setminus \operatorname{cl} G : G \in \mathrm{u}(U)\} \cap \{X \setminus \operatorname{cl} G : G \in \mathrm{l}(X \setminus \operatorname{cl} V)\} \subseteq \mathrm{s}(U) \cap \mathrm{s}(V).$$

Sets $W, X \setminus \operatorname{cl} W$ are disjoint and $V \subseteq X \setminus \operatorname{cl} W$ and $U \subseteq W$. □

Any space with countable weight has the FNS and the FN property.

Proposition 2.5. *If $\mathcal{B}$ is a countable base consisting of regular open sets of X, then the base $\mathcal{B} \cup \{X \setminus \operatorname{cl} V : V \in \mathcal{B}\}$ has the* FNS *property.*

Proof. Let $\mathcal{B}$ be a countable base for X consisting of regular open sets. Put $\mathcal{B}_0 = \mathcal{B} \cup \{X \setminus \operatorname{cl} V : V \in \mathcal{B}\}$. Let $\{U_n : n \in \omega\}$ enumerates the elements of $\mathcal{B}_0$. Define a map $\mathrm{s} : \mathcal{B}_0 \to [\mathcal{B}_0]^{<\omega}$ in the following way

$$\mathrm{s}(U_n) = \{U_k : k \leq n\} \cup \{X \setminus \operatorname{cl} U_k : k \leq n\},$$

for any $U_n \in \mathcal{B}_0$. If $U_k \cap U_i = \emptyset$ and $k < i$, then

$$U_k, X \setminus \operatorname{cl} U_k \in \mathrm{s}(U_k) \cap \mathrm{s}(U_i) \quad \text{and} \quad U_k \subseteq U_k \quad \text{and} \quad U_i \subseteq X \setminus \operatorname{cl} U_k$$

and $U_k \cap (X \setminus \operatorname{cl} U_k) = \emptyset$. □

Proposition 2.6. *If a space has a countable base, then this base has the* FN *property.*

Proof. Enumerate a countable base $\mathcal{B} = \{V_n : n \in \omega\}$ and define maps $\mathrm{u}, \mathrm{l} \colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ in the following way

$$\mathrm{u}(V_n) = \{V_k : k \leq n, V_n \subseteq V_k\} \quad \text{and} \quad \mathrm{l}(V_n) = \{V_k : k \leq n, V_k \subseteq V_n\}.$$

Finally we get that if $V_i \subseteq V_j$, then $V_{\min\{i,j\}} \in \mathrm{u}(V_i) \cap \mathrm{l}(V_j)$. □

Corollary 2.7. *The family* $\operatorname{Clop}(X)$ *of a 0-dimensional compact metrizable space has the* FNS *property.*

One can ask if a 0-dimensional compact space with the FNS must also have the FNS property for the base $\mathrm{Clop}(X)$. We can prove only that if there exists a base $\mathcal{B}$ closed under finite intersections with the FNS property, then one can enlarge the base $\mathcal{B}$ to a base $\mathcal{B}'$ with FNS that contains both families $\mathcal{B}$ and $\mathrm{Clop}(X)$. In fact, we know how to define a map $\bar{\mathrm{s}} : \mathcal{B}' \to [\mathcal{B}']^{<\omega}$ which is a witness on the FNS property, but it is not necessarily that $\bar{\mathrm{s}}$ assigns a clopen set to a family of clopen sets.

Proposition 2.8. *Assume that X is a 0-dimensional compact space. If there exists a base $\mathcal{B}$ closed under finite intersections with the* FNS *property, then there exists a base $\mathcal{B}'$ with the* FNS *property such that $\mathcal{B} \cup \mathrm{Clop}(X) \subseteq \mathcal{B}'$ where $\mathcal{B}'$ is closed under finite unions.*

Proof. Let $\mathcal{B}$ be a base closed under finite intersections and let $\mathrm{s}\colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ be a witness on the FNS property. If $\mathcal{S}$ is a family of subsets of X, then let $\mathcal{S}^\wedge$ be the family of all non-empty intersections of finitely many elements of $\mathcal{S}$. Put

$$\bar{\mathrm{s}}(V_1 \cup \ldots \cup V_n) = \Big\{ \bigcup \mathcal{R} : \mathcal{R} \subseteq (\mathrm{s}(V_1) \cup \ldots \cup \mathrm{s}(V_n))^\wedge \Big\},$$

for each $V_1, \ldots, V_n \in \mathcal{B}$. Let $\mathcal{B}' = \{\bigcup \mathcal{R} : \mathcal{R} \in [\mathcal{B}]^{<\omega}\}$. Observe that $\mathcal{B} \cup \mathrm{Clop}(X) \subseteq \mathcal{B}'$. Note that $\bar{\mathrm{s}}(V_1 \cup \ldots \cup V_n)$ is closed under finite intersections. Indeed, if $\mathcal{R}_1, \mathcal{R}_2 \subseteq (\mathrm{s}(V_1) \cup \ldots \cup \mathrm{s}(V_n))^\wedge$, then

$$\mathcal{R} = \{V \cap U : V \in \mathcal{R}_1 \text{ and } U \in \mathcal{R}_2\} \quad \text{and} \quad \bigcup \mathcal{R}_1 \cap \bigcup \mathcal{R}_2 = \bigcup \mathcal{R}.$$

