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REFINEMENTS OF SOME RECENT INEQUALITIES FOR CERTAIN SPECIAL FUNCTIONS

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Abstract. The aim of this paper is to give some refinements to several inequalities, recently etablished, by P.K. Bhandari and S.K. Bissu in [*Inequalities via Hölder's inequality*, Scholars Journal of Research in Mathematics and Computer Science, **2** (2018), no. 2, 124–129] for the incomplete gamma function, Polygamma functions, Exponential integral function, Abramowitz function, Hurwitz-Lerch zeta function and for the normalizing constant of the generalized inverse Gaussian distribution and the Remainder of the Binet's first formula for $\ln\Gamma(x)$.

1. Introduction

Throughout this section, p, q are conjugate exponents, that is $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. $\mathbb{K}$ is the real or complex field.

For any continuous functions $u, v : [a, b] \to \mathbb{K}$, we recall the integral version of the Hölder's inequality:

$$\int_a^b |u(t)v(t)|\,dt \le \left(\int_a^b |u(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |v(t)|^q\,dt\right)^{\frac{1}{q}}. \tag{1.1}$$

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Modifying an inequality of C. Mortici ([8]), P.K. Bhandari and S.K. Bissu in [3] replaced $u(t)$ and $v(t)$ in (1.1) by

$$[g(t)]^{1/p}[h(t)]^{x/p}[f(t)]^{v/p} \quad \text{and} \quad [g(t)]^{1/q}[h(t)]^{y/q}[f(t)]^{u/q},$$

to obtain the following new inequality:

$$\begin{aligned}(1.2) \quad & \int_a^b g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,dt \\ & \leq \Big(\int_a^b g(t)[h(t)]^x[f(t)]^v dt\Big)^{\frac{1}{p}} \Big(\int_a^b g(t)[h(t)]^y[f(t)]^u dt\Big)^{\frac{1}{q}},\end{aligned}$$

in which $x, y, v, u \in \mathbb{R}$ and g, f, h are nonnegative real integrable functions such that the involved integrals in (1.2) exist.

P.K. Bhandari and S.K. Bissu in [3] applied the inequality (1.2) to establish inequalities for some well-known special functions.

In this paper, we intend to give refinements for inequality (1.1) and (1.2). This is done in Section 2. In section 3, we apply the result obtained in Section 2 to provide refinements to certain inequalities recently obtained by P.K. Bhandari and S.K. Bissu in [3] for the incomplete gamma function, Polygamma functions, Exponential integral function, Abramowitz function, Hurwitz-Lerch zeta function and for the normalizing constant of the generalized inverse Gaussian distribution and the Remainder of the Binet's first formula for $\ln\Gamma(x)$.

2. Some refinements to Hölder's inequality

M. Akkouchi and M.A. Ighachane in [2] proved the following refinements to Hölder's inequality:

THEOREM 2.1 ([2]). *Let f_1 and f_2 be real or complex measurable functions on Ω such that $||f_1||_p \neq 0$ and $||f_2||_q \neq 0$. Then for all integers $n \geq 2$ we have:*

$$\begin{aligned}(2.1) \quad & \int_\Omega |f_1(x)f_2(x)|d\mu(x) \leq \left(\frac{1}{p^n}+\frac{1}{q^n}\right)||f_1||_p||f_2||_q \\ & + \sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^kq^{n-k}}||f_1||_p^{1-\frac{kp}{n}}||f_2||_q^{1-\frac{(n-k)q}{n}}\int_\Omega |f_1(x)|^{\frac{kp}{n}}|f_2(x)|^{\frac{(n-k)q}{n}}d\mu(x) \\ & \leq ||f_1||_p||f_2||_q,\end{aligned}$$

where $\binom{n}{k} := \frac{n!}{(n-k)!k!}$ is the usual binomial coefficient, for all $k \in \{0, 1, 2, \dots n\}$.

As an application of Theorem 2.1, we obtain the following refinements to the inequality (1.1).

THEOREM 2.2. *Let $x, y, v, u \in \mathbb{R}$ and f, h be real and nonnegative integrable functions on Ω. For almost all $t \in \Omega$, set $f_1(t) = [h(t)]^{x/p}[f(t)]^{v/p}$ and $f_2(t) = [h(t)]^{y/q}[f(t)]^{u/q}$. Suppose that $||f_1||_p \neq 0$ and $||f_2||_q \neq 0$. Then we have:*

$$\int_\Omega [h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,d\mu(t) \le \left(\frac{1}{p^n}+\frac{1}{q^n}\right)||f_1||_p||f_2||_q$$
$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}||f_1||_p^{1-\frac{kp}{n}}||f_2||_q^{1-\frac{(n-k)q}{n}}$$
$$\times\int_\Omega \left([h(t)]^x[f(t)]^v\right)^{\frac{k}{n}}\left([h(t)]^y[f(t)]^u\right)^{\frac{(n-k)}{n}}d\mu(t)$$
$$\le \left[\int_\Omega [h(t)]^x[f(t)]^v dt\right]^{\frac{1}{p}}\left[\int_\Omega [h(t)]^y[f(t)]^u d\mu(t)\right]^{\frac{1}{q}}.$$

PROOF. We apply Theorem 2.1 to the measurable functions $f_1(t) = [h(t)]^{x/p}[f(t)]^{v/p}$ and $f_2(t) = [h(t)]^{y/q}[f(t)]^{u/q}$. □

As a consequence, we have the following corollary.

COROLLARY 2.1. *Let $x, y, v, u \in \mathbb{R}$ and let g, f, h be (real) nonnegative and continuous functions on the interval $[a, b]$. For all $t \in [a, b]$, set $f_1(t) = [h(t)]^{x/p}[f(t)]^{v/p}$ and $f_2(t) = [h(t)]^{y/q}[f(t)]^{u/q}$, and suppose that $||f_1||_p \neq 0$ and $||f_2||_q \neq 0$. Then we have:*

$$\int_a^b g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,dt \le \left(\frac{1}{p^n}+\frac{1}{q^n}\right)||f_1||_p||f_2||_q$$
$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}||f_1||_p^{1-\frac{kp}{n}}||f_2||_q^{1-\frac{(n-k)q}{n}}$$
$$\times\int_a^b g(t)\left([h(t)]^x[f(t)]^v\right)^{\frac{k}{n}}\left([h(t)]^y[f(t)]^u\right)^{\frac{(n-k)}{n}}dt$$
$$\le \left[\int_a^b g(t)[h(t)]^x[f(t)]^v dt\right]^{\frac{1}{p}}\left[\int_a^b g(t)[h(t)]^y[f(t)]^u\,dt\right]^{\frac{1}{q}}.$$

3. The Results

In this section, we apply Theorem 2.2 to refine some inequalities established by P.K. Bhandari and S.K. Bissu in [3] for several well-known special functions.

