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A NOTE ON MULTIPLICATIVE (GENERALIZED) (α, β)-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with center $Z(R)$. A map $G\colon R \to R$ is called a multiplicative (generalized) (α, β)-derivation if $G(xy) = G(x)\alpha(y) + \beta(x)g(y)$ is fulfilled for all $x, y \in R$, where $g\colon R \to R$ is any map (not necessarily derivation) and $\alpha, \beta\colon R \to R$ are automorphisms. Suppose that G and H are two multiplicative (generalized) (α, β)-derivations associated with the mappings g and h, respectively, on R and α, β are automorphisms of R. The main objective of the present paper is to investigate the following algebraic identities: (i) $G(xy) + \alpha(xy) = 0$, (ii) $G(xy) + \alpha(yx) = 0$, (iii) $G(xy) + G(x)G(y) = 0$, (iv) $G(xy) = \alpha(y) \circ H(x)$ and (v) $G(xy) = [\alpha(y), H(x)]$ for all x, y in an appropriate subset of R.

1. Introduction

Throughout the present paper, R will denote an associative ring with centre $Z(R)$ and α, β will denote automorphisms on R. For given $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. For any pair $x, y \in R$ we shall write $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for $a \in R$, $aRa = (0)$ implies $a = 0$. An additive map δ from R to R

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is called a derivation of R if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in R$. Let $F\colon R \to R$ be a map associated with another map $\delta\colon R \to R$ such that $F(xy) = F(x)y + x\delta(y)$ holds for all $x, y \in R$. If F is additive and δ is a derivation of R, then F is said to be a generalized derivation of R – a concept introduced by Brešar ([4]). In [9], Hvala gave the algebraic study of generalized derivations of prime rings. We note that if R has the property that $Rx = (0)$ implies $x = 0$ and $\psi\colon R \to R$ is any function, and $\chi\colon R \to R$ is any additive map such that $\chi(xy) = \psi(x)y + x\psi(y)$ for all $x, y \in R$, then χ is uniquely determined by ψ and ψ must be a derivation by [4, Remark 1]. Obviously, every derivation is a generalized derivation of R. Thus, generalized derivations cover both the concept of derivations and left multiplier maps. Following [5], a multiplicative derivation of R is a map $G\colon R \to R$ which satisfies $G(xy) = G(x)y + xG(y)$ for all $x, y \in R$. Of course these maps need not be additive. To the best of our knowledge, the concept of multiplicative derivations appears for the first time in the work of Daif ([5]) and it was motivated by the work of Martindale ([10]). Further, the complete description of those maps was given by Goldmann and Šemrl in [8]. Such maps do indeed exist in the literature (viz. [5] and [8] where further references can be found). Daif and Tammam El-Sayiad ([6]) extended multiplicative generalized derivations as follows: a map $G\colon R \to R$ is called a multiplicative generalized derivation if there exists a derivation g such that $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$. In this definition, if we consider that g is any map that is not necessarily a derivation or additive, then G is said to be multiplicative (generalized)-derivation which was introduced by Dhara and Ali ([7]). Thus, a map $G\colon R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $G(xy) = G(x)y + xg(y)$ holds for all $x, y \in R$, where g is any map (not necessarily a derivation or an additive map). Hence, the concept of a multiplicative (generalized)-derivation covers the concept of a multiplicative derivation. Moreover, multiplicative (generalized)-derivation with $g = 0$ covers the notion of multiplicative centralizers (not necessarily additive). The examples of multiplicative (generalized)-derivations are multiplicative derivations and multiplicative centralizers. Let S be a nonempty subset of R. A mapping $f\colon R \to R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. In this direction, Posner ([11]) was the first who investigate commutativity of the ring. More precisely, he proved that: If R is a prime ring with a nonzero derivation δ on R such that δ is centralizing on R, then R is commutative.

Further, regarding commutativity in prime rings, Ashraf and Rehman ([3]), proved the following: let R be a prime ring and I a non-zero ideal of R. Suppose that δ is a non-zero derivation on R. If one of the following holds: (i) $\delta(xy) + xy \in Z(R)$; (ii) $\delta(xy) - xy \in Z(R)$ for all $x, y \in I$, then R must be commutative. Further, Ashraf et al. ([2]) extended their work, replacing the derivation δ with a generalized derivation F in a prime ring R. More precisely, they proved the following: Let R be a prime ring and I a non-zero ideal of R. Suppose F is a generalized derivation associated with a nonzero derivation δ on R. If one of the following holds: (i) $F(xy) \pm xy \in Z(R)$; (ii) $F(xy) \pm yx \in Z(R)$; (iii) $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then R is commutative. Recently, Albas ([1]) studied the above mentioned identities in prime rings with central values.