We shall prove that $\bar{\mathrm{s}}\colon \mathcal{B}' \to [\mathcal{B}']^{<\omega}$ is a witness on the FNS property. Suppose that $V_1, \ldots, V_n, U_1, \ldots, U_k \in \mathcal{B}$ are such that

$$(V_1 \cup \ldots \cup V_n) \cap (U_1 \cup \ldots \cup U_k) = \emptyset.$$

For each V_i there exist families $\mathcal{T}(V_i), \mathcal{R}(V_i) \subseteq \mathrm{s}(V_i) \cap (\mathrm{s}(U_1) \cup \ldots \cup \mathrm{s}(U_k))$ such that

$$V_i \subseteq \bigcap \mathcal{R}(V_i) \in (\mathrm{s}(U_1) \cup \ldots \cup \mathrm{s}(U_k))^\wedge \cap (\mathrm{s}(V_i))^\wedge$$

and

$$U_1 \cup \ldots \cup U_k \subseteq \bigcup \mathcal{T}(V_i) \in \bar{\mathrm{s}}(V_1 \cup \ldots \cup V_n) \cap \bar{\mathrm{s}}(U_1 \cup \ldots \cup U_k)$$

and $\bigcup\mathcal{T}(V_i)\cap\bigcap\mathcal{R}(V_i)=\emptyset$. Families $\bar{\mathrm{s}}(V_1\cup\ldots\cup V_n),\bar{\mathrm{s}}(U_1\cup\ldots\cup U_k)$ are closed under finite intersections, hence we get

$$\bigcup\mathcal{T}(V_1)\cap\ldots\cap\bigcup\mathcal{T}(V_n)\in\bar{\mathrm{s}}(V_1\cup\ldots\cup V_n)\cap\bar{\mathrm{s}}(U_1\cup\ldots\cup U_k).$$

We have also

$$V_1\cup\ldots\cup V_n\subseteq\bigcap\mathcal{R}(V_1)\cup\ldots\cup\bigcap\mathcal{R}(V_n)\in\bar{\mathrm{s}}(V_1\cup\ldots\cup V_n)\cap\bar{\mathrm{s}}(U_1\cup\ldots\cup U_k)$$

and $(\bigcap\mathcal{R}(V_1)\cup\ldots\cup\bigcap\mathcal{R}(V_n))\cap(\bigcup\mathcal{T}(V_1)\cap\ldots\cap\bigcup\mathcal{T}(V_n))=\emptyset$ and $U_1\cup\ldots\cup U_k\subseteq\bigcup\mathcal{T}(V_1)\cap\ldots\cap\bigcup\mathcal{T}(V_n)$. □

Now we prove that the FNS property is preserved by products.

THEOREM 2.9. *The product of spaces with the* FNS *property has the* FNS *property.*

PROOF. Let $X=\prod\{X_i : i\in A\}$, where each space X_i has the FNS property. Fix a base $\mathcal{B}_i$ and $\mathrm{s}_i\colon\mathcal{B}_i\to[\mathcal{B}_i]^{<\omega}$ which are witnesses on the FNS property for X_i, for each $i\in A$. We shall show that the base

$$\mathcal{B}=\Big\{\bigcap_{k\in E}\mathrm{pr}_k^{-1}(U_k) : E\in[A]^{<\omega}\text{ and }U_k\in\mathcal{B}_k\text{ for every }k\in E\Big\}$$

has the FNS property. Define $\mathrm{s}\colon\mathcal{B}\to[\mathcal{B}]^{<\omega}$ in the following way

$$\mathrm{s}(U)=\{\mathrm{pr}_k^{-1}(U') : U'\in\mathrm{s}_k(U_k)\text{ and }k\in E_U\}$$

for each $U=\bigcap_{k\in E_U}\mathrm{pr}_k^{-1}(U_k)\in\mathcal{B}$. Fix disjoint sets $U,V\in\mathcal{B}$ where

$$U=\bigcap_{k\in E_U}\mathrm{pr}_k^{-1}(U_k)\quad\text{and}\quad V=\bigcap_{k\in E_V}\mathrm{pr}_k^{-1}(V_k).$$