3.1. Refinements of an inequality for the Polygamma function

We use $\Gamma(x)$ to designate the usual gamma function. The Psi function is defined for all $x > 0$, by $\Psi(x) := \frac{d}{dx}\ln(\Gamma(x))$. For every positive integer m, the Polygamma function $\Psi^{(m)}$ (see [9]) has the following integral representation:

$$\Psi^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{t^m}{1-e^{-t}}e^{-xt}dt, \quad m = 1, 2, 3, \ldots$$

For the sequel, for every positive number β, we use the following notation:

$$\Psi_\beta(x) := \int_0^\infty \frac{t^\beta}{1-e^{-t}}e^{-xt}dt, \quad \forall x > 0.$$

In [3], the following result was established.

THEOREM 3.1 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If $\Psi^{(m)}$ denotes the Polygamma function, then for all real numbers $x, y \in (0, \infty)$ and for all integers $u, v \geq 1$ such that $v/p + u/q$ is an integer, we have:*

$$\left|\Psi^{(\frac{v}{p}+\frac{u}{q})}\left(\frac{x}{p}+\frac{y}{q}\right)\right| \leq \left|\Psi^{(v)}(x)\right|^{\frac{1}{p}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}}. \tag{3.1}$$

We point out that the same arguments of proof in [3] show that the following result is true.

COROLLARY 3.1. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. Then for all real numbers $x, y \in (0, \infty)$ and for all positive numbers u, v, we have:*

$$\Psi_{\frac{v}{p}+\frac{u}{q}}\left(\frac{x}{p}+\frac{y}{q}\right) \leq \Psi_v(x)^{\frac{1}{p}}\Psi_u(y)^{\frac{1}{q}}.$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.1):

THEOREM 3.2. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If $\Psi^{(m)}$ denotes the Polygamma function, then for all real number $x, y \in (0, \infty)$, for all integers $u, v \geq 1$ such that $v/p + u/q$ is an integer and all integers $n \geq 2$, we have:*

$$\left|\Psi^{(\frac{v}{p}+\frac{u}{q})}\left(\frac{x}{p}+\frac{y}{q}\right)\right| \leq \left(\frac{1}{p^n}+\frac{1}{q^n}\right)\left|\Psi^{(v)}(x)\right|^{\frac{1}{p}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\left|\Psi^{(v)}(x)\right|^{\frac{1}{p}-\frac{k}{n}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}-\frac{(n-k)}{n}}\left|\Psi_{\frac{k}{n}v+\frac{(n-k)}{n}u}\right|\left(\frac{kx+(n-k)y}{n}\right)$$

$$\leq \left|\Psi^{(v)}(x)\right|^{\frac{1}{p}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}}.$$

PROOF. We apply Theorem 2.2, by taking $\Omega := (0, +\infty)$ equipped with the measure $d\mu(t) := g(t)dt$, where $g(t) = \frac{1}{1-e^{-t}}$, and considering the measurable functions $f(t) = t$ and $h(t) = e^{-t}$. We have the following equalities:

$$\left|\Psi^{(\frac{v}{p}+\frac{u}{q})}\left(\frac{x}{p}+\frac{y}{q}\right)\right| = \int_0^{+\infty}\frac{t^{\frac{v}{p}+\frac{u}{q}}}{1-e^{-t}}e^{-(\frac{x}{p}+\frac{y}{q})t}\,dt$$

$$= \int_0^{+\infty} g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,dt,$$

$$\left|\Psi^{(v)}(x)\right|^{1/p} = \left(\int_0^{\infty}\frac{t^v}{1-e^{-t}}e^{-tx}dt\right)^{1/p} = \left(\int_0^{+\infty} g(t)[h(t)]^x[f(t)]^v dt\right)^{\frac{1}{p}},$$

$$\left|\Psi^{(u)}(y)\right|^{1/q} = \left(\int_0^{\infty}\frac{t^u}{1-e^{-t}}e^{-ty}dt\right)^{1/q} = \left(\int_0^{+\infty} g(t)[h(t)]^y[f(t)]^u dt\right)^{\frac{1}{q}}$$

and

$$\Psi_{\frac{k}{n}v+\frac{(n-k)}{n}u}\left(\frac{k}{n}x+\frac{(n-k)}{n}y\right) = \int_0^{+\infty}\frac{1}{1-e^{-t}}\left(e^{-tx}t^v\right)^{\frac{k}{n}}\left(e^{-ty}t^u\right)^{\frac{(n-k)}{n}}\,dt$$

$$= \int_0^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}}\,dt.$$

Therefore, by Theorem 2.2 we obtain

$$\left|\Psi^{(\frac{v}{p}+\frac{u}{q})}\left(\frac{x}{p}+\frac{y}{q}\right)\right| \leq \left(\frac{1}{p^n}+\frac{1}{q^n}\right)\left|\Psi^{(v)}(x)\right|^{\frac{1}{p}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\left|\Psi^{(v)}(x)\right|^{\frac{1}{p}-\frac{k}{n}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}-\frac{(n-k)}{n}}\Psi_{\frac{k}{n}v+\frac{(n-k)}{n}u}\left(\frac{k}{n}x+\frac{(n-k)}{n}y\right)$$

$$\leq \left|\Psi^{(v)}(x)\right|^{\frac{1}{p}}\left|\Psi^{(u)}(y)\right|^{\frac{1}{q}}.$$

This ends the proof. □

As a consequence, we have the following result concerning the associated functions Ψ_β $(\beta > 0)$.

