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Annales Mathematicae Silesianae **33** (2019), 21–40
DOI: 10.2478/amsil-2018-0008

EXTENDING THE APPLICABILITY OF THE SUPER-HALLEY-LIKE METHOD USING ω-CONTINUOUS DERIVATIVES AND RESTRICTED CONVERGENCE DOMAINS

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Abstract. We present a local convergence analysis of the super-Halley-like method in order to approximate a locally unique solution of an equation in a Banach space setting. The convergence analysis in earlier studies was based on hypotheses reaching up to the third derivative of the operator. In the present study we expand the applicability of the super-Halley-like method by using hypotheses up to the second derivative. We also provide: a computable error on the distances involved and a uniqueness result based on Lipschitz constants. Numerical examples are also presented in this study.

1. Introduction

In this study we consider the super-Halley-like method for approximating a locally unique solution x^* of equation

$$F(x) = 0,$$

where F is a twice Fréchet differentiable operator defined on a subset Ω of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$. In particular, we study

Received: 09.10.2017. Accepted: 25.08.2018. Published online: 27.10.2018.
(2010) Mathematics Subject Classification: 65D10, 65D99.
Key words and phrases: super-Halley-like method, Banach space, local convergence, Fréchet derivative.

the local convergence analysis of the super-Halley method defined for each $n = 0, 1, 2, \dots$ by

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$
$$x_{n+1} = y_n + \frac{1}{2}L_n(I - L_n)^{-1}(y_n - x_n), \tag{1.1}$$

where x_0 is an initial point and $L_n = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$. The efficiency and importance of method (1.1) were discussed in [4–9,14]. The study of convergence of iterative algorithms is usually centered into two categories: semilocal and local convergence analysis. The semilocal convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points. The local convergence of method (1.1) was shown using hypotheses given in non-affine invariant form by

($\mathcal{C}_1$) $F\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a thrice continuously differentiable operator.
($\mathcal{C}_2$) There exists $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $\|F^{-1}(x_0)\| \le \beta$.

There exist $\eta \ge 0$, $\beta_1 \ge 0$, $\beta_2 \ge 0$ and $\beta_3 \ge 0$ such that
($\mathcal{C}_3$) $\|F'(x_0)^{-1}F'(x_0)\| \le \eta$,
($\mathcal{C}_4$) $\|F''(x)\| \le \beta_1$, for each $x \in D$,
($\mathcal{C}_5$) $\|F'''(x)\| \le \beta_2$, for each $x \in D$,
($\mathcal{C}_6$) $\|F'''(x) - F'''(y)\| \le \beta_3\|x - y\|$ for each $x, y \in D$.

The hypotheses for the local convergence analysis of these methods are the same but x_0 is replaced by x^*. Notice however that hypotheses ($\mathcal{C}_5$) and ($\mathcal{C}_6$) limit the applicability of these methods. As a motivational example, let us define function f on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,\ f'(1) = 3,$$
$$f''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$
$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, obviously, function f''' is unbounded on Ω. Hence, the earlier results using the ($\mathcal{C}$) conditions cannot be used to solve equation $f(x) = 0$. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathcal{B}_1$ (see [1–16]). These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial point x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.1) in Section 2. We show in Remark 2.2 and Remark 3.3 as well as in the numerical examples that the radius of convergence of method (1.1) is smaller than the radius of convergence for Newton's method. However method (1.1) is faster than Newton's method. Moreover, the radius of convergence is larger and the error bounds on the distances $\|x_n - x^*\|$ are tighter under our new approach. Furthermore, we show that the equation $f(x) = 0$ mentioned above can be solved under our new technique. Finally, our conditions do not require $(\mathcal{C}_5)$ and $(\mathcal{C}_6)$. Hence, we expand the applicability of method (1.1) in cases not covered before. The same technique can be used to other methods.

We study these methods in the more general setting of a Banach space under hypotheses only on the first and second Fréchet derivatives of F and under the same set of conditions (see conditions $(\mathcal{H})$ that follow). This way we expand the applicability of these methods.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis and Section 3 contains the semi-local convergence analysis of method (1.1). The numerical examples are presented in the concluding Section 4.