There exists $k\in E_U\cap E_V$ such that $U_k\cap V_k=\emptyset$. Hence there exist $U',V'\in\mathrm{s}_k(U_k)\cap\mathrm{s}_k(V_k)$ such that $U_k\subseteq U',V_k\subseteq V'$ and $U'\cap V'=\emptyset$. Therefore

$$U\subseteq\mathrm{pr}_k^{-1}(U')\quad\text{and}\quad V\subseteq\mathrm{pr}_k^{-1}(V')\quad\text{and}\quad\mathrm{pr}_k^{-1}(U')\cap\mathrm{pr}_k^{-1}(V')=\emptyset$$

and $\mathrm{pr}_k^{-1}(U'),\mathrm{pr}_k^{-1}(V')\in\mathrm{s}(U)\cap\mathrm{s}(V)$. □

It turns out that the FN property is also preserved by products, the proof is similar to the proof of Theorem 2.9. We give a proof for the reader's convenience.

THEOREM 2.10. *The product of spaces with the* FN *property has the* FN *property.*

PROOF. Let $X = \prod\{X_i : i \in A\}$ where each space X_i has the FN property. Fix a base $\mathcal{B}_i$ and $\mathrm{u}_i, \mathrm{l}_i \colon \mathcal{B}_i \to [\mathcal{B}_i]^{<\omega}$ which are witnesses on the FN property for X_i, for each $i \in A$ We shall show that the base

$$\mathcal{B} = \Big\{ \bigcap_{k\in E} \mathrm{pr}_k^{-1}(U_k) : E \in [A]^{<\omega} \text{ and } U_k \in \mathcal{B}_k \text{ for every } k \in E \Big\}$$

has the FN property. Define $\mathrm{l}, \mathrm{u} \colon \mathcal{B} \to [\mathcal{B}]^{<\omega}$ in the following way

$$\mathrm{l}(U) = \Big\{ \bigcap_{k\in E} \mathrm{pr}_k^{-1}(U'_k) : U'_k \in \mathrm{l}_k(U_k) \text{ and } k \in E_U \Big\}$$

and

$$\mathrm{u}(U) = \Big\{ \bigcap_{k\in H} \mathrm{pr}_k^{-1}(U'_k) : U'_k \in \mathrm{u}_k(U_k) \text{ and } k \in H \subseteq E_U \Big\}$$

for every set $U = \bigcap_{k\in E_U} \mathrm{pr}_k^{-1}(U_k) \in \mathcal{B}$. Assume that $U, V \in \mathcal{B}, U \subseteq V$ and

$$U = \bigcap_{k\in E_U} \mathrm{pr}_k^{-1}(U_k) \quad \text{and} \quad V = \bigcap_{k\in E_V} \mathrm{pr}_k^{-1}(V_k).$$

Then $E_V \subseteq E_U$ and $U_k \subseteq V_k$ for every $k \in E_V$. Hence there exists $W_k \in \mathrm{u}_k(U_k) \cap \mathrm{l}_k(V_k)$ such that $U_k \subseteq W_k \subseteq V_k$ for every $k \in E_V$. Therefore $U \subseteq \bigcap_{k\in E_V} \mathrm{pr}_k^{-1}(W_k) \subseteq V$. $\square$

Similarly, one can prove the following.

COROLLARY 2.11. *The product of spaces with the* FNS* *property has the* FNS* *property.*

Recall that a topological space satisfies *the countable chain condition* (briefly ccc) if every family of pairwise disjoint non-empty open sets is countable. R. Laver has shown that under the continuum hypothesis, there are two topological spaces satisfying ccc and the product of them does not satisfy ccc. A simple proof of Laver's result was given by F. Galvin in [5]. Following F. Galvin [5] let us add that: K. Kunen, F. Rowbottom and R. M. Solovay independently showed that under Martin's Axiom and the negation of the continuum hypothesis the product of arbitrarily many ccc spaces is ccc.

We will need the Δ-lemma.

Theorem (Δ-lemma). *For any uncountable family $\mathcal{R}$ of finite sets there exists an uncountable family $\mathcal{R}' \subseteq \mathcal{R}$ and a set J such that $A \cap B = J$ for any different sets $A, B \in \mathcal{R}'$.*

Theorem 2.12. *Every space with the* FNS* *property satisfies the countable chain condition.*

Proof. Let $\mathcal{N}$ be a π-network with the FNS property i.e. there exists some $\mathrm{s}\colon \mathcal{N} \to [\mathcal{N}]^{<\omega}$ witnessing on the property FNS. Suppose that there exists an uncountable family $\mathcal{A}$ of pairwise disjoint open sets. For each $V \in \mathcal{A}$ we choose $F_V \in \mathcal{N}$ such that $F_V \subseteq V$. Therefore the family $\mathcal{P} = \{F_V : V \in \mathcal{A}\} \subseteq \mathcal{N}$ is uncountable and consists of pairwise disjoint sets.