THEOREM 3.3. *Let $p, q > 1$ be real numbers satisfying $1/p+1/q = 1$. Then for all real numbers $x, y \in (0, \infty)$ and for all positive numbers u, v, we have:*

$$\Psi_{\frac{v}{p}+\frac{u}{q}}\left(\frac{x}{p}+\frac{y}{q}\right) \leq \left(\frac{1}{p^n}+\frac{1}{q^n}\right)\Psi_v(x)^{\frac{1}{p}}\Psi_u(y)^{\frac{1}{q}}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\Psi_v(x)^{\frac{1}{p}-\frac{k}{n}}\Psi_u(y)^{\frac{1}{q}-\frac{(n-k)}{n}}\Psi_{\frac{k}{n}v+\frac{(n-k)}{n}u}\left(\frac{kx+(n-k)y}{n}\right)$$

$$\leq \Psi_v(x)^{\frac{1}{p}}\Psi_u(y)^{\frac{1}{q}}.$$

3.2. A refined inequality for the incomplete gamma function

We recall (see for example [6]) that the incomplete gamma function is defined for $u, x > 0$ as

$$\gamma(u, x) := \int_0^x t^{u-1}e^{-t}\,dt.$$

It is easy to observe that $\gamma(u, x)$ is given by the following integral:

$$\gamma(u, x) = x^u \int_0^1 t^{u-1}e^{-xt}\,dt.$$

P.K. Bhandari and S.K. Bissu ([3]) proved the following result.

THEOREM 3.4 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If γ denotes the incomplete gamma function, then for all real numbers $x, y \in (0, \infty)$ and for all $u, v \geq 0$, we have:*

$$\gamma\Big(\frac{v}{p} + \frac{u}{q}, \frac{x}{p} + \frac{y}{q}\Big) \leq \frac{(\frac{x}{p} + \frac{y}{q})^{(\frac{v}{p} + \frac{u}{q})}}{x^{\frac{v}{p}} y^{\frac{u}{q}}} \Big[\gamma(v, x)\Big]^{\frac{1}{p}} \Big[\gamma(u, y)\Big]^{\frac{1}{q}}. \tag{3.2}$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.2):

THEOREM 3.5. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If γ denotes the incomplete gamma function, then for all real numbers $x, y \in (0, \infty)$, for all $u, v \geq 0$ and all integers $n \geq 2$, we have:*

$$\begin{aligned}
\gamma\Big(\frac{v}{p} + \frac{u}{q}, \frac{x}{p} + \frac{y}{q}\Big) &\leq \Big(\frac{1}{p^n} + \frac{1}{q^n}\Big) \frac{(\frac{x}{p} + \frac{y}{q})^{(\frac{v}{p} + \frac{u}{q})}}{x^{\frac{v}{p}} y^{\frac{u}{q}}} \Big[\gamma(v, x)\Big]^{\frac{1}{p}} \Big[\gamma(u, y)\Big]^{\frac{1}{q}} \\
&+ \frac{(\frac{x}{p} + \frac{y}{q})^{(\frac{v}{p} + \frac{u}{q})}}{x^{\frac{v}{p}} y^{\frac{u}{q}}} \Bigg[\sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{p^k q^{n-k}} \Big[\gamma(v, x)\Big]^{\frac{1}{p} - \frac{k}{n}} \Big[\gamma(u, y)\Big]^{\frac{1}{q} - \frac{(n-k)}{n}} \\
&\times \frac{x^{\frac{kv}{n}} y^{\frac{(n-k)u}{n}}}{(\frac{k}{n} x + \frac{(n-k)}{n} y)^{\frac{k}{n} v + \frac{(n-k)}{n} u}} \gamma\left(\frac{k}{n} v + \frac{(n-k)}{n} u, \frac{k}{n} x + \frac{(n-k)}{n} y\right)\Bigg] \\
&\leq \frac{(\frac{x}{p} + \frac{y}{q})^{(\frac{v}{p} + \frac{u}{q})}}{x^{\frac{v}{p}} y^{\frac{u}{q}}} \Big[\gamma(v, x)\Big]^{\frac{1}{p}} \Big[\gamma(u, y)\Big]^{\frac{1}{q}}.
\end{aligned}$$

PROOF. We apply Theorem 2.2, by setting $\Omega := (0, 1)$ and considering the measure $d\mu(t) := g(t)dt$ with $g(t) = \frac{1}{t}$ for all $t \in (0, 1)$ and choosing the functions $f(t) = t$ and $h(t) = e^{-t}$. We have the following equalities:

$$\begin{aligned}
\frac{\gamma\Big((\frac{v}{p} + \frac{u}{q}), (\frac{x}{p} + \frac{y}{q})\Big)}{(\frac{x}{p} + \frac{y}{q})^{(\frac{v}{p} + \frac{u}{q})}} &= \int_0^1 \frac{1}{t} e^{-t(\frac{x}{p} + \frac{y}{q})} t^{\frac{v}{p} + \frac{u}{q}} \, dt \\
&= \int_0^1 g(t) [h(t)]^{\frac{x}{p} + \frac{y}{q}} [f(t)]^{\frac{v}{p} + \frac{u}{q}} \, dt,
\end{aligned}$$

$$\Big(\int_0^1 g(t) [h(t)]^x [f(t)]^v dt\Big)^{\frac{1}{p}} = \Big(\int_0^1 \frac{1}{t} e^{-xt} t^v dt\Big)^{1/p} = \frac{1}{x^{\frac{v}{p}}} \Big[\gamma(v, x)\Big]^{1/p},$$

$$\left(\int_0^1 g(t)[h(t)]^y[f(t)]^v dt\right)^{\frac{1}{q}} = \left(\int_0^1 \frac{1}{t}e^{-yt}t^u dt\right)^{1/q} = \frac{1}{y^{\frac{u}{q}}}\Big[\gamma(u,y)\Big]^{1/q}$$

and

$$\int_0^1 g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}} dt$$

$$= \int_0^1 \frac{1}{t}\Big(e^{-xt}t^v\Big)^{\frac{k}{n}}\Big(e^{-yt}t^u\Big)^{\frac{(n-k)}{n}} dt$$

$$= \frac{1}{(\frac{k}{n}x + \frac{(n-k)}{n}y)^{(\frac{k}{n}v + \frac{(n-k)}{n}y)}}\gamma\left(\Big(\frac{k}{n}v + \frac{(n-k)}{n}u\Big), \Big(\frac{k}{n}x + \frac{(n-k)}{n}y\Big)\right).$$