2. Local convergence analysis

Local convergence analysis of method (1.1) is based on some scalar functions and parameters. Let $w_0\colon [0,+\infty) \to [0,+\infty)$ be a continuous and increasing function with $w_0(0) = 0$. Suppose the equation $w_0(t) = 1$ has at least one positive root. Denote by r_0 the smallest such a root, i.e.

$$r_0 = \min\{t > 0 : \ w_0(t) = 1\}. \tag{2.1}$$

Let also $w\colon [0,r_0) \to [0,+\infty)$, $v\colon [0,r_0) \to [0,+\infty)$ and $v_1\colon [0,r_0) \to [0,+\infty)$ be continuous and increasing functions with $w(0) = 0$. Moreover, define functions g_1 and h_1 on the interval $[0,r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

We have $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \longrightarrow r_0^-$. The intermediate value theorem guarantees the existence of zeros for function $h_1(t)$ in $(0, r_0)$. Denote by r_1 the smallest zero. Furthermore, define functions p and h_p on the interval $[0, r_0)$ by

$$p(t) = \frac{v_1(t) \int_0^1 v(\theta t) d\theta t}{(1 - w_0(t))^2}$$

and

$$h_p(t) = p(t) - 1.$$

We get that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \longrightarrow r_0^-$. Denote by r_p the smallest zero of function h_p in the interval $(0, r_0)$. Finally, define functions g_2 and h_2 on the interval $[0, r_p)$ by

$$g_2(t) = g_1(t) + \frac{1}{2} \frac{p(t) \int_0^1 v(\theta t) d\theta}{(1 - p(t))(1 - w_0(t))}$$

and

$$h_2(t) = g_2(t) - 1.$$

We obtain $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \longrightarrow r_p^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, r_p)$. Define the radius of convergence r by

$$r = \min\{r_1, r_2\}. \tag{2.2}$$

Then, for each $t \in [0, r)$

$$0 \leq g_i(t) < 1 \tag{2.3}$$

and

$$0 \leq p(t) < 1. \tag{2.4}$$

Let $U(v, \rho), \bar{U}(v, \rho)$ stand, respectively for the open and closed balls in $\mathcal{B}_1$ with center $v \in \mathcal{B}_1$ and of radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.1) using the preceding notation.

THEOREM 2.1. *Let $F\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a twice continuously Fréchet-differentiable operator. Suppose there exist $x^* \in \Omega$ and a continuous and increasing function $w_0\colon [0,+\infty) \to [0,+\infty)$ with $w_0(0) = 0$ such that for each $x \in \Omega$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.5}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|). \tag{2.6}$$

Let $\Omega_0 = \Omega \cap U(x^*, r_0)$. Suppose there exist functions $w\colon [0,r_0) \to [0,+\infty)$, $v\colon [0,r_0) \to [0,+\infty)$ and $v_1\colon [0,r_0) \to [0,+\infty)$ such that for each $x, y \in \Omega_0$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|), \tag{2.7}$$

$$\|F'(x^*)^{-1}F'(x)\| \le v(\|x - x^*\|), \tag{2.8}$$

$$\|F'(x^*)^{-1}F''(x)\| \le v_1(\|x - x^*\|), \tag{2.9}$$

and $\bar{U}(x^*, r) \subseteq \Omega$, where the radii r_0 and r are defined by (2.1) and (2.2), respectively. Then, sequence $\{x_n\}$ starting from $x_0 \in U(x^*, r) \setminus \{x^*\}$ and given by (1.1) is well defined, remains in $U(x^*, r)$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.11}$$

where the functions $g_i, i = 1, 2$, are defined previously. Furthermore, if there exists $r^* \ge r$ such that

$$\int_0^1 w_0(\theta r^*) d\theta < 1, \tag{2.12}$$

then, the limit point x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap U(x^*, r^*)$.