Suppose that the collection $\{\mathrm{s}(F) : F \in \mathcal{P}\}$ is countable, then there exists a set $F_0 \in \mathcal{P}$ such that a family $\mathcal{P}' = \{G \in \mathcal{P} : \mathrm{s}(G) = \mathrm{s}(F_0)\}$ is uncountable. Let $\mathrm{s}(F_0) = \{H_1, \ldots, H_n\} \subseteq \mathcal{N}$ and for every $G \in \mathcal{P}'$ define $I_G = \{i \in \{1, \ldots, n\} : G \subseteq H_i\}$. By the FNS property the set I_G is non-empty for each $G \in \mathcal{P}'$. There exists $G_0 \in \mathcal{P}'$ such that $I_{G_0} = I_G$ for some uncountable many sets $G \in \mathcal{P}'$. Let $G_1, G_2 \in \mathcal{P}'$ be such that $I_{G_1} = I_{G_2} = I_{G_0}$ and $G_1 \neq G_2$. Since $G_1 \cap G_2 = \emptyset$, then there are $G_1', G_2' \in \mathrm{s}(G_1) = \mathrm{s}(G_2) = \mathrm{s}(F_0)$ such that $G_1 \subseteq G_1', G_2 \subseteq G_2'$ and $G_1' \cap G_2' = \emptyset$. On the other hand $G_1, G_2 \subseteq \bigcap\{H_i : i \in I_{G_0}\} \subseteq G_1' \cap G_2'$, a contradiction. Hence the family $\{\mathrm{s}(F) : F \in \mathcal{P}\}$ is uncountable.

By Δ-lemma there exists an uncountable family $\mathcal{R} \subseteq \{\mathrm{s}(F) : F \in \mathcal{P}\}$ and a set J such that $\mathrm{s}(G_1) \cap \mathrm{s}(G_2) = J$ for any different sets $G_1, G_2 \in \mathcal{P}' = \{F \in \mathcal{P} : \mathrm{s}(F) \in \mathcal{R}\}$. The family $\mathcal{P}'$ is obviously uncountable and the set J is non-empty and finite. Let $J = \{H_1, \ldots, H_n\}$. By the FNS property for each $G \in \mathcal{P}'$ there exists $i \leq n$ such that $G \subseteq H_i$. The rest of the proof is analogous to the first part of the proof, but for the sake of completeness we will repeat it. For every $G \in \mathcal{P}'$ we define $I_G = \{i \in \{1, \ldots, n\} : G \subseteq H_i\}$. There exists $G_0 \in \mathcal{P}'$ such that $I_{G_0} = I_G$ for some uncountable many sets $G \in \mathcal{P}'$. Let $G_1, G_2 \in \mathcal{P}'$ be such that $I_{G_1} = I_{G_2} = I_{G_0}$ and $G_1 \neq G_2$. Since $G_1 \cap G_2 = \emptyset$, then there are $G_1', G_2' \in \mathrm{s}(G_1) \cap \mathrm{s}(G_2) = J$ such that $G_1 \subseteq G_1', G_2 \subseteq G_2'$ and $G_1' \cap G_2' = \emptyset$. On the other hand $G_1, G_2 \subseteq \bigcap\{H_i : i \in I_{G_0}\} \subseteq G_1' \cap G_2'$, a contradiction. Hence the family $\mathcal{P}$ has to be countable. $\square$

By Lemma 2.1 and Theorem 2.12 we get the following corollary.

Corollary 2.13. *Every space with the* FNS *property satisfies the countable chain condition.*

The following corollary is an immediate consequence of Theorem 2.9 and Corollary 2.13.

Corollary 2.14. *The product of topological spaces with the* FNS *property satisfies the countable chain condition.*

We say that a cover $\mathcal{R}$ is a *refinement* of a cover $\mathcal{P}$ if for each $U \in \mathcal{R}$ there is $V \in \mathcal{P}$ such that $U \subseteq V$.