Therefore, by virtue of Theorem 2.2 and after some easy computations, we obtain

$$\frac{1}{(\frac{x}{p}+\frac{y}{q})^{(\frac{v}{p}+\frac{u}{q})}}\gamma\Big(\Big(\frac{v}{p}+\frac{u}{q}\Big),\Big(\frac{x}{p}+\frac{y}{q}\Big)\Big) \le \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\frac{1}{x^{\frac{v}{p}}y^{\frac{u}{q}}}\Big[\gamma(v,x)\Big]^{\frac{1}{p}}\Big[\gamma(u,y)\Big]^{\frac{1}{q}}$$

$$+ \frac{1}{x^{\frac{v}{p}}y^{\frac{u}{q}}}\left[\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\Big[\gamma(v,x)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[\gamma(u,y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}}\right.$$

$$\left.\times \frac{x^{\frac{kv}{n}}y^{\frac{(n-k)u}{n}}}{(\frac{k}{n}x+\frac{(n-k)}{n}y)^{\frac{k}{n}v+\frac{(n-k)}{n}u}}\gamma\Big(\frac{k}{n}v+\frac{(n-k)}{n}u, \frac{k}{n}x+\frac{(n-k)}{n}y\Big)\right]$$

$$\le \frac{1}{x^{\frac{v}{p}}y^{\frac{u}{q}}}\Big[\gamma(v,x)\Big]^{\frac{1}{p}}\Big[\gamma(u,y)\Big]^{\frac{1}{q}},$$

which is the desired inequality. □

3.3. Refinements to certain inequalities for the exponential integral functions

We recall (see [1]) that the Exponential function E_n is given by the following integral representation:

$$E_n(x) = \int_1^{+\infty} e^{-xt}t^{-n}dt, \quad n = 0, 1, 2, \ldots, x > 0.$$

For the sequel, we need to extend the definition above for all nonnegative real numbers. So, we consider the functions E_β ($\beta \in \mathbb{R}^+$) defined by

$$E_\beta(x) = \int_1^{+\infty} e^{-xt} t^{-\beta} dt, \quad \forall \beta \in [0, +\infty), \ \forall x > 0.$$

We call them *Exponential integral type functions* or more simply *Exponential integral functions.*

P.K. Bhandari and S.K. Bissu [3] proved the following result.

THEOREM 3.6 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If E_n denotes the Exponential integral function, then for all real number $x, y \in (0, \infty)$ and for all integers $u, v \geq 0$ such that $v/p + u/q$ is integer, we have:*

$$\left[E_{\frac{v}{p}+\frac{u}{q}}\left(\frac{x}{p} + \frac{y}{q}\right)\right] \leq \left[E_v(x)\right]^{\frac{1}{p}} \left[E_u(y)\right]^{\frac{1}{q}}. \tag{3.3}$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.3):

THEOREM 3.7. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If E_β denotes the Exponential integral function, then for all real number $x, y \in (0, \infty)$, for all nonnegative real numbers u, v and all integers $n \geq 2$, we have:*

$$E_{(\frac{v}{p}+\frac{u}{q})}\left(\frac{x}{p} + \frac{y}{q}\right) \leq \left(\frac{1}{p^n} + \frac{1}{q^n}\right) \left[E_v(x)\right]^{\frac{1}{p}} \left[E_u(y)\right]^{\frac{1}{q}}$$

$$+ \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{p^k q^{n-k}} \left[E_v(x)\right]^{\frac{1}{p}-\frac{k}{n}} \left[E_u(y)\right]^{\frac{1}{q}-\frac{(n-k)}{n}} E_{\frac{k}{n}v+\frac{(n-k)}{n}u}\left(\frac{k}{n}x + \frac{(n-k)}{n}y\right)$$

$$\leq \left[E_v(x)\right]^{\frac{1}{p}} \left[E_u(y)\right]^{\frac{1}{q}}.$$

PROOF. We apply Theorem 2.2, by setting $\Omega := [1, +\infty)$ and considering the measure $d\mu(t) := g(t)dt$, where $g(t) = 1$, for all $t \in [1, +\infty)$ and taking $f(t) = t^{-1}$ and $h(t) = e^{-t}$ for all $t \in [1, +\infty)$. We have:

$$E_{\frac{v}{p}+\frac{u}{q}}\left(\frac{x}{p} + \frac{y}{q}\right) = \int_1^{+\infty} e^{-(\frac{x}{p}+\frac{y}{q})t} t^{-(\frac{v}{p}+\frac{u}{q})}\, dt = \int_1^{+\infty} g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}} [f(t)]^{\frac{v}{p}+\frac{u}{q}}\, dt,$$

$$\left[E_v(x)\right]^{1/p} = \left(\int_1^{\infty} e^{-tx} t^{-v} dt\right)^{1/p} = \left(\int_1^{+\infty} g(t)[h(t)]^x [f(t)]^v dt\right)^{\frac{1}{p}},$$

$$\Big[E_u(y)\Big]^{1/q} = \Big(\int_1^{\infty} e^{-ty} t^{-u}\, dt\Big)^{1/q} = \Big(\int_1^{+\infty} g(t)[h(t)]^y [f(t)]^u dt\Big)^{\frac{1}{q}}$$

and

$$\begin{aligned} E_{\frac{k}{n}v+\frac{(n-k)}{n}u}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big) &= \int_1^{+\infty} \Big(e^{-tx}t^{-v}\Big)^{\frac{k}{n}} \Big(e^{-ty}t^{-u}\Big)^{\frac{(n-k)}{n}} dt \\ &= \int_1^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}} \Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}} dt. \end{aligned}$$

Then by Theorem 2.2, we have

$$\begin{aligned} & E_{\frac{v}{p}+\frac{u}{q}}\Big(\frac{x}{p}+\frac{y}{q}\Big) \le \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\Big[E_v(x)\Big]^{\frac{1}{p}}\Big[E_u(y)\Big]^{\frac{1}{q}} \\ & + \sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\Big[E_v(x)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[E_u(y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}} E_{\frac{k}{n}v+\frac{(n-k)}{n}u}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big) \\ & \le \Big[E_v(x)\Big]^{\frac{1}{p}}\Big[E_u(y)\Big]^{\frac{1}{q}}, \end{aligned}$$

which is the desired inequality. □

3.4. A refined inequality for the Hurwitz-Lerch zeta function

We recall (see [5]) that the integral representation of the Hurwitz-Lerch zeta function $\Phi(\lambda, x, a)$ for all real numbers $x > 0$, $a > 0$ and $|\lambda| \le 1$ with $\lambda \neq 1$ is given by

$$\Phi(\lambda, x, a) = \frac{1}{\Gamma(x)}\int_0^{+\infty} \frac{t^{x-1}}{1-\lambda e^{-t}} e^{-at}\, dt.$$

When $\lambda = 1$ is turns into Hurwitz zeta function, $\Phi(\lambda, x, a) = \zeta(x, a)$.