PROOF. We shall show estimates (2.10) and (2.11) using induction on the integer k. Let $x \in U(x^*, r)$. Using (2.1), (2.2) and (2.5), we get that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \leq w_0(r) < 1. \tag{2.13}$$

In view of (2.13) and the Banach lemma on invertible operators ([1,4,15,16]) we have that $F'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{2.14}$$

In particular, y_0 is well defined by the first substep of method (1.1) for $n = 0$. We can write

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.15}$$

Using (2.2), (2.3) (for $i = 1$), (2.7), (2.14) and (2.15) we obtain in turn that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times \left\| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta \right\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.16}$$

which shows (2.10) for $n = 0$ and $y_0 \in U(x^*, r)$. Next, we must show $(I - L_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. By (2.2), (2.8), (2.9) and (2.14) we have in turn that

$$\begin{aligned} \|L_0\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F''(x_0)\| \\ &\quad \times \|F'(x_0)^{-1}F'(x_0)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \frac{v_1(\|x_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{(1 - w_0(\|x_0 - x^*\|))^2} \\ &= p(\|x_0 - x^*\|) \leq p(r) < 1, \end{aligned} \tag{2.17}$$

so

$$\|(I - L_0)^{-1}\| \leq \frac{1}{1 - p(\|x_0 - x^*\|)}, \tag{2.18}$$

where we also used the estimate

$$
\begin{aligned}
\|F'(x^*)^{-1}F(x_0)\| &= \left\| \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta \right\| \\
&\leq \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|. \qquad (2.19)
\end{aligned}
$$

We also have that x_1 is well defined by the second substep of method (1.1) and (2.18). Moreover, by (2.2), (2.3) (for $i = 2$), (2.14), (2.16)-(2.19), we get in turn that

$$
\begin{aligned}
\|x_1 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0) - \frac{1}{2}L_0(I - L_0)^{-1}F'(x_0)^{-1}F(x_0)\| \\
&\leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\
&\quad + \frac{1}{2}\|L_0\|\|(I - L_0)^{-1}\|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\quad + \frac{p(\|x_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{2(1 - p(\|x_0 - x^*\|))(1 - w_0(t))} \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}
$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r)$. The induction is finished, if we simply replace x_0, y_0, x_1 by x_k, y_k, x_{k+1}, respectively in the preceding estimates. Furthermore, from the estimate

$$\|x_{k+1} - x^*\| \leq c_0\|x_k - x^*\| < r,$$

where $c_0 = g_2(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$, where $y^* \in \Omega_1$ with $F(y^*) = 0$. Using (2.5) and (2.12), we obtain that

$$
\begin{aligned}
\|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\
&\leq \int_0^1 w_0(\theta r^*)d\theta < 1,
\end{aligned}
$$

so $Q^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. $\square$

Remark 2.2.

(a) Let $w_0(t) = L_0 t,\ w(t) = Lt$ and $w^*(t) = L^* t$ (w^* replacing w in (2.7)). In [3], Argyros and Ren used instead of (2.7) the condition

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L^* \|x - y\| \quad \text{for each } x, y \in \Omega. \tag{2.20}$$

But using (2.7) and (2.20) we get that

$$L \leq L^*$$

holds, since $\Omega_0 \subseteq \Omega$. In case $L < L^*$, the new convergence analysis is better than the old one. Notice also that we have by (2.6) and (2.20)

$$L_0 \leq L^*.$$

The advantages are obtained under the same computational cost as before, since in practice the computation of constant L^* requires the computation of L_0 and L as special cases. In the literature (with the exception of our works) (2.20) is only used for the computation of the upper bounds of the inverses of the operators involved.

(b) The radius r_A was obtained by Argyros in [4] as the convergence radius for Newton's method under conditions (2.3)-(2.7). Notice that the convergence radius for Newton's method given independently by Rheinboldt ([15]) and Traub ([16]) is given by

$$\rho = \frac{2}{3L^*} < r_A = \frac{2}{2L_0 + L}.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0,1)$. Then, we have that $L_0 = e - 1 < L^* = e, L = e^{\frac{1}{e-1}}$, so $\rho = 0.24252961 < r_A = 0.3827$.

Moreover, the new error bounds ([4–7]) are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones ([8, 9])

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Moreover, the radius of convergence of method (1.1) given by r is smaller than r_A (see (2.2)).

(c) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy ([4–7]).

(d) The results can be also used to solve equations where the operator F' satisfies the autonomous differential equation ([5, 7]):

$$F'(x) = p(F(x)),$$

where p is a known continuous operator. Since $F'(x^*) = p(F(x^*)) = p(0)$, we can apply the results without actually knowing the solution x^*. As an example, let us consider $F(x) = e^x - 1$. Then, we can choose $p(x) = x + 1$ and $x^* = 0$.