Recently, Dhara and Ali ([7]) studied the following identities related to multiplicative (generalized)-derivations on semiprime rings: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(x)F(y) \pm xy \in Z(R)$, (iv) $F(x)F(y) \pm yx \in Z(R)$ for all x, y in some suitable subset of a semiprime ring R.

In the present paper, we generalize the concept of a multiplicative (generalized)-derivation to a multiplicative (generalized)-(α, β)-derivation. A mapping $G\colon R \to R$ (not necessarily additive) is called a multiplicative (generalized)-(α, β)-derivation of R, if $G(xy) = G(x)\alpha(y) + \beta(x)g(y)$ for all $x, y \in R$, where $g\colon R \to R$ is any map (not necessarily additive) and $\alpha, \beta\colon R \to R$ are automorphisms of R. One can find an example of a multiplicative generalized derivation, which is neither a derivation nor a generalized derivation.

Example 1.1. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta\colon R \to R$ by

$$G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is straightforward to verify that G is not an additive map in R. Hence, G is a multiplicative (generalized)-(α, β)-derivation associated with the mapping g on R, but G is neither a generalized derivation nor a multiplicative (generalized)-derivation of R.

In the present paper, our aim is to investigate some identities with multiplicative (generalized)-$(\alpha; \beta)$-derivations on some suitable subsets in prime rings.

2. Main Results

We begin our discussion with the following lemma.

Lemma 2.1. *Let R be a prime ring and I be a nonzero left ideal of R. Let α,β be automorphisms of R. If $[x,y]_{\alpha,\beta}=0$ for all $x,y\in I$, then R is commutative.*

Proof. We have

$$[x,y]_{\alpha,\beta}=0 \tag{2.1}$$

for all $x,y\in I$. Replacing x by rx in (2.1), $r\in R$, we get

$$r[x,y]_{\alpha,\beta}+[r,\beta(y)]x=0$$

for all $x,y\in I$ and $r\in R$. Application of (2.1) yields that $[r,\beta(y)]x=0$ for all $x,y\in I$ and $r\in R$, that is, $[r,\beta(y)]RI=(0)$ for all $y\in I$ and $r\in R$. Thus, primeness of R forces that $[r,\beta(y)]=0$ for all $y\in I$ and $r\in R$. Now, replace r by $\beta(t)$, $t\in R$, in the above expression, we find that $\beta([r,y])=0$, since β is automorphism, i.e., that $[r,y]=0$. Again replacing y by sy for $s\in R$ in the last expression, we get $[r,s]y=0$ that is, $[r,s]RI=(0)$. Hence, primeness of R gives that $[r,s]=0$ for all $r,s\in R$, so that R is commutative. □

Theorem 2.1. *Let R be a prime ring and I be a nonzero left ideal of R. Suppose that G is a multiplicative (generalized)-(α,β)-derivation on R associated with the map g on R. If $G(xy)+\alpha(xy)=0$ for all $x,y\in I$, then $G(x)=-\alpha(x)$ for all $x\in I$ and $\beta(I)g(I)=(0)$.*

Proof. We have

$$G(xy)+\alpha(xy)=0 \tag{2.2}$$

for all $x,y\in I$. Replacing y by yz in (2.2), we get

$$G(xy)\alpha(z)+\beta(xy)g(z)+\alpha(xy)\alpha(z)=0 \tag{2.3}$$

for all $x,y,z\in I$. Using (2.2) in (2.3), we have

$$\beta(x)\beta(y)g(z)=0 \tag{2.4}$$

for all $x,y,z\in I$. Replacing y by ry in (2.4), $r\in R$, we get $\beta(x)\beta(r)\beta(y)g(z)=0$. Now replacing r by $\beta^{-1}(g(z)r)$ we find that $\beta(x)g(z)R\beta(y)g(z)=(0)$ for all

$x, y, z \in I$. Thus, by primeness of R, we get $\beta(I)g(I) = (0)$. Thus, equation (2.2) implies that $G(x)\alpha(y) + \alpha(xy) = \{G(x) + \alpha(x)\}\alpha(y) = 0$. Replacing y by ry in the last expression and using primeness of R, we conclude that $G(x) = -\alpha(x)$. Thereby the proof is completed. □

THEOREM 2.2. *Let R be a prime ring and I be a nonzero left ideal of R. Suppose that G is a multiplicative (generalized)-(α, β)-derivation on R associated with the map g on R. If $G(xy) + \alpha(yx) = 0$ for all $x, y \in I$, then R is commutative, $\beta(I)g(I) = (0)$ and $G(x) = -\alpha(x)$ for all $x \in I$.*