THEOREM (The Stone theorem [1, 4.4.1]). *Every open cover $\mathcal{R}$ of a metrizable space has an open refinement $\mathcal{B}$ which is locally finite and σ-discrete and if $W \subseteq V$ then $W = V$ for all $W, V \in \mathcal{B}$.*

PROOF. Let $\mathcal{R}$ be an open cover of X. Recalling the standard proof of the Stone theorem [1, 4.4.1], one can define a locally finite refinement

$$\mathcal{B} = \bigcup\{\mathcal{P}_n : n \in \omega\}, \text{ where } \mathcal{P}_n = \{H(U,n) : U \in \mathcal{R}\} \text{ is discrete}$$

and

$$H(U,n) = \bigcup\Big\{K(x, \frac{1}{2^n}) : K(x, \frac{3}{2^n}) \subseteq U, x \notin V \text{ for } V \prec U \\ \text{and } x \notin \bigcup\big\{\bigcup\mathcal{P}_i : i < n\big\}\Big\},$$

where $\prec$ is a well-order relation on $\mathcal{R}$. We say that a point $x \in X$ is an *essential* point for $H(U,n)$ whenever

$$K(x, \frac{3}{2^n}) \subseteq U, x \notin V \text{ for } V \prec U \quad \text{and} \quad x \notin \bigcup\big\{\bigcup\mathcal{P}_i : i < n\big\}.$$

Let $\emptyset \neq H(V,s) \in \mathcal{P}_s, \emptyset \neq H(U,p) \in \mathcal{P}_p$, then $H(V,s) \subseteq H(U,p)$ if and only if $V = U$ and $s = p$. Indeed, let $H(V,s) \subseteq H(U,p)$. Then there exist an essential point c for $H(V,s)$ and an essential point a for $H(U,p)$ such that $c \in K(a, \frac{1}{2^p}) \subseteq H(U,p)$. Suppose that $s < p$. Then $a \in K(c, \frac{1}{2^p}) \subseteq K(c, \frac{1}{2^s}) \subseteq H(V,s) \subseteq \bigcup\mathcal{P}_s$, a contradiction, because $a \notin \bigcup\mathcal{P}_s$. Now suppose that $p < s$. Then $c \in K(a, \frac{1}{2^p}) \subseteq H(U,p) \subseteq \bigcup\mathcal{P}_p$, a contradiction, because $c \notin \bigcup\mathcal{P}_p$. Therefore we get $p = s$. Since $\mathcal{P}_s$ is a discrete family we get $V = U$. □

The next theorem contrasts with Corollary 2.13 and the case of 0-dimensional compact spaces.

THEOREM 2.15. *Every metrizable space has the* FN *property.*

PROOF. We define a base $\mathcal{B} = \bigcup\{\mathcal{B}_k : k \in \omega\}$ such that $\mathcal{B}_{k+1}$ is an open refinement of $\mathcal{B}_k$ consisting of sets with diameter $\leq \frac{1}{k+1}$, each $\mathcal{B}_k$ is locally finite and σ-discrete and if $W, V \in \mathcal{B}_k$ and $W \subseteq V$ then $W = V$. Let $\mathcal{R}_0$ be a cover consisting of balls with radius ≤ 1. By the Stone theorem the cover $\mathcal{R}_0$ has an open refinement $\mathcal{B}_0$ which is locally finite and σ-discrete and if $W \subseteq V$ then $W = V$ for all $W, V \in \mathcal{B}_0$.

Assume that we have just defined $\mathcal{B}_0, \dots, \mathcal{B}_k$ such that for each $i < k$ the family $\mathcal{B}_{i+1}$ is an open refinement of $\mathcal{B}_i$ consisting of sets with diameter $\le \frac{1}{i+1}$, $\mathcal{B}_i$ is locally finite and σ-discrete and if $W, V \in \mathcal{B}_i$ and $W \subseteq V$ then $W = V$.

The cover $\mathcal{B}_k$ has a refinement $\mathcal{R}_{k+1}$ consisting of balls with radius $\le \frac{1}{k+2}$. By the Stone theorem we get an open refinement $\mathcal{B}_{k+1}$ which is locally finite and σ-discrete and if $W, V \in \mathcal{B}_{k+1}$ and $W \subseteq V$ then $W = V$.

A family $\mathcal{B} = \bigcup\{\mathcal{B}_k : k \in \omega\}$ has the required properties. For each $U \in \mathcal{B}$ we define the sets $\mathrm{l}(U)$ and $\mathrm{u}(U)$ by the following formula

$$\mathrm{l}(U) = \{U\} \quad \text{and} \quad \mathrm{u}(U) = \{W \in \mathcal{B}_i : U \subseteq W, i \le \min\{n \in \omega : U \in \mathcal{B}_n\}\}.$$

Assume that $\min\{n \in \omega : U \in \mathcal{B}_n\} = k$. Since $\{W \in \mathcal{B}_k : U \subseteq W\} = \{U\}$ and each $\mathcal{B}_i$ is locally finite the set $\mathrm{u}(U)$ is finite.