(e) It is worth noticing that convergence conditions for method (1.1) are not changing if we use the new instead of the old conditions. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \dots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}, \quad \text{for each } n = 0, 1, 2, \dots$$

instead of the error bounds obtained in Theorem 2.1.

(f) In view of (2.6) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + L_0\|x - x^*\|\end{aligned}$$

condition (2.8) can be dropped and can be replaced by

$$v(t) = 1 + L_0 t$$

or

$$v(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

3. Semi-local convergence analysis

The following conditions $(\mathcal{H})$ in a non affine invariant form have been used to show the semi-local convergence analysis of method (1.1) ([5, 14]):

$(\mathcal{H}_1)$ $\|F'(x_0)^{-1}\| \leq \beta$.

$(\mathcal{H}_2)$ $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

$(\mathcal{H}_3)$ $\|F''(x)\| \leq M$ for each $x \in \Omega$.

$(\mathcal{H}_4)$ There exists a continuous and increasing function $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ such that for each $x, y \in \Omega$ $\|F''(x)-F''(y)\| \leq \varphi(\|x-y\|)$.

$(\mathcal{H}_5)$ There exists a continuous and increasing function $\psi\colon [0,1] \to [0,+\infty)$ such that $\varphi(ts) \leq \psi(t)\varphi(s)$ for each $t \in [0,1]$ and $s \in [0,+\infty)$.

$(\mathcal{H}_6)$ $\bar{U}(x_0,\bar{r}) \subseteq \Omega$, where $\bar{r}$ is a positive zero of some scalar equation.

In many applications the iterates $\{x_n\}$ remain in a neighborhood of Ω_0. If we locate Ω_0 before we find M, q and ψ, then the new semi-local convergence analysis will be weaker. Consequently, the new convergence domain will be at least as large as if we were using Ω. To achieve this goal we consider the weaker conditions $(\mathcal{A})$ in an affine invariant form:

$(\mathcal{A}_1)$ $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

$(\mathcal{A}_2)$ There exists a continuous and increasing function $w_0\colon [0,+\infty) \to [0,+\infty)$ with $w_0(0) = 0$ such that for each $x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq w_0(\|x - x_0\|).$$

Define r_0 the smallest positive solution of equation $w_0(t) = 1$. Set $\Omega_0 := \Omega \cap U(x_0, r_0)$.

$(\mathcal{A}_3)$ There exist continuous and increasing functions $w, w_1, w_2\colon [0, r_0) \to [0,+\infty)$ with $w_1(0) = w_2(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x_0)^{-1}F''(x)\| \leq w(\|x - x_0\|),$$

$$\|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \leq w_1(\|x - x_0\|)$$

and

$$\|F'(x_0)^{-1}(F''(x) - F''(y))\| \leq w_2(\|x - y\|).$$

Define functions q, d, c on the interval $[0, r_0)$ by

$$q(t) = \frac{w(t)\eta}{(1 - w_0(t))^2},$$

$$d(t) = \frac{1}{1-w_0(t)}\Bigg[\frac{\int_0^1 w_2(\theta\eta)(1-\theta)d\theta}{1-q(t)}$$

$$+\frac{\int_0^1 w(t+\theta\eta)(1-\theta)d\theta q(t)}{1-q(t)}$$

$$+\frac{1}{2}\int_0^1 w(1+\theta\eta)d\theta\frac{w(t)}{(1-w_0(t))^2(1-q(t))}$$

$$+\frac{1}{2}\int_0^1 w\left(t+\frac{1}{2}\theta\frac{q(t)}{1-q(t)}\right)(1-\theta)d\theta\frac{w(t)}{(1-w_0(t))^2(1-q(t))}\Bigg]$$

and

$$c(t) = (1+\frac{1}{2}\frac{q(t)}{1-q(t)})d(t)t.$$

($\mathcal{A}_4$) Equation $\frac{\eta}{1-c(t)} - t = 0$ has zeros in the interval $(0, r_0)$. Denote by r the smallest such a zero. Moreover, the zero r satisfies $q(r) < 1$ and $c(r) < 1$.

($\mathcal{A}_5$) $\bar{U}(x_0, r) \subseteq \Omega$.

($\mathcal{A}_6$) There exists $r^* \geq r$ such that $\int_0^1 w_0((1-\theta)r + \theta r^*)d\theta < 1$.