The following inequality was established in [3].

THEOREM 3.8 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If Φ denotes the Hurwitz-Lerch zeta function, then for all real numbers $x, y, u, v, a \in (0, \infty)$ and $|\lambda| \le 1$, $\lambda \neq 1$, we have:*

$$\Phi\Big(\lambda, \frac{x}{p}+\frac{y}{q}, \Big(\frac{v}{p}+\frac{u}{q}\Big)a\Big) \le \frac{\Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}}{\Gamma(\frac{x}{p}+\frac{y}{q})}\Big[\Phi(\lambda, x, va)\Big]^{\frac{1}{p}}\Big[\Phi(\lambda, y, ua)\Big]^{\frac{1}{q}}. \tag{3.4}$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.4):

THEOREM 3.9. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If Φ denotes the Hurwitz-Lerch zeta function, then for all real numbers $x, y, u, v, a \in (0, \infty)$, $|\lambda| \leq 1$, $\lambda \neq 1$, and all integers $n \geq 2$, we have:*

$$\begin{aligned}
&\Phi\Big(\lambda, \frac{x}{p} + \frac{y}{q}, \Big(\frac{v}{p} + \frac{u}{q}\Big)a\Big) \\
&\qquad \leq \Big(\frac{1}{p^n} + \frac{1}{q^n}\Big) \frac{\Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}}{\Gamma(\frac{x}{p} + \frac{y}{q})} \Big[\Phi(\lambda, x, va)\Big]^{\frac{1}{p}} \Big[\Phi(\lambda, y, ua)\Big]^{\frac{1}{q}} \\
&+ \frac{\Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}}{\Gamma(\frac{x}{p} + \frac{y}{q})} \Bigg(\sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{p^k q^{n-k}} \Big[\Phi(\lambda, x, va)\Big]^{\frac{1}{p} - \frac{k}{n}} \Big[\Phi(\lambda, y, ua)\Big]^{\frac{1}{q} - \frac{(n-k)}{n}} \\
&\qquad \times \frac{\Gamma(\frac{k}{n}x + \frac{(n-k)}{n}y)}{\Gamma^{\frac{k}{n}}(x)\Gamma^{\frac{n-k}{n}}(y)} \Phi\Big(\lambda, \frac{k}{n}x + \frac{(n-k)}{n}y, \Big(\frac{k}{n}v + \frac{(n-k)}{n}u\Big)a\Big)\Bigg) \\
&\qquad\qquad \leq \frac{\Big[\Gamma(x)\Big]^{\frac{1}{p}}\Big[\Gamma(y)\Big]^{\frac{1}{q}}}{\Big[\Gamma(\frac{x}{p} + \frac{y}{q})\Big]} \Big[\Phi(\lambda, x, va)\Big]^{\frac{1}{p}} \Big[\Phi(\lambda, y, ua)\Big]^{\frac{1}{q}}.
\end{aligned}$$

PROOF. We apply Theorem 2.2, by setting $\Omega := (0, +\infty)$ and considering the measure $d\mu(t) := g(t)\,dt$ with $g(t) = \frac{1}{t(1-\lambda e^{-t})}$ for all $t \in (0, \infty)$, and choosing $h(t) = t$ and $f(t) = e^{-at}$, for all $t \in (0, \infty)$.

By some easy computations, we have the following successive equalities:

$$\begin{aligned}
\Gamma\Big(\frac{x}{p} + \frac{y}{q}\Big)\Phi\Big(\lambda, \frac{x}{p} + \frac{y}{q}, \Big(\frac{v}{p} + \frac{u}{q}\Big)a\Big) &= \int_0^{+\infty} \frac{t^{(\frac{x}{p} + \frac{y}{q})}}{t(1 - \lambda e^{-t})} e^{-(\frac{v}{p} + \frac{u}{q})a}\,dt \\
&= \int_0^{\infty} g(t)[h(t)]^{\frac{x}{p} + \frac{y}{q}} [f(t)]^{\frac{v}{p} + \frac{u}{q}}\,dt,
\end{aligned}$$

$$\begin{aligned}
\Big[\Gamma(x)\Big]^{1/p}\Big[\Phi(\lambda, x, va)\Big]^{1/p} &= \Big(\int_0^{\infty} \frac{t^x}{t(1 - \lambda e^{-t})} e^{-vat} dt\Big)^{1/p} \\
&= \Big(\int_0^{+\infty} g(t)[h(t)]^x [f(t)]^v dt\Big)^{\frac{1}{p}},
\end{aligned}$$

$$\Big[\Gamma(y)\Big]^{1/q}\Big[\Phi(\lambda,y,ua)\Big]^{1/q} = \Big(\int_0^{\infty}\frac{t^y}{t(1-\lambda e^{-t})}e^{-uat}dt\Big)^{1/q}$$
$$= \Big(\int_0^{+\infty} g(t)[h(t)]^y[f(t)]^u dt\Big)^{\frac{1}{q}}$$

and

$$\Big[\Gamma\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)\Big]\Phi\Big(\lambda,\frac{k}{n}x+\frac{(n-k)}{n}y,\Big(\frac{k}{n}v+\frac{(n-k)}{n}u\Big)a\Big)$$
$$= \int_0^{+\infty}\frac{1}{t(1-\lambda e^{-t})}\Big(t^x e^{-atv}\Big)^{\frac{k}{n}}\Big(t^y e^{-atu}\Big)^{\frac{(n-k)}{n}}dt$$
$$= \int_0^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}}dt.$$