Let $U, V \in \mathcal{B}$ and $U \subseteq V$. Put $n = \min\{j \in \omega : U \in \mathcal{B}_j\}$ and $k = \min\{j \in \omega : V \in \mathcal{B}_j\}$. Suppose that $n < k$. Since $\mathcal{B}_k$ is the refinement of $\mathcal{B}_n$, there exists $V' \in \mathcal{B}_n$ such that $V \subseteq V'$. Then $U = V' = V$ and $U \in \mathrm{l}(V) \cap \mathrm{u}(U)$. If $k \le n$, then $V \in \mathrm{l}(V) \cap \mathrm{u}(U)$, this completes the proof. $\square$

We showed that the properties FN and FNS are not equivalent.

3. Openly generated spaces

L. Heindorf and L.B. Shapiro [6] proved that Boolean algebras with a club consisting of countable relatively complete Boolean subalgebras have the FNS property. We prove this result in the language of topological spaces, namely the family $\mathrm{Clop}(X)$ of 0-dimensional openly generated space X has the FNS property.

We say that X is an *openly generated space* if $X = \varprojlim\{X_\sigma, p^\sigma_\rho, \Sigma\}$ and $\{X_\sigma, p^\sigma_\rho, \Sigma\}$ is a continuous σ-complete inverse system consisting of compact metrizable spaces X_σ and open maps p^σ_ρ. If $X = \varprojlim\{X_\alpha, p^\beta_\alpha, \alpha < \beta < \kappa\}$ is a continuous inverse limit consisting of compact spaces X_α and open maps p^β_α, where $\alpha < \beta < \kappa = \mathrm{w}(X)$, then for any $U \subseteq X$ we define the set

$$\mathrm{d}(U) = \{\alpha < \kappa : (p_{\alpha+1})^{-1}(p_{\alpha+1}(U)) \subsetneq (p_\alpha)^{-1}(p_\alpha(U))\}.$$

The above definition was introduced by E. V. Shchepin [11].

Lemma 3.1 ([6, Lemma 2.1.3]). *Let $X = \varprojlim\{X_\alpha, p^\beta_\alpha, \alpha < \beta < \kappa\}$ be a continuous inverse limit of compact Hausdorff spaces X_α and open maps*

p_α^β*, where* $\kappa = \mathrm{w}(X)$*. If* U, V *are disjoint clopen sets then* $p_0(U) \cap p_0(V) = \emptyset$ *or* $p_{\alpha+1}(U) \cap p_{\alpha+1}(V) = \emptyset$ *for some* $\alpha \in \mathrm{d}(U) \cap \mathrm{d}(V)$*.*

PROOF. Assume that U, V are disjoint clopen sets and $p_0(U) \cap p_0(V) \neq \emptyset$. There exists the minimal $\beta < \kappa$ such that $p_\beta(U) \cap p_\beta(V) = \emptyset$. The ordinal number β is not a limit ordinal. Suppose β is a limit ordinal. Since $p_\beta(U), p_\beta(V)$ are disjoint clopen sets in X_β there is $\gamma < \beta$ such that $p_\gamma(U) \cap p_\gamma(V) = \emptyset$, a contradiction. Let $\beta = \alpha + 1$. Suppose that $\alpha \notin \mathrm{d}(V)$. Thus $(p_\alpha)^{-1}(p_\alpha(V)) = (p_{\alpha+1})^{-1}(p_{\alpha+1}(V))$ and

$$U \subseteq (p_{\alpha+1})^{-1}(p_{\alpha+1}(U)) \subseteq X \setminus (p_{\alpha+1})^{-1}(p_{\alpha+1}(V)) = X \setminus (p_\alpha)^{-1}(p_\alpha(V)).$$

Finally, $U \cap (p_\alpha)^{-1}(p_\alpha(V)) = \emptyset$ and hence $p_\alpha(U) \cap p_\alpha(V) = \emptyset$, a contradiction with the minimality of β. □

LEMMA 3.2. *In a compact space the intersection of any strictly decreasing sequence* $\{U_n : n \in \omega\}$ *of clopen sets is not clopen.*

PROOF. Suppose $\{U_n : n \in \omega\}$ is a sequence of clopen sets such that $U_{n+1} \subsetneq U_n$ for all $n \in \omega$ and $\bigcap\{U_n : n \in \omega\} = U$ is clopen. Then $X = (X \setminus U_0) \cup \bigcup\{U_n \setminus U_{n+1} : n \in \omega\} \cup U$, a contradiction with compactness of X. □

It is easy to prove the following lemma.