It is convenient for the semi-local convergence analysis of method (1.1) that follows to define the scalar sequences $\{p_n\}$, $\{q_n\}$, $\{s_n\}$, $\{t_n\}$ for each $n = 0, 1, 2, \ldots$ by

$$p_n = \frac{w(\|x_n - x_0\|)\|F'(x_0)^{-1}F(x_n)\|}{(1-w_0(\|x_n-x_0\|))^2}, \tag{3.1}$$

$t_0 = 0, s_0 = \eta,$

$$q_n = \frac{w(t_n - t_0)(s_n - t_n)}{(1-w_0(t_n - t_0))^2}, \tag{3.2}$$

$$t_{n+1} = s_n + \frac{1}{2}\frac{q_n}{1-q_n}(s_n - t_n), \tag{3.3}$$

$$s_{n+1} = t_{n+1} + \frac{1}{1-w_0(t_{n+1}-t_0)}\Bigg[\frac{\int_0^1 \bar{w}(\theta(s_n-t_n))(1-\theta)d\theta(s_n-t_n)^2}{1-q_n}$$

$$+\frac{\int_0^1 w(t_n - t_0 + \theta(s_n - t_n))(1-\theta)d\theta q_n(s_n-t_n)^2}{1-q_n} \tag{3.4}$$

$$+\frac{1}{2}\int_0^1 w(t_n - t_0 + \theta(s_n - t_n))d\theta \frac{q_n}{1-q_n}(s_n - t_n)$$
$$+\frac{1}{2}\int_0^1 w\left(s_n - t_0 t + \theta\frac{q_n(s_n - t_n)}{2(1-q_n)}\right)(1-\theta)d\theta\frac{q_n}{1-q_n}(s_n - t_n)\Bigg],$$

where

$$\bar{w} = \begin{cases} w_1, & n = 0, \\ w_2, & n > 0. \end{cases}$$

We need an Ostrowski-type representation of method (1.1).

Assumption 3.1. Suppose that method (1.1) is well defined for each $n = 0, 1, 2, \ldots$ Then the following equality holds

$$\begin{aligned} F(x_{n+1}) = & \int_0^1 [F''(x_n + \theta(y_n - x_n)) - F''(x_n)](1-\theta)(I - L_n)^{-1}(y_n - x_n)^2 d\theta \\ & - \int_0^1 F''(x_n + \theta(y_n - x_n))(1 - L_n)^{-1} L_n(x_n)(1-\theta)d\theta(y_n - x_n)^2 \\ & + \int_0^1 F''(x_n + \theta(y_n - x_n))d\theta(y_n - x_n)(x_{n+1} - y_n) \\ & + \int_0^1 F''(y_n + \theta(x_{n+1} - y_n))(1-\theta)d\theta(x_{n+1} - y_n)^2. \end{aligned} \tag{3.5}$$

Proof. Using method (1.1) and the Taylor series expansion about $y_n \in \Omega$, we can write

$$F(x_{n+1}) = F(y_n) + F'(y_n)(x_{n+1} - y_n) + \int_{y_n}^{x_{n+1}} F''(x)(x_{n+1} - x)dx,$$

which leads to (3.5). □

Next, we present the semi-local convergence analysis of method (1.1) using the preceding notation and conditions $(\mathcal{A})$.

Theorem 3.2. *Let $F\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a twice continuously Fréchet-differentiable operator. Suppose that the conditions $(\mathcal{A})$ are satisfied. Then, sequence $\{x_n\}$ generated for $x_0 \in \Omega$ by method (1.1) is well defined in $U(x_0, r)$, remains in $U(x_0, r)$ for each $n = 0, 1, 2, \ldots$ and converges to solution $x^* \in \bar{U}(x_0, r)$ of equation $F(x) = 0$. Moreover, the following estimates hold*

$$\|x_n - x^*\| \le r - t_n.$$

Furthermore, the limit point x^* *is the only solution of equation* $F(x) = 0$ *in* $\Omega_1 = \Omega \cap \bar{U}(x_0, r)$.