Then by virtue of Theorem 2.2, we have

$$\Phi\Big(\lambda,\frac{x}{p}+\frac{y}{q},\Big(\frac{v}{p}+\frac{u}{q}\Big)a\Big)$$
$$\leq \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\frac{\Big[\Gamma(x)\Big]^{\frac{1}{p}}\Big[\Gamma(y)\Big]^{\frac{1}{q}}}{\Big[\Gamma(\frac{x}{p}+\frac{y}{q})\Big]}\Big[\Phi(\lambda,x,va)\Big]^{\frac{1}{p}}\Big[\Phi(\lambda,y,ua)\Big]^{\frac{1}{q}}$$
$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^kq^{n-k}}\frac{\Big[\Gamma(x)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[\Gamma(y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}}}{\Big[\Gamma(\frac{x}{p}+\frac{y}{q})\Big]}$$
$$\times\Big[\Phi(\lambda,x,va)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[\Phi(\lambda,y,ua)\Big]^{\frac{1}{p}-\frac{(n-k)}{n}}$$
$$\times\Big[\Gamma\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)\Big]\Phi\Big(\lambda,\frac{k}{n}x+\frac{(n-k)}{n}y,\Big(\frac{k}{n}v+\frac{(n-k)}{n}u\Big)a\Big)$$
$$\leq \frac{\Big[\Gamma(x)\Big]^{\frac{1}{p}}\Big[\Gamma(y)\Big]^{\frac{1}{q}}}{\Big[\Gamma(\frac{x}{p}+\frac{y}{q})\Big]}\Big[\Phi(\lambda,x,va)\Big]^{\frac{1}{p}}\Big[\Phi(\lambda,y,ua)\Big]^{\frac{1}{q}},$$

which is the desired inequality. □

3.5. Refinements of an inequality for the Abramowitz function

The Abramowitz function f_n (see [1]) is given for every nonnegative integer and all nonnegative real number $x \geq 0$, by

$$f_n(x) = \int_0^{+\infty} e^{-t^2} t^n e^{-xt^{-1}}\, dt.$$

We point out that the Abramowitz function has been used in many fields of physics, as the field of particle and radiation transform (see [4] for more details).

For the sequel, we need to extend the definition above to all nonnegative real numbers. So, we consider the functions f_β ($\beta \in \mathbb{R}^+$) defined by

$$f_\beta(x) = \int_0^{+\infty} e^{-t^2} t^\beta e^{-xt^{-1}}\, dt, \quad \forall \beta \in [0, +\infty),\ \forall x \geq 0.$$

We call them *generalized Abramowitz functions*, or more simply, *Abramowitz functions*.

The following inequality was established in [3].

THEOREM 3.10 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If f_n denotes the Abramowitz function, then for all real numbers $x, y \geq 0$ and for all nonnegative integers u, v such that $v/p + u/q$ is an integer, we have:*

$$f_{\frac{v}{p}+\frac{u}{q}}\Big(\frac{x}{p} + \frac{y}{q}\Big) \leq \Big[f_v(x)\Big]^{\frac{1}{p}} \Big[f_u(y)\Big]^{\frac{1}{q}}. \tag{3.5}$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.5):

THEOREM 3.11. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If f_β denotes the Abramowitz function, then for all real numbers $x, y \geq 0$, for all nonnegative integers u, v such that $v/p + u/q$ is an integer and all integers $n \geq 2$, we have:*

$$\begin{aligned} f_{\frac{v}{p}+\frac{u}{q}}\Big(\frac{x}{p} + \frac{y}{q}\Big) &\leq \Big(\frac{1}{p^n} + \frac{1}{q^n}\Big) \Big[f_v(x)\Big]^{\frac{1}{p}} \Big[f_u(y)\Big]^{\frac{1}{q}} \\ &+ \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{p^k q^{n-k}} \Big[f_v(x)\Big]^{\frac{1}{p}-\frac{k}{n}} \Big[f_u(y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}} f_{\frac{k}{n}v+\frac{(n-k)}{n}u}\Big(\frac{k}{n}x + \frac{(n-k)}{n}y\Big) \\ &\leq \Big[f_v(x)\Big]^{\frac{1}{p}} \Big[f_u(y)\Big]^{\frac{1}{q}}. \end{aligned}$$

PROOF. We apply Theorem 2.2, by setting $\Omega := (0, +\infty)$ and considering the measure $d\mu(t) := g(t)dt$, where $g(t) = e^{-t^2}$ for all t in $(0, +\infty)$ and choosing the functions: $f(t) = t$ and $h(t) = e^{-t^{-1}}$ for all $t \in (0, +\infty)$. After some easy computations, we have the following equalities:

$$f_{\frac{v}{p}+\frac{u}{q}}\Big(\frac{x}{p}+\frac{y}{q}\Big) = \int_0^{+\infty} e^{-t^2} e^{-(\frac{x}{p}+\frac{y}{q})t^{-1}} t^{(\frac{v}{p}+\frac{u}{q})}\, dt$$

$$= \int_0^{+\infty} g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\, dt,$$

$$\Big(\int_0^{+\infty} g(t)[h(t)]^x[f(t)]^v dt\Big)^{\frac{1}{p}} = \Big(\int_0^{\infty} e^{-t^2}e^{-xt^{-1}}t^v dt\Big)^{1/p} = \Big[f_v(x)\Big]^{1/p},$$

$$\Big(\int_0^{+\infty} g(t)[h(t)]^y[f(t)]^u dt\Big)^{\frac{1}{q}} = \Big(\int_0^{\infty} e^{-t^2}e^{-yt^{-1}}t^u dt\Big)^{1/q} = \Big[f_u(y)\Big]^{1/q}$$

and

$$f_{\frac{k}{n}v+\frac{(n-k)}{n}u}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)$$

$$= \int_0^{+\infty} e^{-t^2}\Big(e^{-xt^{-1}}t^v\Big)^{\frac{k}{n}}\Big(e^{-yt^{-1}}t^u\Big)^{\frac{(n-k)}{n}} dt$$

$$= \int_0^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}} dt.$$

Then by application of Theorem 2.2, we obtain

$$f_{\frac{v}{p}+\frac{u}{q}}\Big(\frac{x}{p}+\frac{y}{q}\Big) \le \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\Big[f_v(x)\Big]^{\frac{1}{p}}\Big[f_u(y)\Big]^{\frac{1}{q}}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^kq^{n-k}}\Big[f_v(x)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[f_u(y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}} f_{\frac{k}{n}v+\frac{(n-k)}{n}u}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)$$

$$\le \Big[f_v(x)\Big]^{\frac{1}{p}}\Big[f_u(y)\Big]^{\frac{1}{q}},$$

which is the required inequality. $\square$

3.6. A refined inequality for the normalizing constant of the generalized inverse Gaussian distribution

The generalized inverse Gaussian distribution function (see [7]) is defined for all $t > 0$ as

$$g(t) = \frac{1}{I(\alpha, \beta, \gamma)} t^{\alpha-1} e^{-t\beta-\gamma t^{-1}},$$

where $-\infty < \alpha < +\infty$, $\beta > 0$, $\gamma > 0$.