LEMMA 3.3. *Let* $f : X \to Y$ *be a continuous surjection between Hausdorff spaces. The map* f *is open if and only if for any open non-empty set* $U \subseteq X$ *there is the minimal open set* $V \subseteq Y$ *with respect to "*$\subseteq$*" such that* $f(U) \subseteq V$*, i.e. if* $W \subseteq Y$ *is an open subset and* $f(U) \subseteq W$ *then* $V \subseteq W$*. Moreover, if* X *and* Y *are 0-dimensional compact spaces then* f *is open if and only if for any clopen non-empty set* $U \subseteq X$ *there is the minimal clopen set* $V \subseteq Y$ *with respect to "*$\subseteq$*" such that* $f(U) \subseteq V$*.*

LEMMA 3.4 ([6, Lemma 2.1.2 and 2.1.3]). *Let* $X = \varprojlim\{X_\alpha, p_\alpha^\beta, \alpha < \beta < \kappa\}$ *be a continuous inverse limit of compact spaces* X_α *and open maps* p_α^β*, where* $\alpha < \beta < \kappa = \mathrm{w}(X)$*. Then the set* $\mathrm{d}(U)$ *is finite for every clopen set* U*.*

PROOF. Let $U \subseteq X$ be a clopen non-empty set. Suppose that $\mathrm{d}(U)$ is infinite. Let $\{\alpha_n : n \in \omega\} \subseteq \mathrm{d}(U)$ be an increasing sequence and $\alpha = \sup\{\alpha_n : n \in \omega\}$. Then $(p_\alpha)^{-1}(p_\alpha(U)) = \bigcap\{(p_{\alpha_n})^{-1}(p_{\alpha_n}(U)) : n \in \omega\}$ and $\{(p_{\alpha_n})^{-1}(p_{\alpha_n}(U)) : n \in \omega\}$ is the strictly decreasing sequence of clopen sets, a contradiction with Lemma 3.2. □

The next lemma we prove only in the 0-dimensional case but it is true without this restriction. This lemma is similar to [13, Lemma 2.6] but we do not assume that B is σ-complete.

Lemma 3.5. *Let* $X = \varprojlim\{X_\sigma, p^\sigma_\rho, \Sigma\}$ *and* $\{X_\sigma, p^\sigma_\rho, \Sigma\}$ *be a continuous σ-complete inverse system consisting of 0-dimensional compact metrizable spaces* X_σ *and open maps* p^σ_ρ. *If* $B \subseteq \Sigma$ *is an upward directed set then* $p_B \colon X \to X_B$ *given by the formula* $p_B((x_a)_{a\in\Sigma}) = (x_a)_{a\in B}$, *where* $X_B = \varprojlim\{X_\sigma, p^\sigma_\rho, B\}$ *is an open map.*

Proof. Suppose that $p_B \colon X \to X_B$ is not open. By Lemma 3.3 there is a clopen set $U \subseteq X$, such that there is no minimal $V \in \mathrm{Clop}(X_B)$ with $p_B(U) \subseteq V$. Inductively we construct a chain $\{b_n : n \in \omega\} \subseteq B$ such that

$$p_B(U) \subseteq (p^B_{b_{n+1}})^{-1}(p_{b_{n+1}}(U)) \subsetneq (p^B_{b_n})^{-1}(p_{b_n}(U)).$$

Assume that we have just constructed $b_0, \ldots, b_n \in B$. According to Lemma 3.3 $(p^B_{b_n})^{-1}(p_{b_n}(U))$ is not the minimal clopen subset of X_B such that $p_B(U) \subseteq (p^B_{b_n})^{-1}(p_{b_n}(U))$, thus there exists a clopen set $V \subseteq X_B$ such that

$$p_B(U) \subseteq V \subsetneq (p^B_{b_n})^{-1}(p_{b_n}(U)).$$

Since V is the clopen set there exists $b_{n+1} \in B$ such that $b_{n+1} > b_n$ and $V = (p^B_{b_{n+1}})^{-1}(p^B_{b_{n+1}}(V))$. Therefore, $p_{b_{n+1}}(U) \subseteq p^B_{b_{n+1}}(V)$ and

$$p_B(U) \subseteq (p^B_{b_{n+1}})^{-1}(p_{b_{n+1}}(U)) \subseteq V \subsetneq (p^B_{b_n})^{-1}(p_{b_n}(U)).$$