Proof. Let $x \in U(x_0, r_0)$, where r_0 is defined in condition $(\mathcal{A}_2)$ from which we have that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le w_0(\|x - x_0\|) \le w_0(r) < 1. \tag{3.6}$$

It follows from (3.6) and the Banach lemma on invertible operators that $F'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(x_0)\| \le \frac{1}{1 - w_0(\|x - x_0\|)}. \tag{3.7}$$

We also have that y_0 is well defined by the first substep of method (1.1) and L_0 exists for $n = 0$. In view of the definition of L_0, $(\mathcal{A}_1)$, $(\mathcal{A}_3)$, $(\mathcal{A}_4)$, (3.1), (3.2) and (3.7), we get that

$$\begin{aligned}
\|L_0\| &= \|F'(x_0)^{-1}F''(x_0)F'(x_0)^{-1}F(x_0)\| \\
&\le \|F'(x_0)^{-1}F''(x_0)\|\|F'(x_0)^{-1}F(x_0)\| \\
&\le w(\|x_0 - x_0\|)\|F'(x_0)^{-1}F(x_0)\| \\
&\le \frac{w(\|x_0 - x_0\|)\|F'(x_0)^{-1}F(x_0)\|}{(1 - w_0(\|x_0 - x_0\|))^2} = p_0 \le q_0 \le q(r) < 1, \qquad (3.8)
\end{aligned}$$

so $(I - L_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|(I - L_0)^{-1}\| \le \frac{1}{1 - q_0}. \tag{3.9}$$

The point x_1 is also well defined by the second substep of method (1.1) for $n = 0$ and

$$\begin{aligned}
\|x_1 - x_0\| &\le \|x_1 - y_0\| + \|y_0 - x_0\| \\
&\le \frac{1}{2}\|L_0\|\|(I - L_0)^{-1}\|\|y_0 - x_0\| + \|y_0 - x_0\| \\
&\le \left(1 + \frac{1}{2}\frac{q_0}{1 - q_0}\right)\|y_0 - x_0\| \\
&\le \left(1 + \frac{1}{2}\frac{q_0}{1 - q_0}\right)\eta < r,
\end{aligned}$$

so $x_1 \in U(x_0, r)$, $\|y_0 - x_0\| \leq s_0 - t_0$ and $\|x_1 - y_0\| \leq t_1 - s_0$. Suppose

$$\|y_k - x_k\| \leq s_k - t_k \tag{3.10}$$

and

$$\|x_{k+1} - y_k\| \leq t_{k+1} - s_k. \tag{3.11}$$

Using (3.5), ($\mathcal{A}_2$), (3.1)–(3.4), (3.7)–(3.9), we get in turn that

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_{k+1})\| \\
&\qquad \leq \frac{\int_0^1 \bar{w}(\|x_k + \theta(y_k - x_k) - x_k\|)(1-\theta)d\theta\|y_k - x_k\|^2}{1 - p_k} \\
&\qquad\quad + \frac{\int_0^1 w(\|x_k + \theta(y_k - x_k) - x_0\|)(1-\theta)d\theta p_k\|y_k - x_k\|^2}{1 - p_k} \\
&\qquad\quad + \int_0^1 w(\|x_k + \theta(y_k - x_k) - x_0\|)d\theta\|x_{k+1} - y_k\| \\
&\qquad\quad + \int_0^1 w(\|y_k + \theta(x_{k+1} - x_k) - x_0\|)(1-\theta)d\theta\|x_{k+1} - y_k\| \\
&\qquad \leq \frac{\int_0^1 \bar{w}(\theta(s_k - t_k))(1-\theta)d\theta(s_k - t_k)^2}{1 - p_k} \\
&\qquad\quad + \frac{\int_0^1 w((t_k - t_0 + \theta(s_k - t_k))(1-\theta)d\theta q_k(s_k - t_k)^2}{1 - q_k} \\
&\qquad\quad + \int_0^1 w(t_k - t_0 + \theta(s_k - t_k))d\theta(t_{k+1} - s_k) \\
&\qquad\quad + \int_0^1 w((s_k - t_0 + \theta(t_{k+1} - s_k) - x_0\|)(1-\theta)d\theta(t_{k+1} - s_k) \\
&\qquad := \alpha_{k+1},
\end{aligned} \tag{3.12}$$

so

$$\begin{aligned}
\|y_{k+2} - x_{k+2}\| &\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \\
&\leq \frac{\alpha_{k+1}}{1 - w_0(\|x_{k+1} - x_0\|)} = s_{k+2} - t_{k+2}
\end{aligned}$$