The number $I(\alpha; \beta, \gamma)$ is the normalizing constant, that is:

$$I(\alpha; \beta, \gamma) = \int_0^{+\infty} t^{\alpha-1} e^{-t\beta-\gamma t^{-1}}\, dt.$$

The following inequality was established in [3].

THEOREM 3.12 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If I denotes the normalizing constant of the generalized inverse Gaussian distrubition, then for all real numbers $x, y, u, v \in (0, \infty)$ and $-\infty < \alpha < +\infty$, we have:*

$$I\Big(\alpha; \frac{v}{p} + \frac{u}{q}, \frac{x}{p} + \frac{y}{q}\Big) \leq \Big[I(\alpha; v, x)\Big]^{\frac{1}{p}} \Big[I(\alpha, u, y)\Big]^{\frac{1}{q}}. \tag{3.6}$$

By using Theorem 2.2, we obtain the following refinements of the inequality (3.6):

THEOREM 3.13. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If I denotes the normalizing constant of the generalized inverse Gaussian distrubition, then for all real number $x, y, u, v \in (0, \infty)$, $-\infty < \alpha < +\infty$ and all integers $n \geq 2$, we have:*

$$\begin{aligned} I\Big(\alpha; \frac{v}{p} + \frac{u}{q}, \frac{x}{p} + \frac{y}{q}\Big) &\leq \Big(\frac{1}{p^n} + \frac{1}{q^n}\Big) \Big[I(\alpha; v, x)\Big]^{\frac{1}{p}} \Big[I(\alpha; u, y)\Big]^{\frac{1}{q}} \\ &+ \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{p^k q^{n-k}} \Big[I(\alpha; v, x)\Big]^{\frac{1}{p} - \frac{k}{n}} \Big[I(\alpha; u, y)\Big]^{\frac{1}{q} - \frac{(n-k)}{n}} \\ &\times I\Big(\alpha; \frac{k}{n} v + \frac{(n-k)}{n} u, \frac{k}{n} x + \frac{(n-k)}{n} y\Big) \\ &\leq \Big[I(\alpha; v, x)\Big]^{\frac{1}{p}} \Big[I(\alpha; u, y)\Big]^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

PROOF. We apply Theorem 2.2, by setting $\Omega := (0, +\infty)$ and considering the measure $d\mu(t) := g(t)dt$ with $g(t) = t^{\alpha-1}$ for all t in $(0, \infty)$, and choosing $f(t) = e^{-t}$ and $h(t) = e^{-t^{-1}}$ for all t in $(0, \infty)$. We have:

$$\begin{aligned}\int_0^{\infty} g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,dt &= \int_0^{+\infty} t^{\alpha-1}e^{-t(\frac{v}{p}+\frac{u}{q})}e^{-(\frac{x}{p}+\frac{y}{q})t^{-1}}\,dt\\ &= I\Big(\alpha; \frac{v}{p}+\frac{u}{q}, \frac{x}{p}+\frac{y}{q}\Big),\end{aligned}$$

$$\Big(\int_0^{+\infty} g(t)[h(t)]^x[f(t)]^v dt\Big)^{\frac{1}{p}} = \Big(\int_0^{\infty} t^{\alpha-1}e^{-vt}e^{-xt^{-1}}dt\Big)^{1/p} = \Big[I(\alpha; v, x)\Big]^{1/p},$$

$$\Big(\int_0^{+\infty} g(t)[h(t)]^y[f(t)]^u dt\Big)^{\frac{1}{q}} = \Big(\int_0^{\infty} t^{\alpha-1}e^{-ut}e^{-yt^{-1}}dt\Big)^{1/q} = \Big[I(\alpha; u, y)\Big]^{1/q}$$

and

$$\begin{aligned}&I\Big(\alpha; \frac{k}{n}v + \frac{(n-k)}{n}u, \frac{k}{n}x + \frac{(n-k)}{n}y\Big)\\ &\qquad= \int_0^{+\infty} t^{\alpha-1}\Big(e^{-vt}e^{-xt^{-1}}\Big)^{\frac{k}{n}}\Big(e^{-ut}e^{-yt^{-1}}\Big)^{\frac{(n-k)}{n}}\,dt\\ &\qquad\qquad= \int_0^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}}\,dt.\end{aligned}$$

Therefore, by Theorem 2.2 we have

$$\begin{aligned}I\Big(\alpha; \frac{v}{p}+\frac{u}{q}, \frac{x}{p}+\frac{y}{q}\Big) &\leq \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\Big[I(\alpha; v, x)\Big]^{\frac{1}{p}}\Big[I(\alpha; u, y)\Big]^{\frac{1}{q}}\\ &\quad+ \sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\Big[I(\alpha; v, x)\Big]^{\frac{1}{p}-\frac{k}{n}}\Big[I(\alpha; u, y)\Big]^{\frac{1}{q}-\frac{(n-k)}{n}}\\ &\quad\times I\Big(\alpha; \frac{k}{n}v + \frac{(n-k)}{n}u, \frac{k}{n}x + \frac{(n-k)}{n}y\Big)\\ &\leq \Big[I(\alpha; v, x)\Big]^{\frac{1}{p}}\Big[I(\alpha; u, y)\Big]^{\frac{1}{q}},\end{aligned}$$

which is the desired inequality (3.7). $\square$

3.7. A refined inequality for the n-th derivative of the remainder of the Binet's first Formula for $\ln\Gamma(x)$

The Binet's first formula for $\ln\Gamma(x)$ is given by

$$\ln\Gamma(x) = (x-1/2)\log(x) - x + \log(\sqrt{2\pi}) + \theta(x)$$

for all number $x > 0$, where the function

$$\theta(x) := \int_0^{+\infty} \Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)\frac{e^{-xt}}{t}\,dt$$

is known as the the remainder of the Binet's first Formula for $\ln\Gamma(x)$ (see for example the handbook [1]).