PROOF. We have the identity

$$G(xy) + \alpha(yx) = 0 \tag{2.5}$$

for all $x, y \in I$. Replacing x by x^2 and y by xy, respectively, in (2.5) and then subtracting one from another, we obtain $\alpha(yx^2) = \alpha(xyx)$ or $[x, y]x = 0$. Replacing y by ry in the last expression, we have $[x, r]yx = 0$, where $r \in R$. Since I is nonzero, so by primeness of R, we have $[x, r] = 0$. Substituting x by sx in the last expression, we obtain $[s, r]x = 0$, where $r, s \in R$. Primeness of R forces that R is commutative. Therefore $G(xy) + \alpha(yx) = 0$ becomes $G(xy) + \alpha(xy) = 0$. Thus, in view of Theorem 2.1, we have $G(x) = -\alpha(x)$ for all $x \in I$ and $\beta(I)g(I) = (0)$. This completes the proof. □

THEOREM 2.3. *Let R be a prime ring and I be a nonzero left ideal of R. Suppose that G is a multiplicative (generalized)-(α, β)-derivation on R associated with the map g on R. If $G(xy) + G(x)G(y) = 0$ for all $x, y \in I$, then either $\alpha(I)[G(x), \alpha(x)] = (0)$ or $\beta(I)[G(x), \beta(x)] = (0)$ for all $x \in I$.*

PROOF. We have the identity

$$G(xy) + G(x)G(y) = 0 \tag{2.6}$$

for all $x, y \in I$. Replacing y by yz in (2.6), we obtain

$$G(xy)\alpha(z) + \beta(xy)g(z) + G(x)G(y)\alpha(z) + G(x)\beta(y)g(z) = 0 \tag{2.7}$$

for all $x, y, z \in I$. Using (2.6) in (2.7), we get

$$\beta(xy)g(z) + G(x)\beta(y)g(z) = 0 \tag{2.8}$$

for all $x, y, z \in I$. Replacing x by xw in (2.8), we have

$$\beta(xwy)g(z) + G(x)\alpha(w)\beta(y)g(z) + \beta(x)g(w)\beta(y)g(z) = 0 \tag{2.9}$$

for all $x, y, z, w \in I$. Again substituting y by wy in (2.8), we obtain

$$\beta(xwy)g(z) + G(x)\beta(w)\beta(y)g(z) = 0 \tag{2.10}$$

for all $x, y, z, w \in I$. Subtracting (2.9) from (2.10), we

$$\{G(x)\alpha(w) + \beta(x)g(w) - G(x)\beta(w)\}\beta(y)g(z) = 0 \tag{2.11}$$

for all $x, y, z, w \in I$. Replacing y by ry in (2.11), where $r \in R$, by primeness of R, we have $G(x)\alpha(w) + \beta(x)g(w) - G(x)\beta(w) = G(xw) - G(x)\beta(w) = 0$ or $\beta(y)g(z) = 0$. From (2.6), we have

$$G(xyz) = -G(xy)G(z) = -G(x)G(yz) \tag{2.12}$$

for all $x, y, z \in I$. Using $G(xy) - G(x)\beta(y) = 0$, equation (2.12) can be written as $G(x)\{\beta(y)G(z) - G(y)\beta(z)\} = 0$. Replacing x by xrw in the last expression, where $w \in I, r \in R$ and using primeness of R, we conclude that $\beta(w)[G(z), \beta(z)] = 0$. Now, the other case $\beta(x)g(y) = 0$ gives $G(xy) = G(x)\alpha(y)$ for all $x, y \in I$, then proceeding in the same way as we have done earlier for $G(xy) = G(x)\beta(y)$, we obtain $\alpha(x)[G(y), \alpha(y)] = 0$. Hence, we get the required result. □

THEOREM 2.4. *Let R be a prime ring and I be a nonzero left ideal of R. Suppose that G and H are multiplicative (generalized)-(α, β)-derivations on R associated with the maps g and h on R, respectively. If $G(xy) = \alpha(y) \circ H(x)$ for all $x, y \in I$, then either R is commutative or $\alpha(I)[\alpha(I), H(I)] = (0)$.*

PROOF. We have the identity

$$G(xy) = \alpha(y) \circ H(x) \tag{2.13}$$

for all $x, y \in I$. Replacing y by yz in (2.13), we obtain

$$G(xy)\alpha(z) + \beta(xy)g(z) = (\alpha(y) \circ H(x))\alpha(z) + \alpha(y)[\alpha(z), H(x)] \tag{2.14}$$

for all $x, y, z \in I$. Using (2.13) in (2.14), we get

$$\beta(xy)g(z) = \alpha(y)[\alpha(z), H(x)] \tag{2.15}$$

for all $x, y, z \in I$. Replacing y by wy in (2.15), we have

$$\beta(xwy)g(z) = \alpha(wy)[\alpha(z), H(x)] \tag{2.16}$$

for all $x, y, z, w \in I$. Left multiply by $\alpha(w)$ to (2.15) and subtract it from (2.16), we obtain