and

$$\begin{aligned}\|x_{k+2}-y_{k+1}\| &\le \frac{1}{2}\|L_{k+1}\|\|(I-L_{k+1})^{-1}\|\|y_{k+1}-x_{k+1}\| \\ &\le \frac{1}{2}\frac{q_{k+1}}{1-q_{k+1}}(s_{k+1}-t_{k+1}) = t_{k+2}-s_{k+1},\end{aligned}$$

and the induction for (3.10) and (3.11) is completed. Moreover, we have that

$$\begin{aligned}\|x_{k+2}-x_{k+1}\| &\le \|x_{k+2}-y_{k+1}\| + \|y_{k+1}-x_{k+1}\| \\ &\le \frac{1}{2}\frac{q_{k+1}}{1-q_{k+1}}(s_{k+1}-t_{k+1}) + (s_{k+1}-t_{k+1}) \\ &= \left(1+\frac{1}{2}\frac{q_{k+1}}{1-q_{k+1}}\right)(s_{k+1}-t_{k+1}) \\ &\le \left(1+\frac{1}{2}\frac{q(r)}{1-q(r)}\right)d(r)r(s_k-t_k) \\ &= c(s_k-t_k) \le \ldots \le c^{k+1}(s_0-t_0) \\ &= c^{k+1}\eta, \quad c=c(r),\end{aligned}$$

so

$$\begin{aligned}\|x_{k+2}-x_0\| &\le \|x_{k+2}-x_{k+1}\| + \ldots + \|x_1-x_0\| \\ &\le c^{k+1}\eta + \ldots + \eta = \frac{1-c^{k+2}}{1-c}\eta < \frac{\eta}{1-c} = r, \qquad (3.13)\end{aligned}$$

hence $x_{k+2} \in U(x_0, r)$ and

$$\begin{aligned}\|y_{k+1}-x_0\| &\le \|y_{k+1}-x_{k+1}\| + \|x_{k+1}-x_0\|, \\ &\le s_{k+1}-t_{k+1}+t_{k+1}-t_0 = s_{k+1} \le r. \qquad (3.14)\end{aligned}$$

Scalar sequences $\{t_k\}, \{s_k\}$ are increasing and bounded from above by r (see (3.10), (3.11), (3.13) and (3.14)), so they converge to their unique least upper bound $r_1 \le r$. In view of (3.10) and (3.11) sequences $\{x_k\}, \{y_k\}$ are Cauchy in a complete space $\mathcal{B}_1$ and as such they converge to some $x^* \in \bar{U}(x_0, r)$ (since $\bar{U}(x_0, r)$ is a closed set). By letting $k \longrightarrow \infty$ in (3.12) we get $F(x^*) = 0$.

To prove the uniqueness part, suppose that there exists $y^* \in \Omega_1$ with $F(y^*) = 0$ and define linear operator $T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Using $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, we obtain in turn that

$$\begin{aligned}
\|F'(x_0)^{-1}(T - F'(x_0))\| &\leq \int_0^1 w_0(\|x^* + \theta(y^* - x^*) - x_0\|)d\theta \\
&\leq \int_0^1 w_0((1-\theta)\|x^* - x_0\| + \theta\|y^* - x_0\|)d\theta \\
&\leq \int_0^1 w_0((1-\theta)r + \theta r^*)d\theta < 1,
\end{aligned}$$

so $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity

$$0 = F(y^*) - F(x^*) = T(y^* - x^*),$$

we conclude that $x^* = y^*$. □

Remark 3.3.

(a) In view of the estimate

$$\begin{aligned}
\|F'(x_0)^{-1}F''(x)\| &= \|F'(x_0)^{-1}[F''(x) - F''(x_0) + F''(x_0)]\| \\
&\leq \|F'(x_0)^{-1}(F''(x) - F''(x_0))\| + \|F'(x_0)^{-1}F''(x_0)\| \\
&\leq w_1(\|x - x_0\|) + \|F'(x_0)^{-1}F''(x_0)\|,
\end{aligned}$$

the first condition in $(\mathcal{A}_3)$ can be dropped, if we choose

$$w(t) = w_1(t) + \|F'(x_0)^{-1}F''(x_0)\|, \quad \textit{for each } t \in [0, r_0).$$

Moreover, the third condition in $(\mathcal{A}_3)$ implies the second condition but

$$w_1(t) \leq w_2(t), \quad \text{for each } t \in [0, r_0). \tag{3.15}$$

Hence, it is important to introduce the second condition, in case strict inequality holds in (3.15).