By making derivatives, we obtain for every positive integer $m \geq 1$:

$$\theta^{(m)}(x) = (-1)^m \int_0^{+\infty} \Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)t^{m-1}e^{-xt}dt := (-1)^m\xi_m(x),$$

where the function

$$\xi_\beta(x) := \int_0^{+\infty} \Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)t^{\beta-1}e^{-xt}\,dt$$

is defined for all number $\beta \geq 1$ and for all positive number $x > 0$.

The following result was established in [3].

Theorem 3.14 ([3]). *Let $p, q > 1$ be real numbers satisfying $1/p+1/q = 1$. If $\theta^{(m)}$ denotes the m-th derivative of the remainder of the Binet's first formula for the logarithm of the function, i.e. $\ln\Gamma(x)$, then for all real numbers $x, y \in (0,\infty)$ and for all integers $u, v \geq 1$ such that $m := v/p + u/q$ is an integer, we have:*

$$\left|\theta^{(\frac{v}{p}+\frac{u}{q})}\Big(\frac{x}{p}+\frac{y}{q}\Big)\right| \leq \left|\theta^{(v)}(x)\right|^{\frac{1}{p}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}}. \tag{3.8}$$

Before giving our result, we need the following lemma.

Lemma 3.1. *For all $t \in (0,+\infty)$, we have $\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} > 0$.*

Proof. For all $t \in (0, +\infty)$, we have

$$\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} = \frac{te^t - 2e^t + t + 2}{2t(e^t - 1)} := \frac{u(t)}{v(t)}.$$

The function u is defined on $[0, +\infty)$ and it is indefinitely differentiable therein. We have $u'(t) = te^t - e^t + 1$ and $u''(t) = te^t > 0$, for all $t > 0$. This shows that u' is increasing on $[0, +\infty)$. Since $u'(0) = 0$, we infer that $u'(t) > u'(0) = 0$ for all $t > 0$. Therefore, u is increasing on $[0, +\infty)$. Then we get $u(t) > 0 = u(0)$ for all $t > 0$. This ends the proof. □

By using Theorem 2.2 and the lemma above, we obtain the following refinements of the inequality (3.8):

Theorem 3.15. *Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$. If $\theta^{(m)}$ denotes the $m-$th derivative of the remainder of the Binet's first formula for the logarithm of the function, i.e. $\ln \Gamma(x)$, then for all real numbers $x, y \in (0, \infty)$, for all integers $u, v \geq 1$ such that $m := v/p + u/q$ is an integer and all integers $n \geq 2$, we have:*

$$\left|\theta^{(\frac{v}{p}+\frac{u}{q})}\Big(\frac{x}{p}+\frac{y}{q}\Big)\right| \leq \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\left|\theta^{(v)}(x)\right|^{\frac{1}{p}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}} \tag{3.9}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^k q^{n-k}}\left|\theta^{(v)}(x)\right|^{\frac{1}{p}-\frac{k}{n}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}-\frac{(n-k)}{n}}\xi_{(\frac{k}{n}v+\frac{(n-k)}{n}u)}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)$$

$$\leq \left|\theta^{(v)}(x)\right|^{\frac{1}{p}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}}.$$

Proof. We apply Theorem 2.2, by setting $\Omega := (0, +\infty)$ and considering the measure $d\mu(t) := g(t)dt$, where $g(t) = \frac{1}{t}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)$, for all t in $(0, +\infty)$ and choosing $f(t) = t$ and $h(t) = e^{-t}$, for all $t \in (0, +\infty)$. Note that

$$\begin{aligned}\left|\theta^{(\frac{v}{p}+\frac{u}{q})}\Big(\frac{x}{p}+\frac{y}{q}\Big)\right| &= \int_0^{+\infty}\frac{1}{t}\Big(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\Big)t^{(\frac{v}{p}+\frac{u}{q})}e^{-t(\frac{x}{p}+\frac{y}{q})}dt \\ &= \int_0^{+\infty} g(t)[h(t)]^{\frac{x}{p}+\frac{y}{q}}[f(t)]^{\frac{v}{p}+\frac{u}{q}}\,dt,\end{aligned}$$

$$\left|\theta^{(v)}(x)\right|^{1/p} = \left(\int_0^{\infty} \frac{1}{t}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)e^{-tx}t^v dt\right)^{1/p}$$

$$= \left(\int_0^{+\infty} g(t)[h(t)]^x[f(t)]^v dt\right)^{\frac{1}{p}},$$

$$\left|\theta^{(u)}(y)\right|^{1/q} = \left(\int_0^{\infty} \frac{1}{t}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)e^{-ty}t^u dt\right)^{1/q}$$

$$= \left(\int_0^{+\infty} g(t)[h(t)]^y[f(t)]^u dt\right)^{\frac{1}{q}}$$

and

$$\xi_{(\frac{k}{n}v+\frac{(n-k)}{n}u)}\Big(\frac{k}{n}x + \frac{(n-k)}{n}y\Big)$$

$$= \int_0^{+\infty} \frac{1}{t}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\Big)\Big(e^{-tx}t^v\Big)^{\frac{k}{n}}\Big(e^{-ty}t^u\Big)^{\frac{(n-k)}{n}} dt$$

$$= \int_0^{+\infty} g(t)\Big([h(t)]^x[f(t)]^v\Big)^{\frac{k}{n}}\Big([h(t)]^y[f(t)]^u\Big)^{\frac{(n-k)}{n}} dt.$$

Therefore, by Theorem 2.2 we have

$$\left|\theta^{(\frac{v}{p}+\frac{u}{q})}\Big(\frac{x}{p}+\frac{y}{q}\Big)\right| \le \Big(\frac{1}{p^n}+\frac{1}{q^n}\Big)\left|\theta^{(v)}(x)\right|^{\frac{1}{p}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}}$$

$$+\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{p^kq^{n-k}}\left|\theta^{(v)}(x)\right|^{\frac{1}{p}-\frac{k}{n}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}-\frac{(n-k)}{n}}\xi_{(\frac{k}{n}v+\frac{(n-k)}{n}u)}\Big(\frac{k}{n}x+\frac{(n-k)}{n}y\Big)$$

$$\le \left|\theta^{(v)}(x)\right|^{\frac{1}{p}}\left|\theta^{(u)}(y)\right|^{\frac{1}{q}}.$$

which is the desired inequality (3.9). □

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