Put $b = \sup\{b_n : n \in \omega\}$. Since $(p^B_{b_{n+1}})^{-1}(p_{b_{n+1}}(U)) \subsetneq (p^B_{b_n})^{-1}(p_{b_n}(U))$ we have $p_{b_{n+1}}(U) \subsetneq (p^{b_{n+1}}_{b_n})^{-1}(p_{b_n}(U))$ for any $n \in \omega$. Hence we have

$$p_b(U) = \bigcap\{(p^b_{b_n})^{-1}(p_{b_n}(U)) : n \in \omega\}$$

and the sequence $\{(p^b_{b_n})^{-1}(p_{b_n}(U)) : n \in \omega\}$ is the strictly decreasing sequence of clopen sets, a contradiction with Lemma 3.2. $\square$

Theorem 3.6 ([6, Theorem 2.2.3]). *Let X be a 0-dimensional openly generated space. Then the family* $\mathrm{Clop}(X)$ *has the* FNS *property.*

PROOF. Let $X = \varprojlim\{X_\sigma, p^\sigma_\rho, \Sigma\}$ and $\{X_\sigma, p^\sigma_\rho, \Sigma\}$ be a continuous σ-complete inverse system consisting of 0-dimensional compact metrizable spaces X_σ and open maps p^σ_ρ. We are going to prove that the family of all clopen sets in $X_B = \varprojlim\{X_\sigma, p^\sigma_\rho, B\}$ has the FNS property by transfinite induction with respect to the cardinality of upward directed sets $B \subseteq \Sigma$. This is true if B is countable by Corollary 2.7. Assume that $\mathrm{Clop}(X_A)$ has the FNS property for any upward directed set A of cardinality less than τ, where $\tau \leq \mathrm{w}(X)$ is an uncountable cardinal. Suppose $B \subseteq \Sigma$ is an upward directed set of cardinality τ. Then, according to [7] (or [8]) there exists a sequence $\{B_\alpha : \alpha < \tau\}$ of upward directed sets such that:

(1) $|B_\alpha| = |\alpha| + \omega$ for $\alpha < \tau$,
(2) $B_\alpha \subseteq B_\beta$ for $\alpha < \beta < \tau$,
(3) $B = \bigcup\{B_\alpha : \alpha < \tau\}$.

Due to our assumption for each $\alpha < \tau$ there exists $\mathrm{s}_\alpha : \mathrm{Clop}(X_{B_\alpha}) \to [\mathrm{Clop}(X_{B_\alpha})]^{<\omega}$ which is witness on the FNS property for $\mathrm{Clop}(X_{B_\alpha})$. Define $\mathrm{s}_B : \mathrm{Clop}(X_B) \to [\mathrm{Clop}(X_B)]^{<\omega}$ in the following way:

$$\mathrm{s}_B(U) = \big\{(p^B_{B_0})^{-1}(V) : V \in \mathrm{s}_0\big(p^B_{B_0}(U)\big)\big\} \cup \bigcup\big\{\{(p^B_{B_{\alpha+1}})^{-1}(V) : V \in \mathrm{s}_{\alpha+1}\big(p^B_{B_{\alpha+1}}(U)\big)\} : \alpha \in \mathrm{d}(U)\big\}.$$

By Lemma 3.5 a map $p^B_{B_\alpha}$ is open for any $\alpha < \tau$. Hence $p^B_{B_{\alpha+1}}(U)$ is clopen for any clopen set U. According to Lemma 3.4 the set $\mathrm{d}(U)$ is finite, so the set $\mathrm{s}_B(U)$ is finite and well defined. Now assume that $U, V \subseteq X_B$ are disjoint clopen sets. By Lemma 3.1 we have

$$p^B_{B_0}(U) \cap p^B_{B_0}(V) = \emptyset \quad \text{or} \quad p^B_{B_{\alpha+1}}(U) \cap p^B_{B_{\alpha+1}}(V) = \emptyset$$

for some $\alpha \in \mathrm{d}(U) \cap \mathrm{d}(V)$. Therefore there exist disjoint sets $V', U' \in \mathrm{s}_B(U) \cap \mathrm{s}_B(V)$ such that $U \subseteq U'$ and $V \subseteq V'$. $\square$

We establish that spaces with the FNS property have some properties that belong to openly generated compact spaces. L. Heindorf and L.B. Shapiro [6] showed that a family of all clopen sets of 0-dimensional compact space X has the FNS property if and only if X is openly generated. This raises questions about the further properties of spaces with the FNS property. For example, the referee suggested the following questions:

(1) Do retracts preserve the FNS?
(2) Do continuous open surjections preserve the FNS?
(3) Do symmetric powers preserve the FNS?
(4) Does the Vietoris hyperspace operation preserve the FNS?

(5) If Y is a continuous image of a compact FNS space, must its π-character be equal to its weight?

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