(b) Clearly conditions $(\mathcal{A})$ are weaker than conditions $(\mathcal{H})$ even, if the former are specialized. Indeed, let us rewrite conditions $(\mathcal{H}_3)$–$(\mathcal{H}_5)$:
$(\mathcal{H}_3')$ $\|F'(x_0)^{-1}F''(x)\| \leq M_1$ for each $x \in \Omega$.

($\mathcal{H}_4'$) $\|F'(x_0)^{-1}(F''(x)-F''(y))\| \leq \varphi_1(\|x-y\|)$ for each $x, y \in \Omega$. Choose $w(t) = M_0$. Then, we have that

$$M_0 \leq M_1$$

and

$$w_2(t) \leq \varphi_1(t),$$

since $\Omega_0 \subseteq \Omega$.

($\mathcal{H}_5'$) $\varphi_1(ts) \leq \psi_1(t)\varphi_1(s)$, for each $t \in [0,1]$ and $s \in [0, +\infty)$.
Then, there exists a continuous and increasing function $\psi_0 \colon [0,1] \to [0, +\infty)$ such that

$$w_2(ts) \leq \psi_0(t) w_2(s), \text{ for each } t \in [0,1] \text{ and } s \in [0, +\infty)$$

and

$$\psi_0(t) \leq \psi(t)$$

leading to a tight convergence analysis (see also the numerical examples).

4. Numerical Examples

The numerical examples are presented in this section.

Example 4.1. Returning back to the motivational example at the introduction of this study, we have $w_0(t) = w(t) = v(t) = 146.6629073t$ and $v_1(t) = 2$. The parameters for method (1.1) are

$$r_1 = 0.6667, \quad r_2 = 0.4384 = r.$$

As already noted in the introduction earlier results using ($\mathcal{C}$) conditions cannot be used.

EXAMPLE 4.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, \Omega = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on Ω for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the conditions (2.5)–(2.9), we get $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t$, $v_1(t) = v(t) = e^{\frac{1}{e-1}}$. The parameters for method (1.1) are

$$r_1 = 0.3827, \quad r_2 = 0.1262 = r.$$

Under the old approach ([2, 3, 5–8, 12, 15, 16]) we have $\tilde{w}_0(t) = \tilde{w}(t) = et$, $\tilde{v}_1(t) = \tilde{v}(t) = e$. The old parameters for method (1.1) are

$$\tilde{r}_1 = 0.2453, \quad \tilde{r}_2 = 0.0540 = \tilde{r}.$$

Here $\tilde{g}_1, \tilde{g}_2$ are g_1, g_2 with $\tilde{w}_0(t) = \tilde{w}(t) = et$, $\tilde{v}_1(t) = \tilde{v}(t) = e$. Table 1 gives the comparison of $g_1(\|x_n - x^*\|)$, $g_2(\|x_n - x^*\|), \tilde{g}_1(\|x_n - x^*\|)$ and $\tilde{g}_2(\|x_n - x^*\|)$.

Table 1. Comparison table

n	$g_1(\|x_n - x^*\|)$	$g_2(\|x_n - x^*\|)$	$\tilde{g}_1(\|x_n - x^*\|)$	$\tilde{g}_2(\|x_n - x^*\|)$
2	0.0032	0.0138	0.0049	0.0213
3	9.4542e-08	3.9732e-07	1.4361e-07	6.0351e-07
4	4.7256e-17	1.9860e-16	7.1779e-17	3.0166e-16

EXAMPLE 4.3. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad \text{for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $w_0(t) = 7.5t$, $w(t) = 15t$, $v(t) = 15$, $v_1(t) = 30$. The parameters for method (1.1) are

$$r_1 = 0.0667, \quad r_2 = 0.0021 = r.$$

Under the old approach $\tilde{w}_0(t) = \tilde{w}(t) = 15t$, $\tilde{v}(t) = 15$, $\tilde{v}_1(t) = 30$. The old parameters for method (1.1) are

$$\tilde{r}_1 = 0.0444, \quad \tilde{r}_2 = 0.0003 = \tilde{r}.$$

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