$$\{\beta(x)\beta(w) - \alpha(w)\beta(x)\}\beta(y)g(z) = 0 \tag{2.17}$$

for all $x, y, z, w \in I$. Replacing y by ry in (2.17), where $r \in R$ and using primeness of R, we get either $\beta(y)g(z) = 0$ or $\beta(x)\beta(w) - \alpha(w)\beta(x) = 0$. If $\beta(x)g(y) = 0$ holds for all $x, y \in I$, then from (2.15), we have $\alpha(I)[\alpha(I), H(I)] = (0)$. For the other case

$$\beta(x)\beta(y) - \alpha(y)\beta(x) = 0 \tag{2.18}$$

for all $x, y \in I$. Replacing y by ry in (2.18), where $r \in R$, we get

$$\beta(x)\beta(ry) - \alpha(ry)\beta(x) = 0. \tag{2.19}$$

Left multiply by $\alpha(r)$ to (2.18) and subtract it from (2.19), we have $\{\beta(x)\beta(r) - \alpha(r)\beta(x)\}\beta(y) = 0$. Since I is nonzero, so primeness of R forces to write $\beta(x)\beta(r) - \alpha(r)\beta(x) = 0$. We can rewrite the last expression as $[\beta(x), r]_{\beta,\alpha} = 0$ for all $x \in I, r \in R$. Application of Lemma 2.1 yields that R is commutative. Thereby the proof is completed. □

THEOREM 2.5. *Let R be a prime ring and I be a nonzero left ideal of R. Suppose that G and H are multiplicative (generalized)-(α, β)-derivations on R associated with the maps g and h on R, respectively. If $G(xy) = [\alpha(y), H(x)]$ for all $x, y \in I$, then either R is commutative or $\alpha(I)[\alpha(I), G(I)] = (0)$.*

PROOF. We have the identity

$$G(xy) = [\alpha(y), H(x)] \tag{2.20}$$

for all $x, y \in I$. Replacing y by yz in (2.20), we obtain

$$G(xy)\alpha(z) + \beta(xy)g(z) = [\alpha(y), H(x)]\alpha(z) + \alpha(y)[\alpha(z), H(x)] \tag{2.21}$$

for all $x, y, z \in I$. Using (2.20) in (2.21), we get

$$\beta(xy)g(z) = \alpha(y)[\alpha(z), H(x)] \tag{2.22}$$

for all $x, y, z \in I$. Note that the equation (2.22) is same as the equation (2.15) in Theorem 2.4; then proceeding in the same way as in Theorem 2.4, we get the required result. □

3. Examples

In this section we construct some examples to show that the primeness condition of the ring in our results are essential.

Example 3.1. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta \colon R \to R$ by

$$G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & -c \end{pmatrix}, \quad g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix},$$

$$\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \quad \beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

It is easy to verify that I is a left ideal on R, G is a multiplicative (generalized)-(α, β)-derivation associated with the map g, α and β are automorphisms on R and $G(xy) + G(x)G(y) = 0$ for all $x, y \in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that $\alpha(I)[G(x), \alpha(x)] \neq (0)$ and $\beta(I)[G(x), \beta(x)] \neq 0$ for all $x \in I$. Hence, the primeness hypothesis in Theorem 2.3 is crucial.

Example 3.2. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let us define $G, g, \alpha, \beta, H, h \colon R \to R$ by

$$G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & -c \end{pmatrix}, \quad g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

$$\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}, \quad \beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

$$H \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad h \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that I is a left ideal on R, G and H are multiplicative (generalized)-(α,β)-derivations associated with the maps g and h, respectively, α, β are automorphisms on R and $G(xy)=[\alpha(y),H(x)]$ fro all $x,y\in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that R is not commutative and $\alpha(I)[\alpha(I),G(I)]\neq 0$. Hence, the primeness hypothesis in Theorem 2.5 is crucial.

Example 3.3. Let

$$R=\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a,b,c\in\mathbb{Z} \right\} \quad \text{and} \quad I=\left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b,c\in\mathbb{Z} \right\}.$$

Let us define $G,g,\alpha,\beta,H,h\colon R\to R$ by

$$G\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix},\quad g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} 0 & -2b \\ 0 & 0 \end{pmatrix},$$

$$\alpha\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},\quad \beta\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

$$H\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix},\quad h\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that I is a left ideal on R, G and H are multiplicative (generalized)-(α,β)-derivations associated with the maps g and h, respectively, α, β are automorphisms on R and $G(xy)=\alpha(y)\circ H(x)$ for all $x,y\in I$. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

R is not a prime ring. We see that R is not commutative and $\alpha(I)[\alpha(I),H(I)]\neq\{0\}$. Hence, the primeness hypothesis in Theorem 2.4 is crucial.

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